Minimum Ply Covering of Points with Unit Disks*

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Abstract

Let P be a set of points and let U be a set of unit disks in the Euclidean plane. A minimum ply cover of P with U is a subset of U that covers P and minimizes the number of disks that share a common intersection. The size of a minimum ply cover is called the minimum ply cover number. Biedl et al. [Comput. Geom., 94:101712, 2020] showed that determining the minimum ply cover number for a set of points by a set of unit disks is NP-hard, and asked whether there exists a polynomial-time $O(1)$-approximation algorithm for this problem. They showed the problem to be 2-approximable in polynomial time for the special case when the minimum ply cover number is constant. In this paper, we settle the question posed by Biedl et al. by providing a polynomial-time $O(1)$-approximation algorithm for the minimum ply cover problem.

1 Introduction

The *minimum set cover problem* is a widely studied optimization problem. The input to the set cover problem is a set P and a collection C of subsets over P. The goal is to identify a subset C' of C with minimum cardinality that contains all the elements of P. The *membership* of an element q in P with respect to a subset C' of C is the number of sets in C' that contain q. The *minimum membership set cover problem* is a variant in which the goal is to find a subset C' of C that minimizes the maximum membership of elements in P. A rich body of literature studies the minimum membership set cover problem [2, 10, 12, 13, 15, 16]. In this paper, we consider a set cover scenario in which the given sets of C may contain elements outside P and membership is evaluated for all elements covered by C', including those outside P. This concept appears in the literature as *ply cover*, which is formalized below.

The *ply* of a collection S of sets, denoted ply(S), is the maximum cardinality of any subset of S that has a non-empty common intersection. The set S *covers* a set P if $P \subseteq \bigcup_{S_i \in S} S_i$. Given a set P and a collection of sets U, a subset $S \subseteq U$ is a *minimum ply cover* of P if S covers P and S minimizes ply(S) over all subsets of U. Formally:

$$\text{plycover}(P,U) = \underset{\substack{S \subseteq U \\ S \text{ covers } P}}{\arg\min}\ \text{ply}(S). \qquad (1)$$

The ply of such a set S is called the *minimum ply cover number* of P with U, denoted ply$^*(P,U)$. For example, if $P = \{1,3,5,7,8\}$ and $U = \{\{1,2,3,4\},\{8\},\{3,4,5\},\{4,5,7\}\}$, then plycover$(P,U) = \{\{1,2,3,4\},\{8\},\{4,5,7\}\}$ and the minimum ply cover number is two.

Motivated by applications in covering problems, including interference minimization in wireless networks, Biedl et al. [3] introduced the *minimum ply cover problem* in the geometric setting: given sets P and U, find a subset $S \subseteq U$ that minimizes (1). When U is a set of unit disks representing transmission ranges of potential locations for placing wireless transmitters and P represents locations of wireless clients, $S \subseteq U$ corresponds to locations to install transmitters that minimize interference at any point in the plane.

Biedl et al. [3] showed that the problem is NP-hard to solve exactly, and remains NP-hard to approximate by a ratio less than two when P is a set of points in $\mathbb{R}^2$ and U is a set of axis-aligned unit squares or a set of unit disks in $\mathbb{R}^2$. They also provided 2-approximation algorithms parameterized in terms of ply$^*(P,U)$ for unit disks and unit squares in $\mathbb{R}^2$. Their algorithm for axis-parallel unit squares runs in $O((k+|P|)(2\cdot|U|)^{3k+1})$ time, where $k = \text{ply}^*(P,U)$, which is polynomial when ply$^*(P,U) \in O(1)$.

Biniaz and Lin [4] generalized this result for any fixed-size convex shape and obtained a 2-approximation algorithm when ply$^*(P,U) \in O(1)$. The problem of finding a polynomial-time approximation algorithm to the minimum ply cover problem remained open for both unit squares and unit disks when the minimum ply cover number, ply$^*(P,U)$, is not bounded by any constant.

Recently, Durocher et al. [11] settled this question affirmatively for unit squares by designing a polynomial-time $(8+\varepsilon)$-approximation algorithm for the problem, where $\varepsilon > 0$. We refer the reader to [19] for subsequent work that achieves faster algorithms, but with larger approximation factors.

Our contribution: In this paper we consider the minimum ply cover problem for a set P of points in $\mathbb{R}^2$

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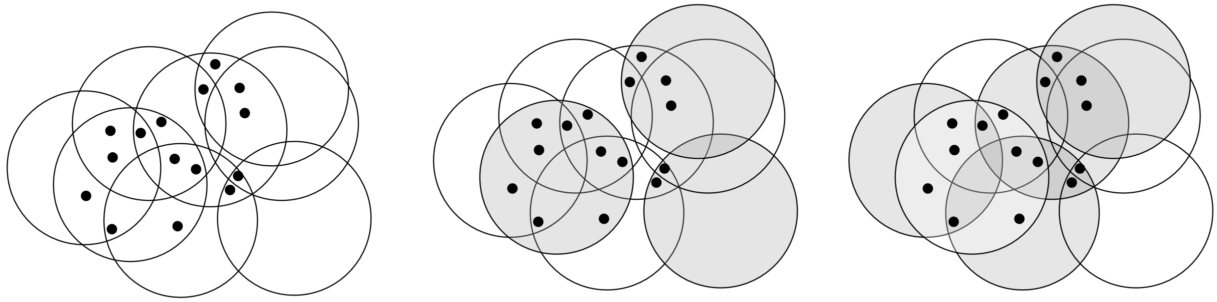

(a) (b) (c)

Figure 1: (a) An input consisting of points and unit disks. (b) A covering of the points with ply 1, which is also the minimum ply cover number for the given input. (c) A covering of the same instance with ply 3.

with a set U of unit disks in $\mathbb{R}^2$. We show that for every $\varepsilon > 0$, the minimum ply cover number can be approximated in polynomial time for unit disks within a factor of $(63+\varepsilon)$. This settles an open question posed in [3] and [4].

Our idea is to leverage the *minimum discrete unit disk cover problem* that seeks to cover a given point set with a smallest cardinality subset of the given disks. We show that there exist instances where the cardinality of the minimum discrete unit disk cover is at least 9.24 times the minimum ply cover. Hence, obtaining an approximation factor of 10 would be interesting, and we believe that achieving an approximation factor smaller than 10 would require a different technique that does not rely predominantly on a discrete unit disk cover.

Recent Developments: Recently, and independently of our work, Bandyapadhyay et al. [1] have shown that minimum ply cover can be approximated within a constant factor in $O(n\cdot\text{polylog}(n))$ time for fat objects, which includes unit disks and unit squares. Their idea is similar to the one that we used for disks. For unit squares, the technique yields an approximation factor of 36. For disks, they only provide a high-level argument for obtaining an $O(1)$-factor approximation rather than aiming for an exact value.

2 Approximating Minimum Ply Covering by Discrete Unit Disk Cover

Let P be a set of points in $\mathbb{R}^2$ and let U be a set of unit disks in $\mathbb{R}^2$. We assume that no three disks in U have boundaries that intersect at a common point. In this section we give a polynomial-time algorithm to approximate the minimum ply cover number for P with U within a factor of $O(1)$. We first give an overview of the algorithm and then describe its details.

2.1 Overview

Consider an axis-aligned grid $\mathcal{G}$ over P, where each grid cell is of size $(1/\sqrt{2}) \times (1/\sqrt{2})$. We choose a grid that is in general position relative to the disks in U, i.e., no disk is tangent to a grid line. A grid cell is called *non-empty* if it contains some point of P, otherwise, we call it *empty*.

We leverage the *minimum discrete unit disk cover problem* that, given a set of points and a set of unit disks on the Euclidean plane, seeks a minimum-cardinality subset of the input disks that covers the input points, for which a PTAS exists [17]. We show that one can first find an approximate solution to the minimum discrete unit disk cover for each non-empty grid cell, and then combine the solutions to obtain an approximate solution to the minimum ply cover for P.

2.2 Details of the Algorithm

We first remove all the disks in U that do not contain any point of P as they are not needed for covering P. Let R be a non-empty grid cell of $\mathcal{G}$. We first provide an upper bound on the cardinality of the minimum discrete unit disk cover in terms of the minimum ply cover number for the points and disks that overlap R (Lemma 1). We then show how to combine the respective solutions from each cell to obtain a cover of P by a subset of U whose ply cover number is at most $(63+\varepsilon)\,\text{ply}^*(P,U)$ (Theorem 2).

Lemma 1 *Let $Q \subseteq P$ be the points that lie in R and let $W \subseteq U$ be the set of unit disks that intersect R. Let S be a set of k points in the plane (i.e., not necessarily in P) such that every disk in W includes at least one point in S (points in S may lie outside R). The cardinality of every minimum discrete unit disk cover of Q by W is at most k times the minimum ply cover number for Q.*

Proof. Let δ be the cardinality of a minimum discrete unit disk cover for covering Q by W. Let β be the

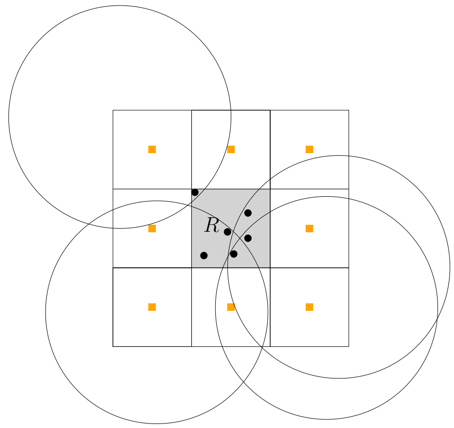

Figure 2: Illustration for Corollary 1.1, where R is shown in gray, Q is shown in black disks and S is shown in orange. Any disk that intersects the center grid cell must cover at least one orange point.

minimum ply cover number for covering Q by W. If $\delta \leq k\beta$, then $\beta \geq \delta/k$.

Suppose for a contradiction that the minimum ply cover number is less than δ/k. Since every disk in the minimum ply cover must hit at least one point in S, the number of disks in the cover is strictly less than δ. This contradicts our initial assumption that δ is the cardinality of a minimum discrete unit disk cover of Q. $\square$

It is straightforward to verify that for Lemma 1, it suffices to choose the centers of the 8 neighbouring cells of R as the point set S (Figure 2). Specifically, let D be a unit disk that intersects R. The unit disks centered at the points of S cover the entire region inside the convex hull of S. Therefore, if the center of D lies inside the convex hull of S, then D must include at least one point from S. The remaining case is when the center of D lies outside of S. If D does not include the points of S, then it can intersect a segment of length at most $1/\sqrt{2}$ from the convex hull boundary of S. However, this chord length is too short for D to reach R, which contradicts the assumption that D intersects R. Hence we obtain the following corollary.

Corollary 1.1 *Let $Q \subseteq P$ be the points that lie in R and let $W \subseteq U$ be set of unit disks that intersect R. The cardinality of a minimum discrete unit disk cover for Q by W is at most 8 times the minimum ply cover number for Q by W.*

In the following theorem we show how to combine the approximate solutions for the cells of $\mathcal{G}$ to obtain an $O(1)$-approximation for the minimum ply cover problem.

Theorem 2 *Let P be a set of points and let U be a set of unit disks, both in $\mathbb{R}^2$. Assume that for every $Q \subseteq P$ and $W \subseteq U$, there exists a $f(Q,W)$-time algorithm $\mathcal{A}$ that can approximate the cardinality of the minimum discrete unit disk cover of Q with W within a factor of γ. Then the minimum ply cover number for P using U can be approximated within a factor of 360γ in $O(|P| \cdot f(P,U))$ time.*

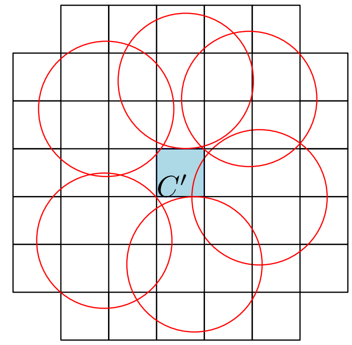

Figure 3: The friend cells for C'. The red circles illustrate that for every friend cell, there is a unit disk that intersects both that cell and C'.

Proof. Let U^* be a minimum ply cover for covering P with U. We consider a grid $\mathcal{G}$ over the point set P where each grid cell is of size $(1/\sqrt{2}) \times (1/\sqrt{2})$. Apply the algorithm $\mathcal{A}$ iteratively to find a γ-approximation for the cardinality of the minimum discrete unit disk cover for each grid cell. Let the maximum cardinality that we attain for a cell be δ_{max}. Let C be the cell that attains δ_{max}, and let Q_C and W_C be the points and unit disks corresponding to C, respectively. By Corollary 1.1, the cardinality of the minimum discrete unit disc cover is at most 8 times the minimum ply cover number for covering Q_C with W_C. Therefore, δ_{max} at most 8γ times the minimum ply cover number for Q_C. Since $Q_C \subseteq P$ and $W_C \subseteq U$, the minimum ply cover number for covering Q_C with W_C is smaller than the minimum ply cover number ($\operatorname{ply}(U^*)$) for covering P with U. Therefore, we have $\delta_{max} \leq 8\gamma \operatorname{ply}(U^*)$.

Let $\mathcal{O}$ be the union of all the approximate discrete unit disk covers obtained by applying the algorithm $\mathcal{A}$ to cells of $\mathcal{G}$, and let r be a point in the plane that does not fall on any grid line of $\mathcal{G}$. Let C' be the cell of $\mathcal{G}$ that contains r. In the following we show that r can belong to at most $45\delta_{max}$ disks in $\mathcal{O}$.

We refer to a cell D to be a *friend* of C' if a solution to the discrete unit disk cover for covering Q_D intersects C'. In other words, for every friend D, there is a unit disk that intersects both D and C'. There are 45 friend cells for C' (see Figure 3). Therefore, the number of disks that contains r in $\mathcal{O}$ is at most $45\delta_{max}$. Since $\delta_{max} \leq 8\gamma \operatorname{ply}(U^*)$, the number of unit disks in $\mathcal{O}$ that may contain r is at most $360\gamma \operatorname{ply}(U^*)$. Thus the ply of $\mathcal{O}$ is at most $360\gamma \operatorname{ply}(U^*)$. $\square$

Since there exists a PTAS for the discrete unit disk cover problem [17], we obtain the following corollary.

Corollary 2.1 *Given a set P of points and a set U of unit disks, both in $\mathbb{R}^2$, a ply cover of P using U can be computed in polynomial time whose ply is within a constant factor of the minimum ply cover number of P by U.*

3 Further Improvements

Note that we have some freedom when choosing the set S in Lemma 1 and the grid resolution in Theorem 2. Therefore, it is natural to leverage such freedom to further lower the approximation factor.

Note that there are several choices for S. For example, consider a regular pentagon inscribed in a unit circle centered at the center of R. Once can choose the corners of the pentagon as the points of S, as illustrated in Figure 4. Specifically, every unit disk with center lying inside the unit circle (shown in red) includes at least one point from S, and every unit disk with center lying outside the unit circle and avoiding S is unable to reach R, as illustrated in blue disks.

If we choose the corners of the pentagon as the points of S, then the approximation factor 8 in Corollary 1.1 improves to 5 and the overall approximation factor in Theorem 2 improves to $45 \cdot 5 \cdot \gamma = 225\gamma$. The factor 225 is determined partly by the number of fried cells, which is 45. To reduce this factor, we choose a hexagonal grid instead of a square grid. This requires us to design a new set of S, but it turns out that the overall approximation factor reduces to 63γ. We now give the details of the construction.

Let H be a regular hexagon that inscribes a unit circle with a side parallel to the x-axis (Figure 5(a)). Consider now a hexagonal grid $\mathcal{H}$ on the point set P where each hexagon is a copy of H. We compute the approximate discrete unit disk cover for each cell of H. Let δ_{max} be the largest approximate discrete unit disk cover that has appeared for a cell C.

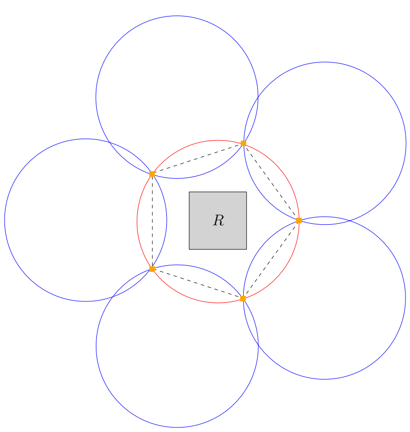

Figure 4: An alternative choice for S.

Observe that each hexagonal cell can be partitioned into 6 triangles by drawing a line segment between opposite corners of the hexagon (Figure 5(b)). While combining the solutions, we consider each triangular region instead of each hexagonal region, as follows.

Let T be a triangular region, as illustrated in Figure 5(c). We first use the idea of Lemma 1 to compute an upper bound on the minimum discrete unit disk cover for the points and unit disks corresponding to T. To obtain such an upper bound, we design a set S of 7 points such that any unit disk intersecting T contains at least one point from S. Let H' be the hexagonal cell that contains T and let o be the center of T. Then S includes the point o and the 6 points obtained from the intersection of the hexagonal grid and the circle of radius 1.5 centered at o. Figure 5(c) illustrates the circle of radius 1.5 in dashed lines and the points of S in orange. To verify that any unit disk D that intersects T contains a point from S, consider two cases. If the center c of D lies inside the hexagon H'' determined by $S \setminus \{o\}$, then c lies in an equilateral triangle with side length 1.5, which is determined by three points of S. Figure 5(c) illustrates the equilateral triangle in green. The radius of the circumscribed circle of this equilateral triangle is $1.5/\sqrt{3} < 1$. Therefore, D must contain a point from S. If the center c of D lies outside H'', then it can reach T only when D passes through two points of S, as illustrated in Figure 5(d).

We now compute the approximation factor using the same proof technique as in Theorem 2. Let $\mathcal{O}$ be the union of all the approximate discrete unit disk covers obtained by applying the algorithm $\mathcal{A}$ to the hexagonal cells of $\mathcal{H}$, and let r be a point in the plane that does not fall on any grid line of $\mathcal{H}$. Let C' be a triangular region that contains r. We now count the hexagonal cells that are within unit distance to the C'. In other words, the discrete unit disk cover solution for only these cells may contain r. There are 9 friend cells for C' (see Figure 5(e)). Therefore, the number of disks that contains r in $\mathcal{O}$ is at most $9\delta_{max}$. Since $|S| = 7$, we have $\delta_{max} \leq 7\gamma \operatorname{ply}(U^*)$, where γ is the approximation factor for the minimum discrete unit disk cover and U^* is the minimum ply cover. Consequently, the number of unit disks in $\mathcal{O}$ that may contain r is at most $9 \cdot 7 \cdot \gamma \operatorname{ply}(U^*)$. Thus the ply of $\mathcal{O}$ is at most $63\gamma \operatorname{ply}(U^*)$. Since there is a polynomial-time $(1+\varepsilon')$-approximation for the minimum discrete unit disk cover [17], we obtain a $(63+\varepsilon)$-approximation for the minimum ply cover number where we choose ε' to be $\varepsilon/63$.

The following theorem summarizes the result of this section.

Theorem 3 *Given a set P of points and a set U of unit disks, both in $\mathbb{R}^2$, and a constant $\varepsilon > 0$, a ply cover of P*

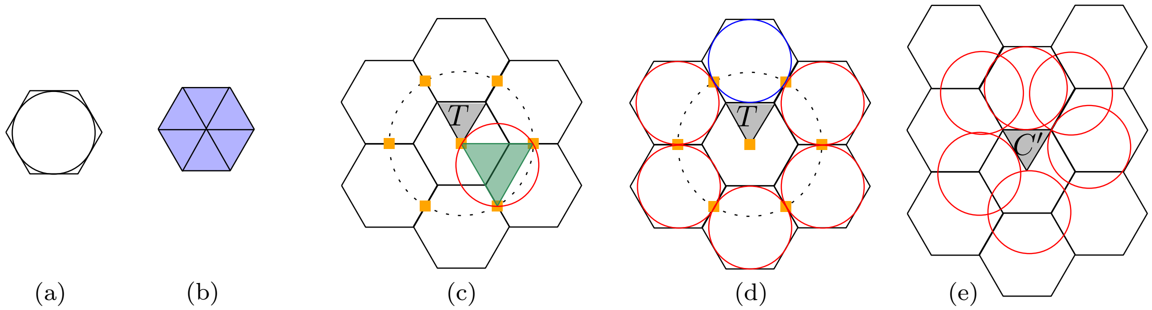

Figure 5: Improving the approximation factor by choosing a hexagonal grid.

using U can be computed in polynomial time whose ply is at most $(63+\varepsilon)$ times the minimum ply cover number of P by U.

The bottleneck of the running time of our algorithm is the time to compute the discrete unit disk cover. In 1995, Brönnimann and Goodrich gave an $O(1)$-approximation algorithm for minimum discrete unit disk cover [5]. A rich body of research attempted to lower the approximation factor since then [6, 18, 7, 8]. The $(1+\varepsilon)$-approximation result for the minimum discrete unit disk cover [17] has a running time of $O(m^{2(c/\varepsilon)^2+1}n)$, where m and n are the numbers of disks and points, respectively, and c is a constant. This running time is large, i.e., the fastest achievable running time is $O(m^{65}n)$ by setting $\varepsilon = 2$, which gives a 3-approximation [14]. Das et al. [9] gave an 18-approximation algorithm that runs in $O(n \log n + m \log m + mn)$ time, which may be used to compute an approximate solution to the minimum ply cover problem faster, but the approximation factor would increase to 1134.

4 Lower Bound

Our approximation algorithm for the minimum ply cover problem relies heavily on finding a discrete unit disk cover. In this section, we construct instances where the cardinality of the minimum discrete unit disk cover is at least 9.2444 times the minimum ply cover number. The bound 9.24 is constructed to complement our approach, i.e., in general, the number of disks in a discrete unit disk cover could be unbounded compared to the minimum ply cover number. This 9.24 lower bound indicates that achieving an approximation factor less than 10 may be unlikely using our approach.

Choose any $n \geq 2$. We construct a set $\{D_1, \ldots, D_n\}$ of n unit disks such that the boundary of each disk is tangent to a common point o (each disk center is a unit distance from o), and the disks are positioned uniformly around o. Figure 6 shows these disks in gray. Consider a circle C of radius 2 centered at o (shown in orange in Figure 6). For each $i \in \{1, \ldots, n\}$, we add a point p_i (shown in red in Figure 6) at the intersection of the boundaries of C and D_i, and place a unit disk D'_i (shown in black in Figure 6) such that p_i is the midpoint of the centers of D_i and D'_i.

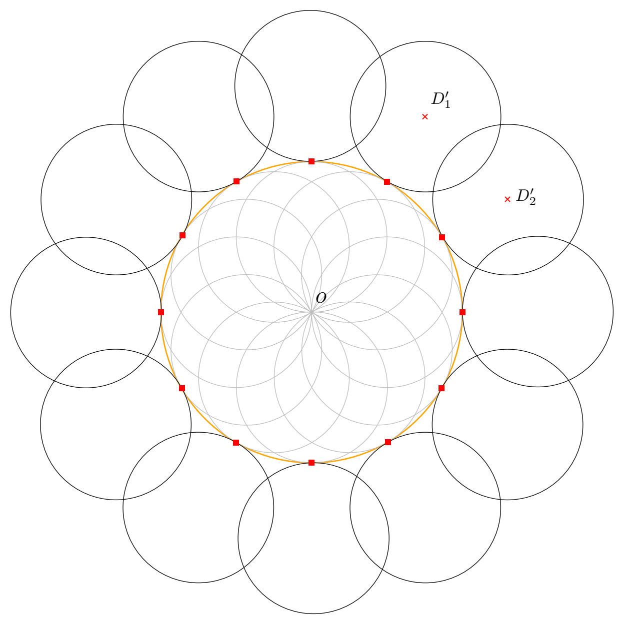

Figure 6: Illustration for the construction of a ply cover instance (P, U) when $n = 12$. The points of P are shown in red, and U consists of the black and gray disks.

Consider an instance of the minimum ply cover problem (P, U), where $P = \{p_1, \ldots, p_n\}$ and $U = \{D_1, \ldots, D_n, D'_1, \ldots, D'_n\}$. For each $i \in \{1, \ldots, n\}$, the point $p_i \in P$ is covered by exactly two disks in U, D_i and D'_i; furthermore, D_i and D'_i cover no points in $P \setminus \{p_i\}$. Therefore, any disk cover of P by U must contain at least n disks and must contain either D_i or D'_i for each $i \in \{1, \ldots, n\}$.

The set $U' = \{D'_1, \ldots, D'_n\}$ covers P and $|U'| = n$. Therefore, U' is a minimum discrete unit disk cover of P. Similarly, the set $U'' = \{D_1, \ldots, D_n\}$ covers P, $|U''| = n$, and U'' is also a minimum discrete unit disk cover of P. U'' has ply n. We now calculate the ply of U'.

See Figure 7, illustrating the point o and disks D_i and D'_i, for some $i \in \{1, \ldots, n\}$. The segment $\overline{oc_i}$ is the diameter of D_i plus the radius of D'_i; therefore it has length 3. Consequently, $\theta = 2\sin^{-1}(1/3)$, and the ply of U' is $\lceil \frac{n2\sin^{-1}(1/3)}{2\pi} \rceil$.

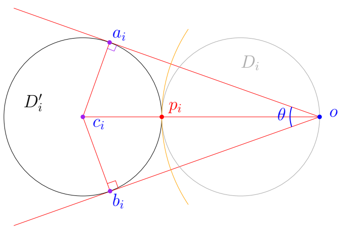

Figure 7: The sector rooted at o with boundary tangent to the disk D'_i forms an angle $\theta = 2\sin^{-1}(1/3)$ at o.

An adversarial choice of minimum discrete unit disk cover of P by U selects U'. Consequently, no minimum discrete unit disk cover can guarantee to approximate the minimum ply by less than

$$\lim_{n\to\infty} \frac{\text{ply}(U'')}{\text{ply}(U')} = \lim_{n\to\infty} \frac{n}{\lceil 2n\sin^{-1}(1/3)\rceil/(2\pi)} = \frac{\pi}{\sin^{-1}(1/3)} > 9.2444.$$

The following theorem summarizes the result of this section.

Theorem 4 *For sufficiently large n, there exists a set of n points and $2n$ disks for which the ply of a minimum discrete unit disk cover is at least 9.24 times the minimum ply cover.*

5 Conclusion

We have shown that given a set of points and a set of unit disks in the Euclidean plane, one can compute a ply cover whose ply is within a constant factor of the minimum ply cover number. The approximation factor we obtain is large (i.e., $63 + \varepsilon$), whereas only a 2-inapproximability result is known [3]. Therefore, a natural direction of future research is to narrow down this gap.

Our approximation algorithm relies on finding an approximate discrete unit disk cover and we have constructed instances where a minimum discrete unit disk cover is at least 9.24 times the minimum ply cover number. This raises the question of whether the approximation factor could be brought down closer to 10, or whether the existing 2-inapproximability result could be strengthened further using the disk configurations that we used in this paper.

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