Watchtower for k-crossing Visibility

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Abstract

Given a 1.5D terrain T, consisting of an x-monotone polygonal chain with n vertices in the plane, and a positive integer k, we propose an algorithm to place one point, called a *watchtower*, whose vertical height above T is minimized, such that every point x on T is k-crossing visible from the watchtower w. That is, the line segment from w to any point x on T crosses T at most k times. Our algorithm runs in $O((n^2 + h) \log n)$ time, where h denotes the number of vertices on the boundary of the k-kernel of T. For arbitrary k, $h \in O(n^4)$, and for $k = 2$, $h \in O(n^2)$. We present an $O(n^3)$-time algorithm for the discrete version of the problem, in which the watchtower is restricted to being positioned over vertices of T.

1 Introduction

A *terrain* T in $\mathbb{R}^2$ is an x-monotone polygonal chain consisting of a sequence of vertices $v_0, v_1, \ldots, v_{n-1}$, each of which is a point in $\mathbb{R}^2$, such that v_i is to the left of v_j for all $i < j$ and $v_i v_{i+1}$ is an edge for $i \in \{0, \ldots, n-2\}$. See Figure 1. As defined by Chang et al. [5], "two paths [polygonal chains], P and Q, are *weakly disjoint* if, for all sufficiently small $\epsilon > 0$, there are disjoint paths $\tilde{P}$ and $\tilde{Q}$ such that $d_{\mathcal{F}}(P, \tilde{P}) < \epsilon$ and $d_{\mathcal{F}}(Q, \tilde{Q}) < \epsilon$", where $d_{\mathcal{F}}(A, B)$ denotes the Fréchet distance between A and B. As also defined by Chang et al. [5], "two paths [polygonal chains] *cross* if they are not weakly disjoint." We say two polygonal chains P and Q *cross k times*, if there exist partitions $P_1, \ldots, P_k$ of P and $Q_1, \ldots, Q_k$ of Q such that P_i and Q_i cross, for all $i \in \{1, \ldots, k\}$. Two points p and q are *k-crossing visible* if and only if the line segment pq crosses T at most k times. When $k = 0$, k-crossing visibility corresponds to the traditional definition of visibility.

A *watchtower* w is a point on or above T. Given a terrain T and a positive integer k, the goal in the *1-watchtower* problem is to place a watchtower w with minimum height on or above T (length of the vertical line segment from w to T) such that the entire terrain T is k-crossing visible from w. This definition can be generalized to the *m-watchtower* problem where the goal is to assign positions to a set $W = \{w_1, \ldots, w_m\}$ of m watchtowers, such that each w_i is a point on or above T, and for each point p on T, there exists a watchtower $w \in W$ such that p is k-crossing visible from w.

The watchtower problem presents itself in two forms: discrete and continuous. In the discrete version, the watchtower must be located on a vertical line through a vertex of the terrain, while in the continuous version the watchtower can be located anywhere above the terrain. Solutions to the discrete and continuous watchtower problems can vary significantly. Figure 2 shows an instance for which the solution to the continuous 1-watchtower problem has height zero (on the terrain), whereas the solution to the discrete 1-watchtower problem on the same terrain requires a watchtower to be positioned significantly higher.

This paper examines algorithms for the 1-watchtower problem, for both the discrete and continuous cases, under k-visibility. We also describe faster algorithms for the case $k = 2$ and $k = 0$.

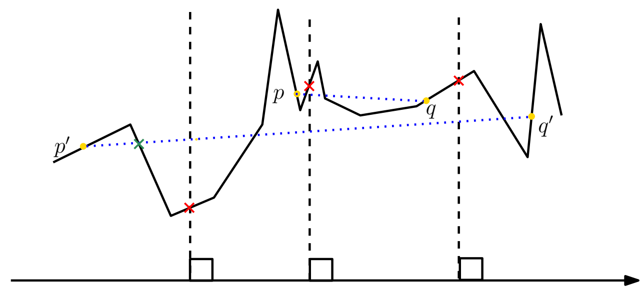

Figure 1: The points p and q mutually 2-crossing visible, while p' and q' are not.

2 Related Work

The original terrain watchtower problem was introduced by Sharir for polyhedral terrains [11]. The minimum height for one watchtower can be found in $O(n \log n)$ time for both the continuous and discrete problems under 0-crossing visibility on an x-monotone polyhedral terrain in $\mathbb{R}^3$ [12].

Bespamyatnikh et al. [4] proposed an $O(n^4)$-time algorithm for the discrete 2-watchtower problem under 0-crossing visibility on a 2.5D terrain. They also gen-

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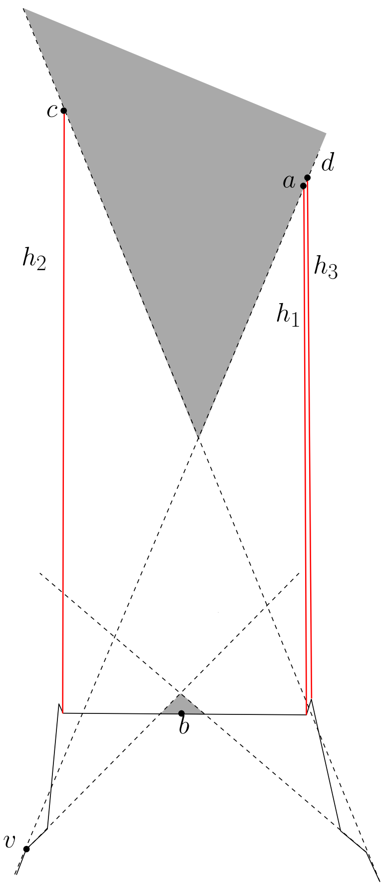

Figure 2: When $k = 2$, the solution of the 1-watchtower problem for the continuous version is much smaller than the discrete version. The points b and d represent the locations of the watchtower in the continuous and discrete versions, respectively (suppose $h_3 < h_1$). In the continuous version, the tower is located on the edge of the terrain with height zero, while in the discrete version it must be located above the terrain with height h_3, significantly bigger than zero. Notice that the points below d cannot see the edges adjacent to the vertex v.

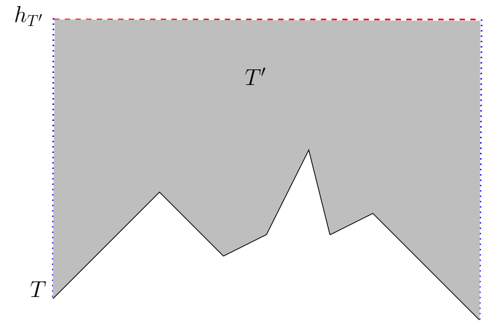

Figure 3: The shaded region is a simple polygon T' constructed for a given terrain.

eralized their approach to the continuous version of the problem with assumptions on the time required to solve a specific cubic equation with three bounded variables. Under the assumption that the equation can be solved in $O(f_3)$ time, their approach takes $O(n^4 + n^3 f_3)$ time. Using parametric search, they show that the discrete and continuous versions of the problem can be solved in $O(n^3 \log^2 n)$ and $O(n^4 \log^2 n)$ time, respectively. Ben-Moshe et al. [2] improved the time to $O(n^{3/2}\sqrt{m'(n)})$ for the discrete 2-watchtower problem, where $m'(n)$ denotes the time required to multiply two $n \times n$ matrices, resulting in a time of $O(n^{2.37+\epsilon})$ using the current fastest matrix multiplication algorithm [8]. Using parametric search, Agarwal et al. [1] improved the time complexity of the discrete and continuous 2-watchtower problems for 0-crossing visibility to $O(n^2 \log^4 n)$ and $O(n^3 \alpha(n) \log^3 n)$ respectively, where $\alpha(n)$ denotes the inverse Ackermann function.
The watchtower problem generalizes to the setting of k-crossing visibility for any k. We consider the problem of placing one watchtower. In Section 3, we present an algorithm for the continuous problem, and then propose an algorithm for the discrete problem in Section 4. For both algorithms we describe how the running time can be decreased when $k = 2$ and $k = 0$.

3 The Continuous Case

In this section, we solve the continuous 1-watchtower problem under k-visibility for general k, and then describe how the running time can be reduced when $k = 2$ and $k = 0$.

Consider a simple polygon T', bounded from above by a horizontal line segment $h_{T'}$ that lies above T, and on its sides by vertical line segments aligned with the respective left and right endpoints of T; see Figure 3.
We first find the k-kernel of T'. The k-kernel of a given polygon P is the set of all points p such that every point in P is k-crossing visible from p; see Figure 4. The algorithm of Evans and Sember [6] finds the k-kernel of T' in $O(n^2 \log n + h)$ time, where h denotes the complexity

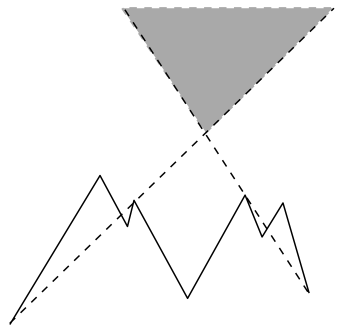

Figure 4: 2-kernel

(the number of boundary vertices) of the k-kernel. The k-kernel consists of $O(n^4)$ disjoint simple polygons. The worst-case number of vertices of the k-kernel is $\Theta(n^4)$. For $k = 2$, the complexity of the k-kernel is $\Theta(n^2)$, and for $k = 3$, the complexity of the k-kernel is $O(n^4)$ and $\Omega(n^2)$ [6].

The lower envelope of the portion of the k-kernel above T is the locus of feasible locations for the top of the watchtower from which the entire terrain T is k-crossing visible. Finding the minimum-length vertical line segment between this lower envelope and T yields the optimal solution for the 1-watchtower problem; see Figure 6. Notice that given line segments s_1 and s_2 that intersect a vertical line, the distance between s_1 and s_2 along the vertical line is minimized at a vertex of s_1 or a vertex of s_2. Hence, to find the optimal height for the continuous 1-watchtower problem, it suffices to examine vertical line segments from the vertices of the lower envelope of the k-kernel to T, and vertical line segments from the vertices of T to the lower envelope of the k-kernel. The minimum length of these line segments is the minimum height of the continuous 1-watchtower problem.

The minimum height of a watchtower can be found by partitioning the edges of the k-kernel into those that lie above T and those that lie below T. Following this partition, the lower envelope of the edges above T is computed. By sweeping a vertical line across T and the lower envelope, we stop at all vertices to evaluate the distance on the sweep line between these two x-monotone chains, maintaining the minimum distance thus far. These steps can be implemented in a single sweep using a modification of the algorithm of Bentley and Ottmann [3]. At each event during the sweep, it suffices to measure the distance along the sweep line between T and the closest line segment above T. If this distance is less than the previously recorded minimum, we update the minimum distance and the current x-coordinate of the sweep line. Observe that no two edges of the k-kernel cross, and that no two edges of T cross. Furthermore, if any edge of the k-kernel crosses T, then this point of intersection corresponds to the location of a watchtower of height zero: this is the solution, and the algorithm terminates. Consequently, the number of intersection events processed is at most 1. Since the number of edges in the k-kernel is $h \in O(n^4)$ and the number of edges in T is n, the total running time of the algorithm is $O((n^2 + h)\log n))$.

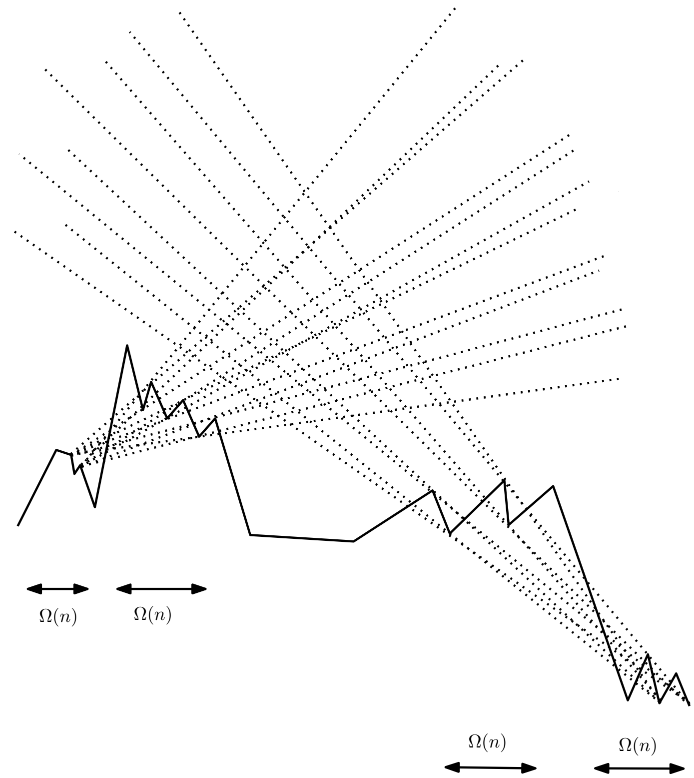

Figure 5: The 4-kernel of a monotone chain has $\Omega(n^4)$ vertices. There are $\Omega(n^2)$ cells in the arrangement of dotted lines that form the v-regions of the vertices on the terrain. These lines have $\Omega(n^2)$ points of intersection.

Although we seek the k-kernel in a restricted type of polygon, i.e., a monotone polygon, the k-kernel for a monotone polygon has $\Theta(n^4)$ complexity in the worst case when $k \geq 4$; see Figure 5. The complexity of the k-kernel when $k = 3$ is unknown [6]. When $k = 2$ its complexity is $O(n^2)$. Since the watchtower must be located above the terrain, it must be inside T'.

When $k = 0$, the 0-kernel corresponds to the kernel of the polygon T'. This kernel is a convex polygon with $O(n)$ vertices from which the entire polygon is 0-crossing visible. Additionally, the kernel is the feasible region for the watchtower, and can be determined in $O(n)$ time [9, 10]; see Figure 6. As mentioned above, to find the solution for the continuous 1-watchtower problem, it is sufficient to examine the vertical line segments from the vertices of the kernel to T, and the vertical line segments from the vertices of T to the kernel. The boundary of the 0-kernel is an x-monotone chain consisting of $O(n)$ vertices given in order. The terrain T is

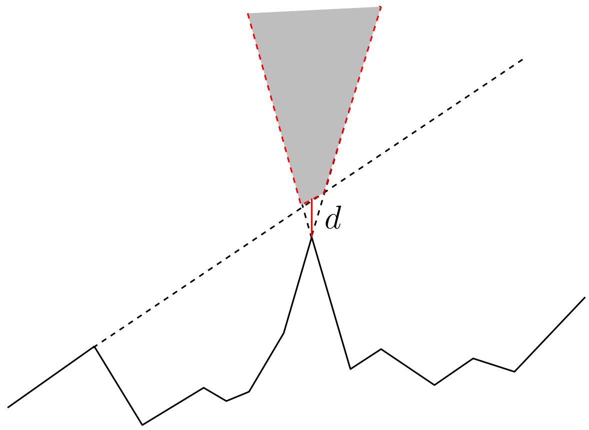

Figure 6: The shaded region is the intersection of the visible part of the plane for each vertex when $k = 0$; dotted lines show the boundaries of some of these regions.

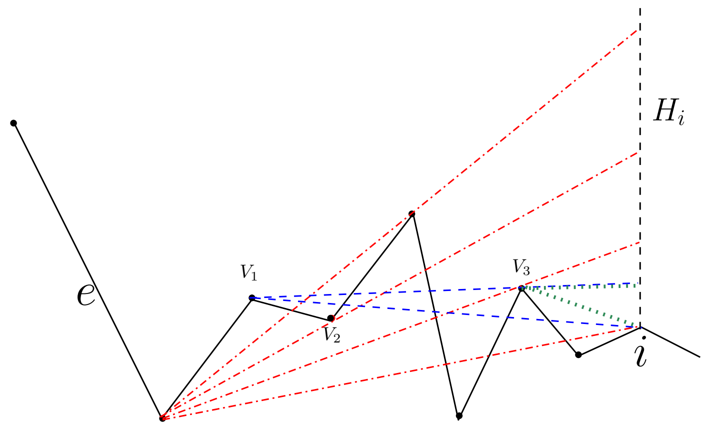

Figure 7: The v-regions and their intersection with H_i for three vertices V_1, V_2 and V_3 are shown in dashed, dotted, dashed and dotted respectively.

an x-monotone chain of n vertices given in order. By merging the two sets of sorted vertices of T and of the kernel in O(n) time, for each vertex in the merged sorted list the corresponding edge intersected by the vertical line segment can be found in $O(1)$ time by comparing the current vertex against the previous vertex in the list. If the previous vertex is on the same chain, then the current vertex intersects the same edge as the previous vertex. Otherwise, if the previous vertex is not on the same chain, then the edge that starts from the previous vertex is the intersected edge. At each step, the minimum vertical line segment encountered is maintained. Thus, the minimum length segment can be found in $O(n)$ time.

When $k = 2$, the boundary of the 2-kernel has $O(n^2)$ vertices [6]. Consequently, we can find the minimum length vertical line segment between the 2-kernel and the terrain T in $O(n^2 \log n)$ time, so the continuous 1-watchtower problem for 2-visibility can be solved in $O(n^2 \log n)$ time.

Theorem 1 *The continuous 1-watchtower problem can be solved in $O((n^2 + h) \log n)$ time under k-crossing visibility, where $h \in O(n^4)$ is the size of the k-kernel. For $k = 0$ and $k = 2$, the continuous 1-watchtower problem can be solved in $O(n)$ and $O(n^2 \log n)$ time, respectively.*

4 The Discrete Case

In this section, we propose an $O(n^3)$-time algorithm for the discrete k-crossing visible 1-watchtower problem on a terrain T.

As defined by Evans and Sember [6], "The v-region for vertex v of a polygon P, is the set of points q for which q is k-visible to every point of P on ray $\rightarrow qv$". The boundary of each v-region is a simple polygon with $O(n)$ vertices [6]. Computing the v-region of each vertex of the polygon takes $O(n \log n)$ time. We compute the v-region for each vertex of T' in $O(n \log n)$ time per vertex using the algorithm of Evans and Sember [6], using $O(n^2 \log n)$ total time. The intersection of v-regions of the polygon P is the k-kernel of the polygon P [6]. In other words, the intersection of v-regions of the vertices of P is the locus for the watchtower.

Observation 1 *The intersection of the v-regions of the vertices of T corresponds to the set of feasible locations for the top of the watchtower.*

Proof. The intersection of the v-regions is the k-kernel of T' [6], which is the region where the entire T' including T is k-crossing visible from. So, T is k-crossing visible from a watchtower located in this region. □

In the discrete problem, the watchtower must be located on a vertical line emanating from a vertex of the terrain. Consider a vertical line passing through a vertex of the terrain. We find the intersection of the v-regions of the vertices of T with this vertical line.

Lemma 2 *Any vertical line crosses the boundaries of the v-regions of the vertices of T $O(n^2)$ times.*

Proof. The number of vertices on the boundary of each v-region is $O(n)$. So each v-region may intersect a vertical line $O(n)$ times. As there exist n v-regions, so the number of intersections between v-regions and any given vertical line is $O(n^2)$. □

Let V_i denote the v-region of vertex v_i in T. We have the following lemma:

Lemma 3 *The intersection of any v-region with any vertical line is a set of at most n disjoint intervals on the line, where the topmost interval is open.*

Proof. Considering a bounding box around T'. The v-region of a vertex v_i is a closed Jordan curve with $O(n)$ complexity. The intersection between the vertical line and the inside of this closed Jordan curve is a set of $O(n)$ intervals. The last interval is open as after moving sufficiently high above the terrain T all of T will be visible while looking toward the vertex v_i. □

Consider a vertical line ℓ_i passing through a given vertex v_i of T, and the intersections with the v-regions $V_1, \ldots, V_n$ for the vertices $v_1, \ldots, v_n$ of T. Let each v-region be determined by a specific color i. As a result, we have n different colors of intervals on the line ℓ_i. Each color is a set of $O(n)$ pairwise disjoint intervals. If the optimal watchtower lies on this vertical line, it is in the interval which intersects all n v-regions with the lowest y-coordinate. We define depth-n intervals as the intervals on ℓ_i on which all n v-regions intersect.

Lemma 4 *The minimum height of a watchtower located above the vertex v_i is the closest depth-n interval.*

Proof. Intervals with the same color do not intersect each other. So, the maximum number of intersection is n where n v-regions intersect. So, a depth-n interval is in the k-kernel and T is k-crossing visible from such intervals. Among all such depth-n intervals we look for the one that has the smallest distance from the terrain. □

As a result of Lemma 4, we can remove the color on the intervals. This transforms the problem to that of finding the depth-n intervals among $O(n^2)$ intervals.

Lemma 5 *Given a v-region of a vertex of the terrain T, finding and sorting the intersections of this v-region with a given vertical line takes $O(n)$.*

Proof. We can find the intersection of a v-region with the vertical line ℓ_i in $O(n)$ time. This gives a set of $O(n)$ intervals on ℓ_i. We can sort these intervals in $O(n)$ time as the v-region is a Jordan arc [7]. □

We find the sorted list of the intersections of each polygon V_i with a line ℓ_i in $O(n^2)$ time by Lemma 5. So we have n sorted lists each containing $O(n)$ intervals. Let these lists be labeled as $L_1, L_2, ..., L_n$. We have the following lemma:

Lemma 6 *The deepest interval with the minimum height for a set of $O(n^2)$ intervals on a given line ℓ_i can be found in $O(n^2)$ time.*

Proof. As mentioned in Lemma 5, each set of n intervals in the list L_i can be sorted in linear time. There exist n lists, so it takes $O(n^2)$ time to sort all $L_1, \ldots, L_n$. Consider two list L_1 and L_2. First, we find the intersections between L_1 and L_2. Given two sets of sorted

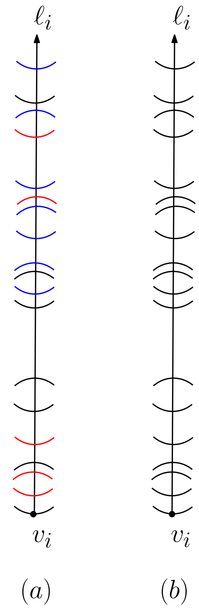

Figure 8: a. Colored intervals on a vertical line ℓ_i. b. Intervals can be considered as a set of $O(n^2)$ intervals without color.

intervals X and Y, their intersection can be found in $O(|X|+|Y|+h)$, where h denotes the number of output intervals [13]. As X and Y are of size $O(n)$ for the lists L_1 and L_2. h is also of size $O(n)$. This is because if an interval in L_1 intersects m intervals of L_2, remaining intervals in L_1 can intersect at most $n-m+2$ intervals in L_2. As a result, finding the intersection between L_1 and L_2 takes $O(n)$ time; let the output list be called L'_1. Next, we find the intersection of L'_1 and L_3 (called L'_2) in $O(n)$ time. Repeating this process, the intersection between L'_{n-1} and L_n results in the intersections of $L_1, L_2, ..., L_n$. There are n steps, each taking $O(n)$ time. The algorithm takes $O(n^2)$ total time. □

Theorem 7 *The discrete 1-watchtower problem can be solved in $O(n^3)$ time under k-crossing visibility.*

Proof. There are n vertices in T corresponding to n vertical lines as the candidates for the location of the watchtower. By Lemmas 4 and 6, finding the minimum height of a watchtower located at the vertex v_i takes $O(n^2)$ time. So, the total required time is $O(n^3)$. □

Considering 0-crossing visibility, the kernel is the potential location of the top of the watchtower as described for the continuous version. The difference between the discrete and continuous versions is that in the discrete version, the algorithm restricts the possible watchtowers to those whose x-coordinates coincide with a vertex of T. As a result, the discrete 1-watchtower problem under 0-crossing visibility can also be solved in $O(n)$ time.

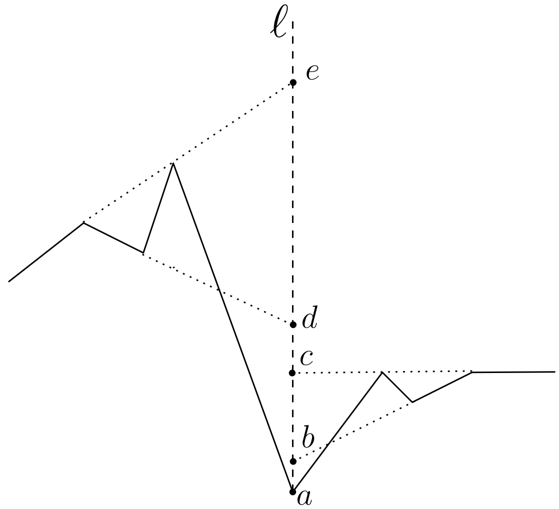

Figure 9: Going up and losing visibility: On point a, the entire terrain T is 2-crossing visible. At point b, the rightmost edge of T is not 2-crossing visible anymore. At point c, the entire terrain T becomes 2-crossing visible, while on d, the leftmost edge of T is not 2-crossing. At point e, T is 2-crossing visible again.

In the case of 2-crossing visibility, we apply the same approach as for the continuous version. The key difference is that only the vertical line segments emanating from vertices of the terrain are of interest as the possible location for the watchtower. As a result, the discrete version of the 2-watchtower problem can also be solved in $O(n^2 \log n)$ time.

4.1 Comparison between k-visibility and 0-visibility

As mentioned, both the discrete and continuous versions of the 1-watchtower problem for 0-crossing visibility can be solved in $O(n)$ time, while for k-crossing visibility the time complexity increases significantly when $k > 0$. The main reason is the fact that when $k \neq 0$, the k-kernel can be disconnected. Under 0-visibility, increasing the height of a watchtower always increases its visibility; that is, if p and q are two points on a vertical line above T, where p lies above q, then the region of T visible to q is contained in the region of T visible to p. This property does not hold when $k > 0$; q could see all of T (i.e., q is in the k-kernel), whereas p does not see all of T, even though p lies above q. See Figure 9.

5 Possible Directions for Future Research

The 1-watchtower problem generalizes to the m-watchtower problem, where instead of positioning a single watchtower to guard the terrain T, an algorithm must select positions for m watchtowers. The goal is to minimize the maximum height of any watchtower, while ensuring that each point on T is k-crossing visible from at least one watchtower. To solve the continuous 1-watchtower problem, it suffices to consider candidate locations for the watchtower whose x-coordinate coincides with that of a vertex of T or a vertex of the k-kernel of T. This property is not true in general for the continuous m-watchtower problem, even when $m = 2$; see Figure 10. It remains open to find an efficient algorithm to solve the (discrete or continuous) m-watchtower problem under k-crossing visibility, even for $m = 2$.

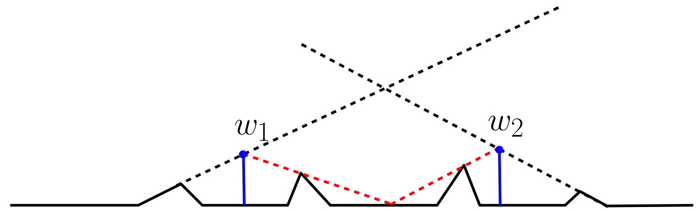

Figure 10: Even when $k = 0$ in the 2-watchtower problem, the x-coordinates of watchtowers do not coincide with those of vertices of the terrain, vertices of the k-kernel, nor of the intersections of the $\Theta(n^2)$ lines determined by pairs of vertices of the terrain.

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