Guarding Orthogonal Terrains

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Abstract

A *1.5-dimensional terrain* T with n vertices is an x-monotone polygonal chain in the plane. A point guard p on T guards a point q of T if the line segment connecting p to q lies on or above T; p is a *vertex guard* if it is a vertex of T. In the *Optimal Terrain Guarding (OTG) problem* on T, the objective is to guard the vertices of T by the minimum number of vertex guards. King and Krohn [9] showed that the OTG problem is NP-hard on arbitrary terrains, and Gibson et al. [6] gave a PTAS for this problem. In this paper, we introduce *directed visibility* in which the visibility is directed only at adjacent vertices. We give an $O(n)$-time algorithm that solves the OTG problem exactly on *orthogonal* terrains under directed visibility.

1 Introduction

A *1.5-dimensional terrain* T is an x-monotone polygonal chain in the plane, where $V(T) = \{v_1, \dots, v_n\}$ is the set of vertices of T ordered from left to right, and $E(T) = \{e_1 = (v_1, v_2), \dots, e_{n-1} = (v_{n-1}, v_n)\}$ is the set of edges of T induced by the vertex set $V(T)$. Terrain T is called an *orthogonal terrain* if each edge $e \in E(T)$ is either horizontal or vertical. Let p be a point guard on T; p is called a *vertex* guard if $p \in V(T)$. A point q on T is seen/guarded by p (or, p sees/guards q) if and only if every point of the line segment $\overline{pq}$ lies either on or above T.

Given a (not necessarily orthogonal) terrain T, two common types of guarding problems are defined on T. In the *continuous* terrain guarding problem, the objective is to find a minimum-cardinality set S of points on T that *guards* T; that is, for every point $p \in T$, either p is in S or p is guarded by at least one point in S. In the *discrete* terrain guarding problem, on the other hand, two sets P and G of points on T are given along the terrain T as input and the objective is to find a subset $G' \subseteq G$ of minimum cardinality such that G' guards the points in P.

Related Work. The terrain guarding problem belongs to the well-known family of art gallery problems. The objective of the art gallery problem is to guard the interior of a polygon using the minimum number of point guards. The problem was first introduced by Klee in 1973 [12] and Chvátal [2] was the first to answer Klee's art gallery question by giving an upper bound proving that $\lfloor n/3 \rfloor$ point guards are always sufficient and sometimes necessary to guard a simple polygon with n vertices. The orthogonal art gallery problem was first studied by Kahn et al. [7] who proved that $\lfloor n/4 \rfloor$ guards are always sufficient and sometimes necessary to guard the interior of a simple orthogonal polygon with n vertices. In terms of the complexity of the art gallery problem, Lee and Lin [11] showed that the art gallery problem is NP-hard on simple polygons. Moreover, the problem is also NP-hard on simple orthogonal polygons [13] and it remains NP-hard even for monotone polygons [10]. Eidenbenz et al. [3] proved that the art gallery problem is APX-hard on simple polygons. They also showed that if the input polygon is allowed to have holes, then the problem cannot be approximated by a polynomial-time algorithm with factor $((1-\epsilon)/12)\ln n$ for any $\epsilon > 0$, where n is the number of the vertices of the polygon.

Ben-Moshe et al. [1] gave the first constant-factor approximation algorithm for the terrain guarding problem and left the complexity of the problem open. King and Krohn [9] showed that both continuous and discrete versions of the terrain guarding problem are NP-hard on arbitrary terrains. A 4-approximation algorithm for the terrain guarding problem was given by Elbassioni et al. [4], and Katz and Roisman [8] gave a 2-approximation algorithm for the OTG problem on orthogonal terrains. Gibson et al. [6] gave a polynomial-time approximation scheme (PTAS) for the discrete version of the terrain guarding problem, and a PTAS for the continuous version of the problem was recently given by Friedrichs et al. [5]. To the best of our knowledge, however, the complexity of the OTG problem on orthogonal terrains remains open. We note that the hardness result of King and Krohn [9] does not apply to the OTG problem on orthogonal terrains due to a number of essential differences between arbitrary and orthogonal terrains (e.g., see Lemma 4).

Problem Definition and Our Result. In this paper, we consider the discrete terrain guarding problem on an orthogonal terrain T under *directed visibility* such that $P = G = V(T)$; let $n = |V(T)|$. Directed visibility is defined as follows.

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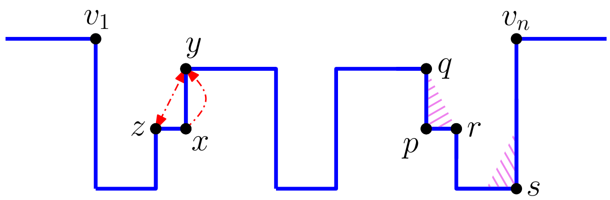

Figure 1: An orthogonal terrain T; throughout the paper, we assume that the leftmost and rightmost edges of T are two horizontal rays starting from v_1 and v_n, respectively. (a) An illustration of directed visibility: neither vertex y nor z can see vertex x under directed visibility, but they can see each other. The vertex x can see vertex y, but it cannot see vertex z because the line segment $\overline{xz}$ is horizontal. (b) The vertices q and r are reflex while the vertices p and s are convex. Moreover, q and r are both right reflex, p is left convex and s is right convex; vertex y is a left reflex vertex.

Definition 1 ***(Directed Visibility).*** *Let u be a vertex of T. If u is a reflex vertex, then u sees a vertex v of T if and only if every point in the interior of the line segment $\overline{uv}$ lies* strictly *above T. If u is a convex vertex, then u sees a vertex v of T if and only if $\overline{uv}$ is a non-horizontal line segment that lies on or above T.*

It is possible, under directed visibility, that a vertex u of T sees a vertex v, but vertex v cannot see u; see Figure 1(a) for an example. Therefore, we consider the following problem:

Definition 2 ***(The Directed Terrain Guarding (DTG) Problem on Orthogonal Terrains).*** *Given an orthogonal terrain T, compute a subset $S \subseteq V(T)$ of minimum cardinality that guards the vertices of T under directed visibility. That is, for every vertex $u \in V(T)$, either $u \in S$ or u is guarded by at least one other vertex in S under directed visibility.*

We give an $O(n)$-time algorithm for the DTG problem on orthogonal terrains under directed visibility. To this end, we first reduce the DTG problem to two subproblems such that an exact solution for the DTG problem reduces to the union of exact solutions of the two subproblems. We then give an $O(n)$-time greedy algorithm for solving each of the subproblems. To the best of our knowledge, this is the first exact algorithm for a non-trivial instance of the art gallery problem on terrains and partially answers a question posed by Ben-Moshe et al. [1] for orthogonal terrains.

1.1 Paper Organization

The paper is organized as follows. Section 2 presents preliminaries and some definitions. In Section 3, we give a characterization for an exact solution of the DTG problem: we define two subproblems and show that an exact solution for the DTG problem reduces to the union of the exact solutions of the subproblems. In Section 4, we show how to solve each subproblem in $O(n)$ time by a simple greedy algorithm. We conclude the paper in Section 5.

2 Preliminaries and Definitions

We denote the x- and y-coordinates of a point p on an orthogonal terrain T by $x(p)$ and $y(p)$, respectively. We use terms "terrain" and "guard" to refer to an orthogonal terrain and a vertex guard, respectively, unless otherwise stated. Moreover, we simply use "guarding" to mean "guarding under directed visibility" unless otherwise stated.

A vertex u of T is convex if the angle formed by the edges incident to u above T is $\pi/2$ degrees, otherwise u is reflex. We partition the vertices of T into 4 equivalences classes right or left endpoints of a horizontal edge of T, and whether the vertex is reflex or convex. We use $V_{LC}(T)$, $V_{RC}(T)$, $V_{LR}(T)$ and $V_{RR}(T)$ to respectively denote the left convex, right convex, left reflex, and right reflex subsets of the vertices of T. See Figure 1(b) for an example of these definitions.

For consistency, we assume that the leftmost and rightmost edges of T are two horizontal rays starting from v_1 and v_n, respectively; see Figure 1 for an illustration. For a reflex vertex u of T, we denote the convex vertex directly below u by $B(u)$. We say that a subset M of vertices of T guards a subset M' of vertices of T, where $M \cap M' = \emptyset$, if every vertex in M' is guarded by at least one vertex in M. We first describe some properties of orthogonal terrains.

Observation 1 *Let u and v be two reflex vertices of T. If vertex u sees $B(v)$, then u must also see v; see Figure 2 for an illustration.*

Let u and v be two convex vertices of T. If $y(u) = y(v)$, then clearly u and v cannot see each other under directed visibility because the line segment $\overline{uv}$ is horizontal. If $y(u) \neq y(v)$, then depending on the x-coordinates of u and v the line segment $\overline{uv}$ will pass through the region below the horizontal edge incident to either u or v and, therefore, u and v cannot see each other. This leads to the following lemma.

Lemma 1 *No two convex vertices of T can see each other under directed visibility.*

Observation 2 *Let u be a reflex vertex of a terrain T. If u is right reflex and sees a right convex vertex v of T, then $x(u) < x(v)$ and $y(u) > y(v)$. Similarly, if u is left reflex and sees a left convex vertex v of T, then $x(u) > x(v)$ and $y(u) > y(v)$.*

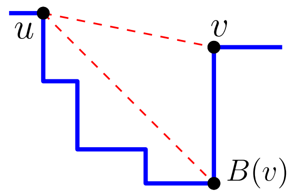

Figure 2: If a reflex vertex u sees $B(v)$, for some reflex vertex v, then u must also see vertex v itself.

Since directed visibility imposes a constraint relative to the standard visibility, the visibility graph of the vertices of T under directed visibility is a subgraph of that of the vertices of T under standard visibility. Therefore, the following property, called *the order claim*, still holds under directed visibility:

Lemma 2 (Ben-Moshe et al. [1]) *Let p, q, r and s be four vertices of a terrain T such that $x(p) < x(q) < x(r) < x(s)$. If p sees r and q sees s, then p sees s.*

Lemma 3 *Let u be a reflex vertex of a terrain T. If u is right reflex (resp., left reflex), then u cannot see any left convex (resp., right convex) vertex of T.*

Proof. We prove the lemma for when u is right reflex; the other case is proved by a symmetric argument. Let v be a left convex vertex of T. If $x(v) = x(u)$, then $v = B(u)$ and, therefore, u cannot see v under directed visibility. If $x(v) \neq x(u)$, then there are three cases.

- If $y(v) = y(u)$, then v is the adjacent vertex to the left of u and so u cannot see v under directed visibility.
- If $y(v) > y(u)$, then the line segment $\overline{uv}$ passes through the region below the horizontal edge incident to v and, therefore, vertex u cannot see v.
- If $y(v) < y(u)$, then there are two cases: (i) if $x(v) < x(u)$, then the line segment $\overline{uv}$ passes through the region below the horizontal edge incident to u and, therefore, vertex u cannot see v. (ii) If $x(v) > x(u)$, then the line segment $\overline{uv}$ passes through the region to the left of the vertical edge incident to v and, therefore, vertex u cannot see v.

The three cases above complete the proof of the lemma. □

In an arbitrary terrain, it is possible that a reflex vertex can guard both a left and a right convex vertex. For orthogonal terrains, however, this is not the case. This property is stated in the following lemma.

Lemma 4 *Let u be a right convex vertex and v be a left convex vertex of a terrain T. Then, there is no reflex vertex of T that sees both u and v under directed visibility.*

Proof. By Lemma 3, (i) no left reflex vertex of T can see u, and (ii) no right reflex vertex of T can see v. Therefore, no reflex vertex of T can see both u and v. This completes the proof of the lemma. □

3 An Exact Algorithm for the DTG Problem

In this section, we present our exact $O(n)$-time algorithm for the DTG problem on orthogonal terrains. Let T be an orthogonal terrain with n vertices. To solve the DTG problem on T, we first show that the DTG problem on T can be reduced to two subproblems such that an exact solution for the DTG problem is equivalent to the union of the exact solutions for the two subproblems. The subproblems are defined as follows.

Definition 3 ***(The Left-Convex Guarding (LCG(M)) Problem).*** *Given a set $M \subseteq V_{LC}(T)$, the objective of the LCG(M) problem is to compute a minimum-cardinality set $M' \subseteq V(T)$ such that for every vertex $u \in M$, either $u \in M'$ or u is guarded by at least one vertex in M'.*

Definition 4 ***(The Right-Convex Guarding (RCG(M)) Problem).*** *Given a set $M \subseteq V_{RC}(T)$, the objective of the RCG(M) is to compute a minimum-cardinality set $M' \subseteq V(T)$ such that for every vertex $u \in M$, either $u \in M'$ or u is guarded by at least one vertex in M'.*

To compute an exact solution for the DTG problem on T, we first show that we can restrict our attention to solutions that are in standard form. A feasible solution S to the DTG problem on T is in *standard form* if and only if every reflex vertex in S sees at least one convex vertex of T.

Lemma 5 *For any orthogonal terrain T, there exists an exact solution S for the DTG problem on T that is in standard form.*

Proof. Take any exact solution S_0 for the DTG problem on T. We construct a feasible solution S from S_0 such that $|S| \leq |S_0|$ and S is in standard form. To this end, for each reflex vertex $u \in S_0$ that does not see any convex vertex of T, replace u with $B(u)$ (i.e., the convex vertex directly below u). Let S be the resulting set. Clearly, $|S| \leq |S_0|$ and every reflex vertex in S sees at least one convex vertex of T. We now show that S is a feasible solution for the DTG problem on T. Consider a reflex vertex $u \in S_0$ that was replaced by $B(u)$ in S and let $Vis(u)$ be the set of vertices of T that are seen by u. We next prove that every vertex in $Vis(u)$ is still guarded by at least one vertex in S. First, note that every vertex in $Vis(u)$ is a reflex vertex. Let $v \in Vis(u)$ and consider $B(v)$. If $B(v) \in S$, then v is guarded by at least one vertex in S (i.e., the vertex

$B(v)$). If $B(v) \notin S$, then there must be a reflex vertex $w \in S_0$ that guards $B(v)$ because no two convex vertices of T can guard each other by Lemma 1. We note that $w \in S$ because w sees at least one convex vertex of T and so we have not replaced it with $B(w)$ in S. By Observation 1, vertex $w \in S$ guards v and, therefore, S is a feasible solution. Since $|S| \leq |S_0|$, the set S is an exact solution for the DTG problem on T that is in standard form. This completes the proof of the lemma. □

The following lemma, whose proof is given in Appendix A due to space constraints, states a necessary and sufficient condition for solving the DTG problem on T.

Lemma 6 *Let S be a feasible solution for the DTG problem on T. The set S is an exact solution if and only if there exists a partition $\{S_L, S_R\}$ of S such that (i) the set S_L is an exact solution for the LCG($V_{LC}(T)$) problem on T, and (ii) the set S_R is an exact solution for the RCG($V_{RC}(T)$) problem on T.*

By Lemma 6, we have the following theorem.

Theorem 7 *To solve the DTG problem on T, it is sufficient to solve the LCG($V_{LC}(T)$) and the RCG($V_{RC}(T)$) problems on T.*

4 Solving the LCG($V_{LC}(T)$) Problem

In this section, we present an $O(n)$-time exact algorithm for the LCG($V_{LC}(T)$) problem on T; an exact algorithm for the RCG($V_{RC}(T)$) problem can be derived analogously. First, by Lemma 1 (no convex vertex of T can see one other convex vertex of T) and Lemma 3 (no left reflex vertex of T can see a right convex vertex of T), we have the following result.

Lemma 8 *If M is a feasible solution for the LCG($V_{LC}(T)$) problem on T, then $M \subseteq \{V_{LC}(T) \cup V_{LR}(T)\}$.*

Next, we show that we can restrict our attention to solutions that are in a standard form. A feasible solution M for the LCG($V_{LC}(T)$) problem on T is in *standard form* if and only if a left convex vertex u is in M if and only if no reflex vertex of T can see u.

Lemma 9 *For any orthogonal terrain T, there exists an exact solution M for the LCG($V_{LC}(T)$) problem on T that is in standard form.*

Proof. Take any exact solution M_0 for the LCG($V_{LC}(T)$) problem on T. We construct a feasible solution M from M_0 such that $|M| \leq |M_0|$ and M is in standard form. For every left convex vertex $u \in M_0$ that is seen by at least one left reflex vertex v of T, replace u with v; let M be the resulting set. Clearly, $|M| \leq |M_0|$. Moreover, M is a feasible solution for the LCG($V_{LC}(T)$) problem on T because (i) the vertex u is now guarded by v, and (ii) the vertex u, which is left convex, cannot see any other left convex vertex of T. Therefore, every left convex vertex of T is still guarded by at least one vertex in M. Since $|M| \leq |M_0|$ and no left convex vertex of T that is in M is seen by a left reflex vertex of T, we conclude that M is an exact solution for the LCG($V_{LC}(T)$) problem on T that is in standard form. □

4.1 A Characterization

To solve the LCG($V_{LC}(T)$) problem on T, we give a characterization for an exact solution of the LCG($V_{LC}(T)$) problem on T. The following lemma, whose proof is given in Appendix B due to space constraints, is similar to the one given in Lemma 6 for the DTG problem.

Lemma 10 *Let M be a feasible solution for the LCG($V_{LC}(T)$) problem on T. The set M is an exact solution if and only if there exists a partition $\{A, B\}$ of M such that (i) $u \in A$ if and only if u is a left convex vertex and no reflex vertex of T can see u, and (ii) $B = M \setminus A$ is a minimum-cardinality subset of $V_{LR}(T)$ that guards $V_{LC}(T) \setminus A$.*

A similar result can be derived for an exact solution of the RCG($V_{RC}(T)$) problem analogously.

Lemma 11 *Let M be a feasible solution for the RCG($V_{RC}(T)$) problem on T. The set M is an exact solution if and only if there exists a partition $\{P, Q\}$ of M such that (i) $u \in P$ if and only if u is a right convex vertex and no reflex vertex of T can see u, and (ii) $Q = M \setminus P$ is a minimum-cardinality subset of $V_{RR}(T)$ that guards $V_{RC}(T) \setminus P$.*

By Lemma 10 and Lemma 11, we have the following theorem.

Theorem 12 *To solve the LCG($V_{LC}(T)$) problem on T, it is sufficient to first find the subset A of $V_{LC}(T)$, where $u \in A$ if and only if no reflex vertex of T can see u, and then compute a minimum-cardinality subset B of $V_{LR}(T)$ that guards $V_{LC}(T) \setminus A$. Similarly, to solve the RCG($V_{RC}(T)$) problem on T, it is sufficient to first find the subset P of $V_{RC}(T)$, where $u \in P$ if and only if no reflex vertex of T can see u, and then compute a minimum-cardinality subset Q of $V_{RR}(T)$ that guards $V_{RC}(T) \setminus P$.*

4.2 A Greedy Algorithm

In this section, we show how to compute an exact solution for the LCG($V_{LC}(T)$) problem on T; an exact solution for the RCG($V_{RC}(T)$) problem on T can be

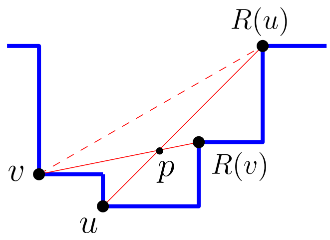

Figure 3: An illustration in support for the proof of Lemma 13.

computed analogously. By Theorem 12, we first compute the set A, where $u \in A$ if and only if u is a left convex vertex and it is not seen by any reflex vertex of T. In Section 4.3, we give a linear-time algorithm for computing $R(u)$ for all the left convex vertices of u, where $R(u)$ is the rightmost left reflex vertex of T that sees u (see Lemma 14). Therefore, we can use that algorithm to determine whether a left convex vertex u of T is seen by any reflex vertex of T at all and, therefore, the set A can be computed in $O(n)$ time overall. Now, let $C = V_{LC}(T) \setminus A$. In the following, we give an $O(n)$-time greedy algorithm for the problem of guarding C with the minimum-cardinality subset B of $V_{LR}(T)$.

For each left convex vertex $u \in C$, let $R(u)$ be the righmost left reflex vertex of T (i.e., the rightmost vertex in $V_{LR}(T)$) that sees u. Consider the left convex vertices of C from right to left: for each left convex vertex u in order, if u is not yet guarded by a reflex vertex in B, then we add $R(u)$ into B. Clearly, the set B is a feasible solution for guarding the vertices in C. Let B' be the set of convex vertices that force the algorithm to add a new guard into B. Clearly, $|B'| = |B|$. We now show that no left reflex vertex of T can see two vertices in B', which proves that the set B is an exact solution. Suppose for a contradiction that there exists a left reflex vertex v that sees two vertices w_i and w_j in B'. Without loss of generality, assume that $x(w_i) > x(w_j)$; that is, vertex w_i is guarded before vertex w_j in the ordering. Since v sees w_i, we must have that $x(R(w_i)) \geq x(v)$. Note that $x(R(w_i)) \neq x(v)$ because otherwise we would have not added a new guard for w_j. Therefore, we have the ordering $x(w_j) < x(w_i) < x(v) < x(R(w_i))$ such that w_j sees v and w_i sees $R(w_i)$. But, by Lemma 2, this means that w_j is seen by $R(w_i)$ which is a contradiction. This proves that no left reflex vertex of T can see two convex vertices in B' and so the set B is an exact solution for guarding the vertices in C.

4.3 Algorithmic Details

In this section, we show how to implement the algorithm in time linear in n, the number of vertices of T. Our implementation of the algorithm uses the following result.

Lemma 13 *Let u and v be two left convex vertices of T such that $x(v) < x(u)$. Then, the line segments $\overline{uR(u)}$ and $\overline{vR(v)}$ do not intersect at an interior point.*

Proof. Suppose for a contradiction that the line segments $\overline{uR(u)}$ and $\overline{vR(v)}$ intersect at an interior point p. Since $x(v) < x(u)$, we must have that $x(R(v)) < x(R(u))$. Therefore, we have the ordering $x(v) < x(u) < x(R(v)) < x(R(u))$; see Figure 3 for an example. By Lemma 2, the vertex v must see vertex $R(u)$, which is a contradiction to the fact that $R(v)$ is the righmost left reflex vertex of T that sees v. This completes the proof of the lemma. □

Consider the left convex vertices of T from right to left and let u and v be two left convex vertices such that $x(v) < x(u)$. By Lemma 13, vertex $R(v)$ cannot lie between the vertices u and $R(u)$; that is, vertex $R(v)$ is either $R(u)$ or a vertex to the right of $R(u)$, or it is a vertex to the left of vertex u. This property leads us to a linear-time algorithm for computing $R(u)$ for all the left convex vertices u in C as follows. Consider the vertices in $\{C \cup V_{LR}(T)\}$ from right to left in order. Note that the first vertex must be a left reflex vertex r. Moreover, we assume that the second vertex is also left reflex; otherwise, we set $R(u)$ to r for every visited left convex vertex until we reach to a left reflex vertex s; we push r and s into a stack $\texttt{S}$ in the order they have been visited. In the following, let s and r be the vertices on top of the stack $\texttt{S}$. Moreover, let t be the next visited vertex and let α be the angle formed by the line segments $\overline{ts}$ and $\overline{sr}$ that faces above T:

- if t is left reflex, then we pop the two vertices s and r from $\texttt{S}$. If $\alpha > \pi$, then we push the three vertices r, s and t into the stack $\texttt{S}$; otherwise, we ignore vertex s and push only vertex r into $\texttt{S}$. Now, we repeat the same procedure with the current two top vertices s' and r' of $\texttt{S}$ until α becomes greater than π in which case we push the three vertices r', s' and t into $\texttt{S}$.

- if t is left convex, then we pop the two vertices s and r from $\texttt{S}$. If $\alpha > \pi$, then we set $R(t)$ to s and push vertices r and s back into the stack $\texttt{S}$; otherwise, we ignore vertex s and push only vertex r into $\texttt{S}$. Now, we repeat the same procedure with the current two top vertices s' and r' of $\texttt{S}$ until α becomes greater than π in which case we set $R(t)$ to s' and push r' and s' into the stack $\texttt{S}$.

See Figure 4 for an example of the algorithm. Let u be a left reflex vertex of T. If $\alpha > \pi$, then we process u in $O(1)$ time and move to the next vertex. If $\alpha \leq \pi$, then one vertex is removed from the stack $\texttt{S}$ and we then repeat the same procedure which may consist of removing further vertices from $\texttt{S}$. Therefore, at each left reflex vertex u, either we perform an $O(1)$-time operation or

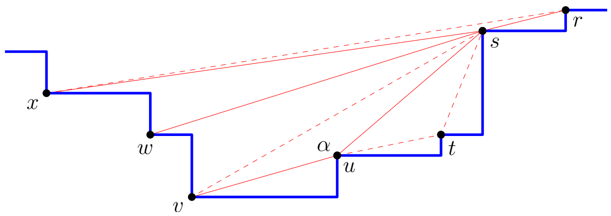

Figure 4: An example illustrating the computation of $R(v)$, $R(w)$ and $R(x)$. After processing vertex u, the status of the stack $\mathtt{S}$ from top to bottom is: $[u, s, r]$. When processing vertex v, vertex u is removed from $\mathtt{S}$ since $\alpha < \pi$ for the line segments $\overline{vu}$ and $\overline{us}$; then $R(v)$ is set to s. Vertex $R(w)$ is also set to s because $\alpha > \pi$ for the line segments $\overline{ws}$ and $\overline{sr}$. Finally, vertex s is removed from $\mathtt{S}$ and $R(x)$ is set to r. The final status of $\mathtt{S}$ is: $[r]$.

we remove a set S_u of vertices from $\mathtt{S}$ permanently. Note that by Lemma 13, the vertices in S_u will not be pushed back into $\mathtt{S}$ in the future. We can show using an analogous argument that at each left convex vertex, either we perform an $O(1)$-time operation or we remove a set of vertices from $\mathtt{S}$ permanently.

Although this procedure was described for computing $R(u)$ for all the left convex vertices in C, in fact it can be used to compute $R(u)$ for all the left convex vertices of T in $O(n)$ time. This leads us to the following lemma:

Lemma 14 *Given an orthogonal terrain T, the overall procedure of computing $R(u)$ for all the left convex vertices u of T can be completed in $O(n)$ time, where $|V(T)|$.*

By Lemma 14, we have the following theorem.

Theorem 15 *The $LCG(V_{LC}(T))$ problem on T can be solved exactly in $O(n)$ time, where $n = |V(T)|$.*

We note that the RCG($V_{RC}(T)$) problem on T can be solved analogously in $O(n)$ time. Let S_1 and S_2 be the exact solutions for the LCG($V_{LC}(T)$) and the RCG($V_{RC}(T)$) problems on T, respectively. By Theorem 7, the set $S = \{S_1 \cup S_2\}$ is an exact solution for the DTG problem on T. Therefore, by Theorem 15, we have the main result of this paper.

Theorem 16 *There exists an $O(n)$-time exact algorithm for the DTG problem on any orthogonal terrain T with n vertices.*

5 Conclusion

In this paper, we considered the problem of guarding the vertices of an orthogonal terrain T with the minimum number of vertex guards under directed visibility (i.e., the DTG problem). We showed that the DTG problem on T is linear-time tractable by first reducing the problem to two subproblems (i.e., the $LCG(V_{LC}(T))$ and $RCG(V_{RC}(T))$ problems) and then solving each subproblem by a greedy algorithm that runs in $O(n)$ time, where n is the number of the vertices of T. Our algorithm assumes the directed visibility and it does not apply to the DTG problem under standard visibility. The complexity of the problem remains open without the directed visibility constraint.

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Appendix A: Proof of Lemma 6

Proof. ($\Rightarrow$) Let S be an exact solution for the DTG problem on T; by Lemma 5, we assume that S is in standard form. Let $S_L \subseteq S$ such that $u \in S_L$ if and only if u is either a left convex vertex or it is a left reflex vertex of T. Similarly, let $S_R \subseteq S$ such that $v \in S_R$ if and only if v either is a right convex vertex or it is a right reflex vertex of T that sees at least one right convex vertex. Since S is in standard form, $\{S_L, S_R\}$ is a partition of S.

We first prove that S_L is a feasible solution for the LCG($V_{LC}(T)$) problem on T. Let a be a left convex vertex of T. If $a \in S$, then $a \in S_L$. If $a \notin S$, then by Lemma 3 and the fact that no convex vertex can see another convex vertex (see Lemma 1), we conclude that there must be a left reflex vertex $b \in S$ that guards a and, therefore, $b \in S_L$. This means that for every left convex vertex a of T, we have either $a \in S_L$ or a is guarded by at least one vertex in S_L. Therefore, S_L is a feasible solution for the LCG($V_{LC}(T)$) problem on T. By an analogous argument, we can show that S_R is a feasible solution for the RCG($V_{RC}(T)$) problem on T.

We next prove that S_L is an exact solution for the LCG($V_{LC}(T)$) problem on T. Suppose for a contradiction that there exists a feasible solution S'_L for the LCG($V_{LC}(T)$) problem on T such that $|S'_L| < |S_L|$. In the following, we prove that the set $\{S'_L \cup S_R\}$ is a feasible solution for the DTG problem on T, which is a contradiction to the fact that S is an exact solution for the DTG problem on T because $|S'_L \cup S_R| \leq |S'_L| + |S_R| < |S_L| + |S_R| = |S|$ (the last equality follows from the fact that $\{S_L, S_R\}$ is a partition of S). Let u be a vertex of T. If u is left convex, then u is either in S'_L or it is guarded by a left reflex vertex in S'_L because S'_L is a feasible solution for the LCG($V_{LC}(T)$) problem on T. Similarly, if u is a right convex vertex, then u is either in S_R or it is guarded by a right reflex vertex in S_R because S_R is a feasible solution for the RCG($V_{RC}(T)$) problem on T. Now, suppose that u is a reflex vertex that is not in $S'_L \cup S_R$. Then, consider the vertex $B(u)$. If $B(u) \in \{S'_L \cup S_R\}$, then u is guarded by at least one vertex in $S'_L \cup S_R$ (i.e., the vertex $B(u)$). If $B(u) \notin \{S'_L \cup S_R\}$, then it must be guarded by a reflex vertex $w \in \{S'_L \cup S_R\}$ because no two convex vertices of T can see each other by Lemma 1. By Observation 1, vertex w must also guard the vertex u. This proves that every vertex of T that is not in $S'_L \cup S_R$ is guarded by at least one vertex in $S'_L \cup S_R$ and, therefore, $S'_L \cup S_R$ is a feasible solution for the DTG problem on T. By an analogous argument, we can show that S_R is an exact solution for the RCG($V_{RC}(T)$) problem on T.

($\Leftarrow$) Suppose that there exists a partition $\{S_L, S_R\}$ of S such that S_L is an exact solution for the LCG($V_{LC}(T)$) problem on T and S_R is an exact solution for the RCG($V_{RC}(T)$) problem on T. We now prove that $S = \{S_L \cup S_R\}$ is an exact solution for the DTG problem on T. Suppose for a contradiction that there exists a feasible solution S' for the DTG problem on T such that $|S'| < |S|$; by Lemma 5, we assume that S' is in standard form. Let X be a subset of S' such that $u \in X$ if and only if u is either a left convex vertex or it is a left reflex vertex of T. Similarly, let Y be a subset of S' such that $v \in Y$ if and only if v is either a right convex vertex or it is a right reflex vertex of T. Since S' is in standard form, $\{X, Y\}$ is a partition of S'. Since $|S'| < |S|$, we must have $|X| < |S_L|$ or $|Y| < |S_R|$. Without loss of generality, assume that $|X| < |S_L|$. In the following, we show that X is a feasible solution for the LCG($V_{LC}(T)$) problem on T, which is a contradiction to the fact that S_L is an exact solution for the LCG($V_{LC}(T)$) problem on T. To show the feasibility of X, let x be a left convex vertex of T. If $x \in S'$, then $x \in X$. If $x \notin S'$, then we conclude by Lemma 3 that there must be a left reflex vertex $y \in S'$ that guards x. Since y guards at least one left convex vertex of T, we have $y \in X$. This means that every left convex vertex of T is either in X or it is guarded by at least one left reflex vertex in X. Therefore, the set X is a feasible solution for the LCG($V_{LC}(T)$) problem on T.

We have proved that it is not possible that $|S'| < |S|$ and, therefore, the set S is an exact solution for the DTG problem on T. This completes the proof of the lemma. $\square$

Appendix B: Proof of Lemma 10

Proof. ($\Rightarrow$) Suppose that M is an exact solution for the LCG($V_{LC}(T)$) problem on T; by Lemma 9, we assume that M is in standard form. Let A be the subset of M such that $u \in A$ if and only if u is a left convex vertex of T, and let $B = M \setminus A$. Clearly, $\{A, B\}$ is a partition of M. Also, no reflex vertex of T can see a vertex in A because M is in standard form and, by Lemma 8, we have that $B \subseteq V_{LR}(T)$. Moreover, since M is a feasible solution for the LCG($V_{LC}(T)$) problem, every left convex vertex of T that is not in A is guarded by at least one left reflex vertex in B. Therefore, it only remains to show that B has minimum cardinality among all subsets of $V_{LR}(T)$ that guard $V_{LC}(T) \setminus A$. Suppose for a contradiction that $B' \subseteq V_{LR}(T)$ guards $V_{LC}(T) \setminus A$ such that $|B'| < |B|$. Then, $\{A \cup B'\}$ is a feasible solution for the LCG($V_{LC}(T)$) problem on T, but $|A \cup B'| \leq |A| + |B'| < |A| + |B| = |M|$ (the last equality is due to the fact that $\{A, B\}$ is a partition of M); this is a contradiction to the fact that M is an exact solution for the LCG($V_{LC}(T)$) problem on T.

($\Leftarrow$) Suppose that there exists a partition $\{A, B\}$ of M such that (i) $u \in A$ if and only if u is a left convex vertex and no reflex vertex of T can see u, and (ii) $B = M \setminus A$ is a minimum-cardinality subset of $V_{LR}(T)$ that guards $V_{LC}(T) \setminus A$. We now show that $M = \{A \cup B\}$ is an exact solution for the LCG($V_{LC}(T)$) problem on T. Suppose for a contradiction that there exists a feasible solution M' for the LCG($V_{LC}(T)$) problem on T such that $|M'| < |M|$. By Lemma 8, we have that $M' \subseteq \{V_{LC}(T) \cup V_{LR}(T)\}$. Partition M' into two sets X and Y such that $x \in X$ if and only if x is a left convex vertex that is not seen by any left reflex vertex of T, and let $Y = M' \setminus X$. We can assume that $Y \subseteq V_{LR}(T)$ because otherwise we can replace every left convex vertex y in Y with a left reflex vertex of T that sees y.[1] Recall that if $x \in X$, then no left reflex vertex of T can see x and, by Lemma 3, no right reflex vertex of T can see x. Therefore, $x \in A$ because no reflex vertex of T can see x and M is a feasible solution for the LCG($V_{LC}(T)$) problem on T. By an analogous argument, we can show that if $x \in A$, then

[1]Note that at least one such left reflex vertex of T exists because otherwise we would have added y into X.

$x \in X$. Therefore, $X = A$. This means that Y is a subset of $V_{LR}(T)$ that guards $V_{LC}(T) \setminus X = V_{LC}(T) \setminus A$. Since $X = A$ and $|M'| < |M|$, we must have that $|Y| < |B|$, which is a contradiction to the fact that B is a minimum-cardinality subset of $V_{LR}(T)$ that guards $V_{LC}(T) \setminus A$. This completes the proof of the lemma. $\square$