Modelling Gateway Placement in Wireless Networks: Geometric k-Centres of Unit Disc Graphs*

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ABSTRACT

Motivated by the gateway placement problem in wireless networks, we consider the *geometric k-centre* problem on unit disc graphs: given a set of points P in the plane, find a set F of k points in the plane that minimizes the maximum graph distance from any vertex in P to the nearest vertex in F in the unit disc graph induced by $P \cup F$. We describe exact and approximate polynomial-time solutions to this problem for any fixed k and show that the problem is NP-hard when k is an arbitrary input parameter.

Categories and Subject Descriptors

F.2.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity—*Nonnumerical Algorithms and Problems*; G.2.2 [**Mathematics of Computing**]: Discrete Mathematics—*Graph Theory*; C.2.1 [**Computer Systems Organization**]: Computer-Communication Networks—*Network Architecture and Design*

General Terms

Algorithms, Theory

Keywords

unit disc graph, k-centre, gateway placement, facility location, wireless networks

*Work of the first, third, and fourth authors was supported in part by NSERC.

DialM-POMC'08, August 22, 2008, Toronto, Ontario, Canada.

1. INTRODUCTION

1.1 Motivation

In a wireless sensor network, sensor nodes collect and send data to *sink* nodes, which may either be the users of the data, or gateways to another (possibly wired) network through which a remote user can access the data. Sensor nodes perform a sensing function as well as a routing and forwarding function to move data to sink nodes. Since sensor nodes are battery powered, conserving and making efficient use of energy is an important consideration for all network protocols. In particular, forwarding packets depletes battery power at all nodes on a routing path, a problem that is made worse if sink nodes are poorly positioned, resulting in longer path lengths to sink nodes. Similarly, much of the traffic in a wireless mesh network passes through *gateway* nodes that provide connectivity to exterior networks such as the Internet [1]. To optimize bandwidth usage, it is important to minimize the path length between nodes and gateways [1].

This motivates the problem of optimal sink placement in a wireless sensor network or gateway placement in a wireless mesh network. In this paper, we model these problems as a *facility location* problem, in which network nodes correspond to clients, and gateways or sink nodes correspond to facilities. A wireless network is often modelled by a unit disc graph (e.g., [4, 11, 12, 16, 17]) where the nodes are represented by points on the plane and a node u is connected to every node located in the unit disc centred at u. Given a set of points P in the plane, we consider the problem of finding a set F of k points in the plane that minimizes the maximum graph distance between any point in P and the nearest point in F in the unit disc graph induced by $P \cup F$. Although this problem is similar to the Euclidean k-centre and vertex k-centre problems (see Section 3), this version of the problem incorporates both geometric and graph-theoretic constraints, resulting in a new problem which we call the *geometric k-centre* problem.

In the geometric k-centre problem, facilities may be selected from anywhere in the plane (as in the Euclidean k-centre problem) whereas the distance between clients and facilities is measured by graph distance (as in the vertex k-

centre problem). Thus the geometric k-centre problem is neither set solely in the host metric space nor on a graph. Given this new dual setting, existing solutions to the k-centre problem on graphs or in Euclidean space do not necessarily provide solutions to the geometric k-centre problem.

1.2 Overview of Results

We show that the vertex 1-centre provides a 5-approximation of the geometric 1-centre; this bound is tight. We describe polynomial-time algorithms for finding exact and approximate geometric 1-centres of a unit disc graph. Our technique generalizes to finding a geometric k-centre for any fixed k. When k is an arbitrary input parameter, we show that the geometric k-centre problem is NP-hard on unit disc graphs.

2. DEFINITIONS

Given a continuous metric space S, let $d_S(p, q)$ denote the metric distance between points p and q in S. Given a graph $G = (V, E)$, let $d_G(u, v)$ denote the unweighted graph distance between vertices u and v in V. Region $R_i \subseteq S$ is a *ball* of radius ρ if there is a central point $c \in R_i$ such that $R_i = \{p \mid p \in S \text{ and } d_S(c, p) \leq \rho\}$. Let $\partial(R_i)$ denote the boundary of region R_i in S and let $\text{int}(R_i)$ denote its interior. We say a set of regions $R = \{R_1, \ldots, R_n\}$ is *uniform* if there exists a $\rho \geq 0$ such that every $R_i \in R$ is a ball of radius ρ in S. Examples of uniform sets of regions include a set of unit intervals in $\mathbb{R}$ under any ℓ_p metric, a set of unit discs in $\mathbb{R}^2$ under the ℓ_2 metric, and a set of unit cubes in $\mathbb{R}^3$ under the ℓ_∞ metric.

In this paper we consider the geometric k-centre problem on intersection graphs of uniform discs. We remind the reader of the definition of an intersection graph:

Definition 1 (Intersection Graph). *Given a set of regions $R = \{R_1, \ldots, R_n\}$ in S, the* intersection graph *induced by R has vertex set R and edge set $\{(R_i, R_j) \mid R_i \cap R_j \neq \varnothing\}$.*

Next we define a geometric k-centre on an intersection graph:

Definition 2 (Geometric k-Centre). *Given a set of regions $R = \{R_1, \ldots, R_n\}$ in a metric space S, a positive integer k, and a non-negative real number ρ, a* geometric k-centre *of R is a set of regions $F = \{F_1, \ldots, F_k\}$ in S, such that each $F_i \in F$ is a ball of radius at most ρ and F minimizes the* eccentricity *of F in R, denoted $\text{ecc}_G(R, F)$, where*

$$\text{ecc}_G(R, F) = \max_{R_i \in R} \min_{F_j \in F} d_G(R_i, F_j), \tag{1}$$

and G denotes the intersection graph of $R \cup F$.

For a given R, we refer to the minimum value of (1) as the *geometric k-radius* of R. In the facility location literature, R typically represents a set of *clients* (the input defining a problem instance) and F represents a set of *facilities* (a solution to the problem instance); we use these terms to differentiate between regions in R and regions in F. The geometric k-centre problem is closely related to the vertex k-centre problem:

Definition 3 (Vertex k-Centre). *Given a graph $G = (V, E)$ and a positive integer k, a* vertex k-centre *of G is a set of vertices $F = \{v_1, \ldots, v_k\} \subseteq V$ that minimizes*

$$\max_{u \in V} \min_{v_j \in F} d_G(u, v_j). \tag{2}$$

We refer to the value of (2) as the *vertex k-radius* of G. A vertex k-centre is often called simply a *k-centre*; we add the prefix "vertex" to distinguish it from a geometric k-centre. The vertex k-centre problem has been studied extensively (see Section 3).

Although the geometric k-centre problem can be applied to several classes of intersection graphs, we primarily focus on graphs commonly used to model the topology of wireless networks: unit disc graphs.

Given a point $p \in \mathbb{R}^2$, let $\text{Disc}_r(p)$ denote the disc of radius r centred at p. Similarly, given a set of points $P \subseteq \mathbb{R}^2$, let $\text{Disc}_r(P)$ denote the corresponding set of discs. When $r = 1$ we omit the subscript r.

Definition 4 (Unit Disc Graph). *Given a set of points P in $\mathbb{R}^2$ under the ℓ_2 metric, the* unit disc graph *induced by P, denoted $\text{UDG}(P)$, is an embedded graph with vertex set P and edge set $\{(u, v) \mid d_S(u, v) \leq 1\}$.*

That is, vertices p and q in P are adjacent in $\text{UDG}(P)$ if and only if $q \in \text{Disc}(p)$. See the example in Figure 1. Equivalently, vertices p and q in P are adjacent in $\text{UDG}(P)$ if and only if $\text{Disc}_{1/2}(p) \cap \text{Disc}_{1/2}(q) \neq \varnothing$. Thus, a unit disc graph is an intersection graph. With respect to our discussion of geometric k-centres on unit disc graphs, we fix $\rho = 1/2$ and identify the location of a client or facility by the point p at the centre of the corresponding disc.

If $P \subseteq \mathbb{Z}^2$, then $\text{UDG}(P)$ is a *grid graph*. A unit disc graph is not necessarily planar and its maximum degree can be as large as $|P| - 1$. A grid graph, on the other hand, is planar and has maximum degree at most four. Naturally, the definition of a unit disc graph generalizes to three or higher dimensions as a *unit ball graph* and to one dimension as a *unit interval graph*, both of which can be considered with respect to the geometric k-centre problem.

The *arrangement* induced by a set of regions R in S, denoted $\mathcal{A}_R$, is a set of *cells*, each of which is a maximal connected region such that $\mathcal{A}_R$ forms a partition of S and for all $C \in \mathcal{A}_R$ and all $R_i \in R$, $\text{int}(C) \cap \partial(R_i) = \varnothing$. When $S = \mathbb{R}^2$, we define the *arrangement graph* of R as the planar multigraph $G = (V, E)$ whose vertex set V corresponds to points at which the boundaries of three or more cells intersect and whose edge set E corresponds to the simple curves in $\cup_{C \in \mathcal{A}_R} \partial(C) \setminus V$. The edges of the dual of the arrangement graph can be directed such that $(C_a, C_b) \in E$ if and only if for every $p \in P$, $C_a \in \text{Disc}(p) \Rightarrow C_b \in \text{Disc}(p)$. See Figures 2A and 2B.

Given a graph $G = (V, E)$, we employ standard graph-theoretic notation, where for each vertex $v \in V$, $\text{Adj}(v) = \{u \mid (u, v) \in E\}$ denotes the set of vertices adjacent to v, $\deg(v) = |\text{Adj}(v)|$ denotes its degree, and $N(v) = \text{Adj}(v) \cup \{v\}$ denotes its neighbourhood.

3. RELATED WORK

3.1 Vertex k-Centre

Given a graph $G = (V, E)$, Hakimi and Kariv [13] give an algorithm to find a vertex 1-centre in $O(mn + n^2 \log n)$ time, where $n = |V|$ and $m = |E|$. A vertex 1-centre

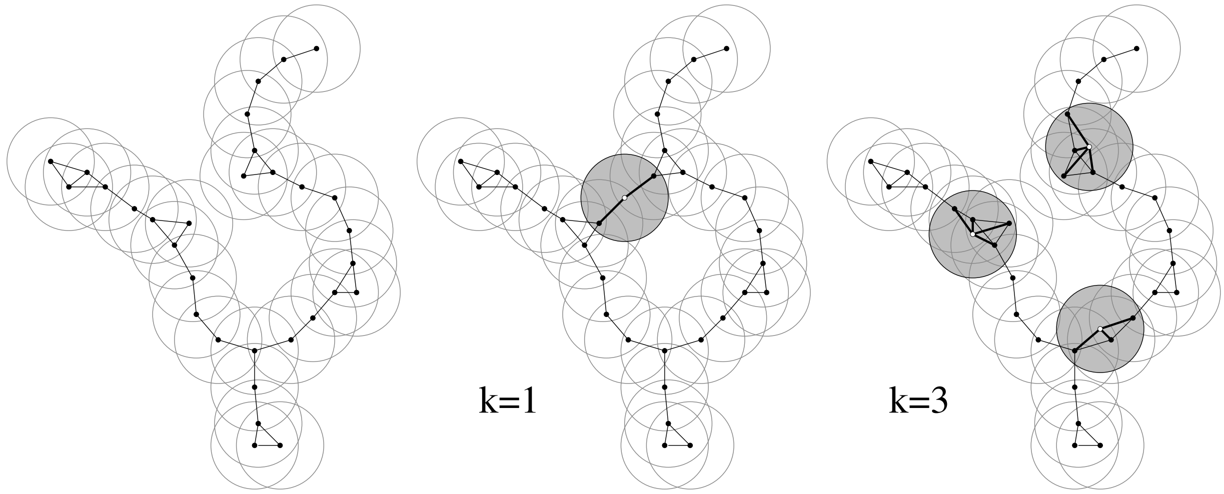

Figure 1: (Left) A set of points P, the corresponding set $\mathrm{Disc}(P)$, and $\mathrm{UDG}(P)$. (Middle) The point at the centre of the shaded unit disc is a geometric 1-centre of P. The corresponding graph $\mathrm{UDG}(P \cup F)$ is illustrated. (Right) The set of points at the centres of the three shaded unit discs is a geometric 3-centre of P. The corresponding graph $\mathrm{UDG}(P \cup F)$ is illustrated.

can also be found by calculating the unweighted all-pairs shortest path distances and identifying the vertex for which the maximum distance is minimized; as shown by Chan [7], this can be done in $O(mn/\log n)$ time if $m > \log^2 n$, $O(mn \log\log n/\log n)$ time if $m > n \log\log n$, and $O(n^2 \log^2 \log n/\log n)$ time if $m \leq n \log\log n$. When k is fixed, a vertex k-centre can be found in $O(m^k n^k \log n)$ time [21]. When k is an input parameter, the problem is NP-hard [13].

3.2 Unit Disc Graphs

Clark et al. [9] give hardness results for several problems on unit disc graphs, including the minimum dominating set problem (which we use as the basis for our hardness reduction in Section 4.5). They mention an earlier result by Masuyama et al. [19] regarding hardness of the vertex k-centre problem on unit disc graphs. Marathe et al. [18] describe approximation algorithms for NP-hard problems on unit disc graphs, including a 5-approximation for the minimum dominating set problem. They observe that any independent set in the neighbourhood of a vertex v has cardinality at most five. Given $P \subseteq \mathbb{R}^2$, Breu [5] describes an $O(m + n \log n)$-time algorithm for constructing $\mathrm{UDG}(P)$ and an $O(n \log n)$-time algorithm for enumerating the connected components of $\mathrm{UDG}(P)$. Breu and Kirkpatrick [6] show it is NP-hard to decide whether a graph is a unit disc graph. That is, given only the combinatorial description for a UDG it is NP-hard to find a unit disc embedding in the plane. This result is extended by Kuhn et al. [15] who show that it is NP-hard to approximate within a factor of $1/\sqrt{2}$. The difficulty in finding a geometric k-centre of a unit disc graph arises from the geometric constraints implied by an embedding; given only a combinatorial description for a graph, the addition of a universal vertex trivially solves the problem. As such, we assume knowledge of the graph's planar embedding in a problem instance.

3.3 Geometric Sink/Relay Placement

Similar to the geometric k-centre problem in which k is fixed and the objective is to minimize the geometric k-radius, Mihandoust and Narayanan [20] consider the related h-hop covering set problem on a unit disc graph, in which the maximum k-radius is fixed and the objective is to minimize k. They provide PTASs for several variations of this problem. Aoun et al. [1] follow a similar approach for gateway placement in wireless mesh networks. Efrat et al. [10] consider the related relay placement problem, in which the objective is to add the minimum number of facilities (relays) such that the resulting network is connected. They consider a more general model in which the range of communication of relays and network nodes may differ.

4. FINDING A GEOMETRIC K-CENTRE OF A UNIT DISC GRAPH

We begin by examining properties of arrangements of unit discs. We then establish bounds on the ratio of the geometric 1-radius to the vertex 1-radius. We describe polynomial-time algorithms for finding exact and approximate solutions to the geometric 1-centre problem on unit disc graphs and discuss how to generalize the solution for any fixed k. Finally, we show that the problem is NP-hard when k is an arbitrary input parameter. Throughout Section 4, P denotes an arbitrary set of points in $\mathbb{R}^2$, $R = \mathrm{Disc}(P)$, $\mathcal{A}_R$ denotes the arrangement induced by $\mathrm{Disc}(P)$, $n = |P|$, and m denotes the number of edges in $\mathrm{UDG}(P)$.

4.1 The Arrangement of a Set of Discs

Definition 4 and the definition of an arrangement imply the following observation:

Observation 1. *Given a set $P \subseteq \mathbb{R}^2$ and points f_1 and f_2 in the same cell of the arrangement of* $\mathrm{Disc}(P)$,

$$\mathrm{ecc}_{\mathrm{UDG}(P \cup \{f_1\})}(P, \{f_1\}) = \mathrm{ecc}_{\mathrm{UDG}(P \cup \{f_2\})}(P, \{f_2\}).$$

Therefore, if point f_1 is a geometric 1-centre of P, then any point in the same cell as f_1 is also a geometric 1-centre of P. Consequently, to identify a geometric 1-centre of P it suffices to consider one point from every cell in $\mathcal{A}_R$.

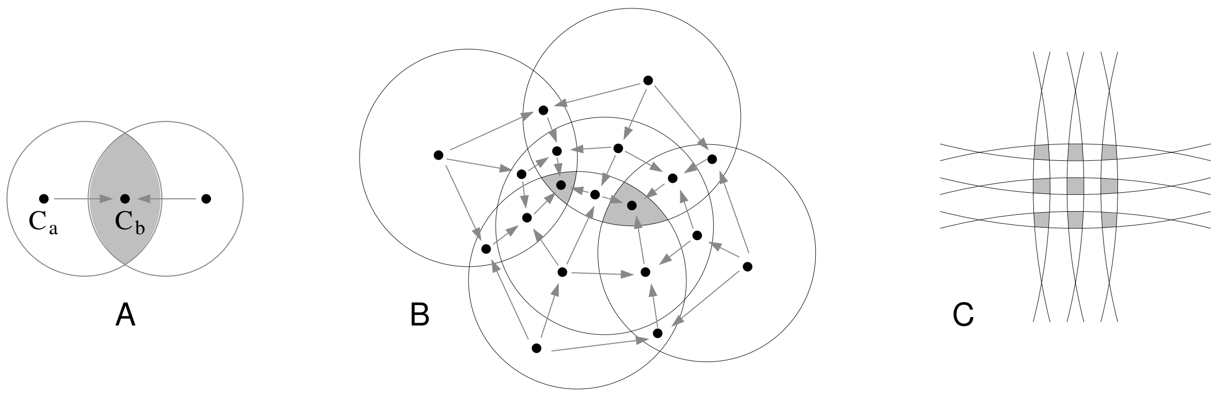

Figure 2: A. The edges of the dual of the arrangement graph can be directed such that $(C_a, C_b) \in E$ if and only if for every $p \in P$, $C_a \in \mathrm{Disc}(p) \Rightarrow C_b \in \mathrm{Disc}(p)$. B. The arrangement induced by these four unit discs partitions the plane into twenty cells. The partial order of the corresponding dual graph has four sources and two sinks. To select locations for a facility, it suffices to consider the sinks, which correspond to convex cells (shaded). C. This example due to Tóth [22] shows an arrangement induced by n unit discs that has $\Omega(n^2)$ convex cells.

By Propositions 2 and 3, the number of cells (and the number of convex cells) in any arrangement of discs in the plane is $\Theta(n^2)$ in the worst case; this value is directly proportional to the run times of algorithms we describe in Sections 4.3 and 4.4.

PROPOSITION 2 (KONHAUSER ET AL. 1996 [14]). *An arrangement of n discs in $\mathbb{R}^2$ contains at most $n^2 - n + 2$ cells. This bound is tight.*

The geometric dual of the arrangement graph of $\mathcal{A}_R$ is a planar graph $G = (V, E)$ whose vertex set is $\mathcal{A}_R$ and whose edges connect adjacent cells in $\mathcal{A}_R$. The edges of G can be directed such that $(C_a, C_b) \in E$ if and only if for every $p \in P$, $C_a \in \mathrm{Disc}(p) \Rightarrow C_b \in \mathrm{Disc}(p)$. Since any facility located within it will be disconnected from $\mathrm{UDG}(P)$, we omit any face not contained in a unit disc (e.g., the exterior face) from V. See Figures 2A and 2B. Observe that G is a partial order relation. Furthermore, for any cells $\{C_a, C_b\} \subseteq \mathcal{A}_R$ and any points $f_a \in C_a$ and $f_b \in C_b$, if $(C_a, C_b) \in E$, then $\mathrm{UDG}(P \cup \{f_a\})$ is a subgraph of $\mathrm{UDG}(P \cup \{f_b\})$. That is,

$$\mathrm{ecc}_{\mathrm{UDG}(P\cup\{f_b\})}(P, f_b) \leq \mathrm{ecc}_{\mathrm{UDG}(P\cup\{f_a\})}(P, f_a).$$

Consequently, when selecting a position for a 1-centre, it suffices to consider only cells in $\mathcal{A}_R$ that are sinks with respect to the partial order induced by $\mathcal{A}_R$. The sinks correspond exactly to the convex cells in $\mathcal{A}_R$. One might hope that the number of sinks is asymptotically less than the total number of cells; this is not the case, as shown by the following proposition based on an example suggested by Tóth [22].

PROPOSITION 3. *For any $n \in \mathbb{Z}^+$, there exists an arrangement of n unit discs in $\mathbb{R}^2$ for which the number of convex cells is at least $\lfloor n/4 \rfloor^2$.*

PROOF. Choose any n.

Case 1. Suppose $n \bmod 4 = 0$. Position two unit discs such that their centres are distance $2 - 64/(16 + n^2)$ apart. It is straightforward to show that their intersection is a lune of width $64/(16 + n^2)$ and height $16n/(16 + n^2)$. Observe that the height is $n/4$ times the width. Therefore, n discs can be positioned such that $n/4$ vertical lunes each intersect $n/4$ horizontal lunes. See Figure 2C. Each lune is convex and, therefore, the intersection of two lunes is also convex, resulting in at least $n^2/16$ convex cells.

Case 2. Suppose $n = 4k + i$ for some $k \in \mathbb{Z}$ and some $i \in \{1, 2, 3\}$. Given any sets of unit discs R_1 and R_2, the number of convex cells in $\mathcal{A}_{R_1 \cup R_2}$ is greater than or equal to the number of convex cells in $\mathcal{A}_{R_1}$. The result follows by Case 1 since $\lfloor (4k + i)/4 \rfloor^2 = \lfloor (4k)/4 \rfloor^2$. $\square$

4.2 Approximating a Geometric 1-Centre by a Vertex 1-Centre

Facilities in a geometric 1-centre can be positioned anywhere in the plane while facilities in a vertex 1-centre must coincide with clients. Consequently, the geometric 1-radius of a unit disc graph is at most the vertex 1-radius. Of course, the geometric 1-radius can be less than the vertex 1-radius. Theorem 4 bounds the ratio between the two radii.

THEOREM 4. *If $\mathrm{UDG}(P)$ is connected, then the vertex 1-radius of $\mathrm{UDG}(P)$ is at most five times its geometric 1-radius. This bound is tight.*

PROOF. Choose any finite set $P \subseteq \mathbb{R}^2$. Let $f \in \mathbb{R}^2$ be a geometric 1-centre of P and let r denote the corresponding geometric 1-radius. Let $\{c_1, \ldots, c_t\}$ be an independent set of $\mathrm{Adj}(f)$ in $\mathrm{UDG}(P \cup \{f\})$. It follows that $t \leq 5$ [18]. Partition P into $P_1, \ldots, P_t$ such that for all $\{i, j\} \subseteq \{1, \ldots, t\}$,

$$\forall p \in P_i,\ d_{\mathrm{UDG}(P\cup\{f\})}(p, c_i) \leq d_{\mathrm{UDG}(P\cup\{f\})}(p, c_j).$$

Therefore, for any i and any $\{p, q\} \subseteq P_i$, $d_{\mathrm{UDG}(P)}(p, q) \leq 2r - 1$. Since $\mathrm{UDG}(P)$ is connected and $k \leq 5$, it follows that for some i and all $p \in P$, $d_{\mathrm{UDG}(P)}(c_i, p) \leq 5r$. Therefore, the vertex 1-radius of $\mathrm{UDG}(P)$ is at most $5r$.

This bound is realized in the limit as $s \to \infty$ by the graph G_s illustrated in Figure 3. For any $s \geq 2$, G_s has geometric 1-radius $2 + \lceil s/2 \rceil$ (realized by the geometric 1-centre located at f) and vertex 1-radius $\lceil 5s/2 \rceil$. $\square$

In other words, a vertex k-centre of P provides a 5-approximation of its geometric k-centre when $\mathrm{UDG}(P)$ is connected.

4.3 Finding a Geometric 1-Centre

Building on our observations from Section 4.1, we describe algorithms for finding a geometric 1-centre in $\Theta(n^2 m)$

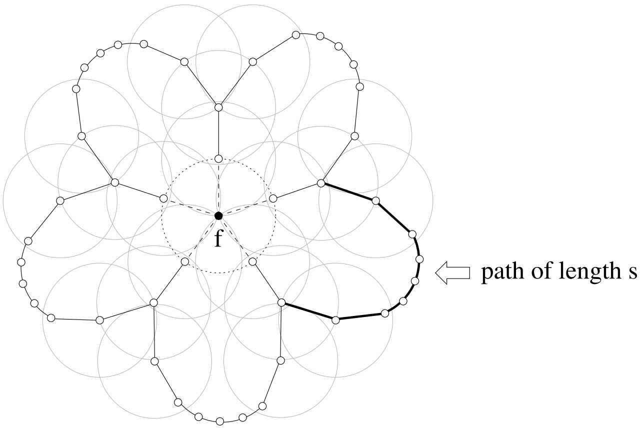

Figure 3: illustration in support of Theorem 4

worst-case time and a nearly-optimal approximate geometric 1-centre in $\Theta(n^3)$ time; the resulting approximate solution has eccentricity at most one greater than the geometric 1-radius, corresponding to an additive approximation factor of at most one.

4.3.1 An Exact Solution using Breadth-First Search

Chazelle and Lee [8] describe how to build the arrangement graph of R in $O(n^2)$ time. As the graph is constructed, for each cell C we maintain a list of discs within which C is contained; these correspond to the neighbours of f in $\mathrm{UDG}(P \cup \{f\})$, where f is any point in C. Since a disc can be contained in $\Theta(n)$ other discs, this increases the run time to $O(n^3)$. A traversal of this graph can be used to enumerate the cells of $\mathcal{A}_R$ (faces of the graph) in $O(|\mathcal{A}_R|)$ time. A geometric 1-centre of P can be found by considering one point f from each cell in $\mathcal{A}_R$ and using breadth-first search to compute the eccentricity of f in $\mathrm{UDG}(P \cup \{f\})$. The minimum such value is the geometric 1-radius of $\mathrm{UDG}(P)$ and the corresponding point f is a geometric 1-centre. In the pseudocode below, BFS-DEPTH(G, v) calls a standard queue-based breadth-first search algorithm to calculate the distance from v to the furthest vertex in G.

GEOMETRIC 1-CENTRE(P)

$\quad$ $radius \leftarrow \infty$
$\quad$ **for** each cell $C \in \mathcal{A}_R$
$\quad\quad$ $f \leftarrow$ any point in C
$\quad\quad$ $ecc \leftarrow$ BFS-DEPTH$(\mathrm{UDG}(P \cup \{f\}), f)$
$\quad\quad$ **if** $ecc < radius$
$\quad\quad\quad$ $radius \leftarrow ecc$
$\quad\quad\quad$ $centre \leftarrow f$
$\quad$ **return** $centre$

Adding vertex f increases the number of edges in $\mathrm{UDG}(P)$ by at most n. Therefore, each call to breadth-first search on $\mathrm{UDG}(P \cup \{f\})$ takes $\Theta(n + m)$ time. By Proposition 2, $|\mathcal{A}_R| \in O(n^2)$. Therefore, GEOMETRIC 1-CENTRE has worst-case run time $\Theta(n^2(m+n))$. Recall that $\mathrm{UDG}(P \cup \{f\})$ must be connected for a geometric 1-centre to exist. Therefore, $m \geq n - 1$ and the run time simplifies to $\Theta(n^2 m)$. In the worst case, therefore, this algorithm is quartic in n.

4.3.2 A Faster Approximate Solution

Although it suffices to consider only convex cells in $\mathcal{A}_R$, the number of such cells remains $\Omega(n^2)$ in the worst case by Proposition 3. Therefore, the worst-case run time of GEOMETRIC 1-CENTRE is not improved by considering only convex cells. As we now show, a faster algorithm is possible if we relax constraints on optimality and allow the eccentricity of a solution to exceed the geometric 1-radius by at most one.

As with the previous algorithm, we begin by constructing $\mathcal{A}_R$ and the corresponding lists of discs in which each cell is contained. A point f is selected within each cell and each of these lists is partitioned according the corresponding regions $R_1(f)$ through $R_6(f)$. These regions correspond to six symmetric sectors whose union forms the unit disc centred at f. See Figure 4A. The algorithm computes the approximate eccentricity by iteratively calculating

$$\min_{C \in \mathcal{A}_R} \max_{p \in P} \min_{i \in \{1,\ldots,6\}} \left(1 + d_{\mathrm{UDG}(P)}(q_i, p)\right),$$

where f is any point in C and q_i is any point in $R_i(f)$. By Lemma 5, to compute the approximate eccentricity of a point f it suffices to iterate over all $p \in P$ and compare the graph distance between p and a point q_i in $P \cap R_i(f)$ for each nonempty region $R_i(f)$. Adding one to the minimum of these (at most) six distances gives either $d_{\mathrm{UDG}(P\cup\{f\})}(f, p)$ or $d_{\mathrm{UDG}(P\cup\{f\})}(f, p) + 1$, depending on whether a shortest path from f to p passes through the point q_i that was selected. The algorithm makes use of unweighted all-pairs shortest-path distances on the vertices of $\mathrm{UDG}(P)$. This distance function can be precomputed in $o(mn)$ time (e.g., see [7]).

APPROXIMATE GEOMETRIC 1-CENTRE(P)

$\quad$ $approxRadius \leftarrow \infty$
$\quad$ **for** each cell $C \in \mathcal{A}_R$
$\quad\quad$ $f \leftarrow$ any point in C
$\quad\quad$ $approxEcc \leftarrow 0$
$\quad\quad$ **for** each point $p \in P$
$\quad\quad\quad$ $dist \leftarrow \infty$
$\quad\quad\quad$ **for** $i \leftarrow 1$ **to** 6
$\quad\quad\quad\quad$ $q_i \leftarrow$ any point in $R_i(f) \cap P$
$\quad\quad\quad\quad$ **if** $d_{\mathrm{UDG}(P)}(q_i, p) + 1 < dist$
$\quad\quad\quad\quad\quad$ $dist \leftarrow d_{\mathrm{UDG}(P)}(q_i, p) + 1$
$\quad\quad\quad$ **if** $dist > approxEcc$
$\quad\quad\quad\quad$ $approxEcc \leftarrow dist$
$\quad\quad$ **if** $approxEcc < approxRadius$
$\quad\quad\quad$ $approxRadius \leftarrow approxEcc$
$\quad\quad\quad$ $approxCentre \leftarrow f$
$\quad$ **return** $approxCentre$

LEMMA 5. *For any set of points P in $\mathbb{R}^2$, any point $f \in \mathbb{R}^2$, and any point $p \in P$,*

$$\left(1 + \min_{i \in \{1,\ldots,6\}} d_{\mathrm{UDG}(P)}(q_i, p) - d_{\mathrm{UDG}(P\cup\{f\})}(f, p)\right) \in \{0, 1\},$$

where q_i is any point in $R_i(f) \cap P$.

PROOF. For any $i \in \{1, \ldots 6\}$, any two points a and b in $R_i(f)$ are at most unit distance apart. Consequently, a and b are adjacent in $\mathrm{UDG}(P)$ and $d_{\mathrm{UDG}(P)}(a, p) \leq d_{\mathrm{UDG}(P)}(b, p) + 1$ for any $p \in P$. See Figures 4B and 4C. Any shortest path from f to p must pass through a vertex in $P \cap R_i(f)$, for some $i \in \{1, \ldots, 6\}$. The result follows. $\square$

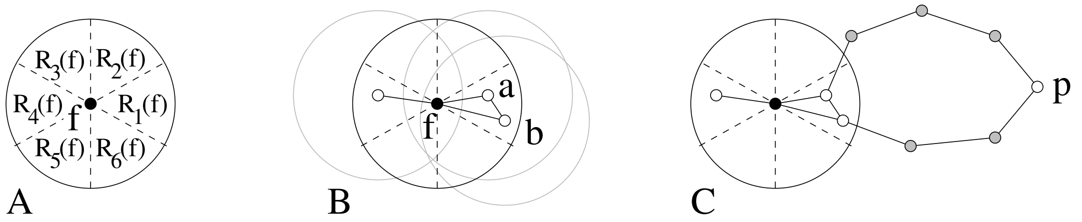

Figure 4: If a and b are in the same sector, $d_{\mathrm{UDG}(P)}(a,p)$ and $d_{\mathrm{UDG}(P)}(b,p)$ differ by at most one.

For every point f, the sets $R_1(f) \cap P$ through $R_6(f) \cap P$ are precomputed in $O(n^3)$ time. Thus, a point can be selected from each set in $O(1)$ time, giving the following theorem.

THEOREM 6. *Given a set of points P in $\mathbb{R}^2$, algorithm* APPROXIMATE GEOMETRIC 1-CENTRE *identifies a point $f \in \mathbb{R}^2$ in $O(n^3)$ time such that $\mathrm{ecc}_{\mathrm{UDG}(P \cup \{f\})}(P, \{f\}) \leq r + 1$, where r denotes the geometric 1-radius of P and $n = |P|$.*

4.4 Finding a Geometric k-Centre for a Fixed k

When k is fixed, our GEOMETRIC 1-CENTRE algorithm generalizes to give an $O(mn^{2k})$-time algorithm for finding a geometric k-centre of a unit disc graph. We begin with the following observation:

OBSERVATION 7. *Given a set of points $P \subseteq \mathbb{R}^2$ and a set of points $F \subseteq \mathbb{R}^2$ that forms a geometric k-centre of P, for every client $p \in P$, some shortest path in $\mathrm{UDG}(P \cup F)$ from p to a facility $f \in F$ nearest to p does not contain any facility $f' \in F$, where $f' \neq f$.*

An analogous property also holds for a vertex k-centre of any graph. As a consequence of Observation 7, edges connecting two facilities need not be considered when selecting locations for a geometric k-centre. Any two or more facilities located in a cell of $\mathcal{A}_R$ serve the same set of clients in P, resulting in redundant facilities. Therefore, by Proposition 2, it suffices to consider at most $\binom{n^2-n+2}{k}$ combinations for assigning k facilities to cells in $\mathcal{A}_R$. For each combination of cells, we calculate the corresponding eccentricity. Thus, the GEOMETRIC 1-CENTRE algorithm is modified such that the outer loop considers all combinations of k cells. In this case, BFS-DEPTH(G, V) begins breadth-first search at the vertices in the set V, returning the eccentricity of V in graph G. The corresponding run time is at most

$$(n+m)\binom{n^2-n+2}{k} \in O(mn^{2k}).$$

This gives the following theorem.

THEOREM 8. *For any fixed $k \in \mathbb{Z}^+$, a geometric k-centre of a set of n unit discs in $\mathbb{R}^2$ can be found in $O(mn^{2k})$ time.*

GEOMETRIC k-CENTRE(P)

$\quad$ $radius \leftarrow \infty$
$\quad$ **for** each combination of cells $C = \{C_1, \ldots, C_k\} \subseteq \mathcal{A}_R$
$\quad\quad$ $F \leftarrow \varnothing$
$\quad\quad$ **for** each $C_i \in C$
$\quad\quad\quad$ $f_i \leftarrow$ any point in C_i
$\quad\quad\quad$ $F \leftarrow F \cup f_i$
$\quad\quad$ $ecc \leftarrow$ BFS-DEPTH$(\mathrm{UDG}(P \cup F), F)$
$\quad\quad$ **if** $ecc < radius$
$\quad\quad\quad$ $radius \leftarrow ecc$
$\quad\quad\quad$ $Centres \leftarrow F$
$\quad$ **return** $Centres$

4.5 Finding a Geometric k-Centre for an Arbitrary k

In the last section we described an $O(mn^{2k})$-time algorithm for finding a geometric k-centre of a unit disc graph. Of course this run time is exponential if k is an arbitrary input parameter to the problem. In this section we show that GEOMETRIC k-CENTRE is NP-hard on unit disc graphs when k is not fixed. This implies NP-hardness for the more general problem, that is, on intersection graphs of sets of regions in two or more dimensions.

THEOREM 9. *When k is an arbitrary input parameter, the geometric k-centre problem on unit disc graphs is NP-hard.*

PROOF. Given a graph $G = (V, E)$ and an integer k, the DOMINATING SET decision problem is to determine whether there exists a set $D \subseteq V$ such that $|D| \leq k$ and every vertex in V is adjacent to some vertex in D. DOMINATING SET remains NP-hard if G is a grid graph [9, 19]. Choose any finite set of points $P \subseteq \mathbb{Z}^2$ and any integers $k \geq 1$ and $i \geq 0$. Let $s = 2i + 1$ and $r = 3i + 1$. Let $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ denote the uniform scaling function defined by $f((p_x, p_y)) = (sp_x, sp_y)$. Similarly, let $f^{-1}((p_x, p_y)) = (p_x/s, p_y/s)$. If A is a set, let $f(A) = \{f(p) \mid p \in A\}$. Let

$$\begin{aligned} P' = {} & f(P) \\ & \cup \{(sx+i, sy) \mid 1 \leq i \leq s-1 \text{ and } \{(x,y),(x+1,y)\} \subseteq P\} \\ & \cup \{(sx, sy+i) \mid 1 \leq i \leq s-1\} \text{ and } \{(x,y),(x,y+1)\} \subseteq P\}. \end{aligned}$$

For each $p \in P'$, let $g(p)$ denote the unique point in $f(P)$ that is nearest to p in $\mathrm{UDG}(P')$ by graph distance. Therefore,

$$\forall p \in P' \; d_{\mathrm{UDG}(P')}(p, g(p)) \leq \lfloor s/2 \rfloor. \tag{3}$$

Since the points of P' lie on the unit grid, $\mathrm{GG}(P') = \mathrm{UDG}(P')$. Furthermore, $\mathrm{GG}(P)$ is a minor of $\mathrm{UDG}(P')$. That is, $\mathrm{GG}(P)$ is equal to $\mathrm{UDG}(P')$ upon scaling the grid by a factor of s and replacing each edge by a path of length s. See Figure 5. We claim that $\mathrm{GG}(P)$ has a dominating set of cardinality

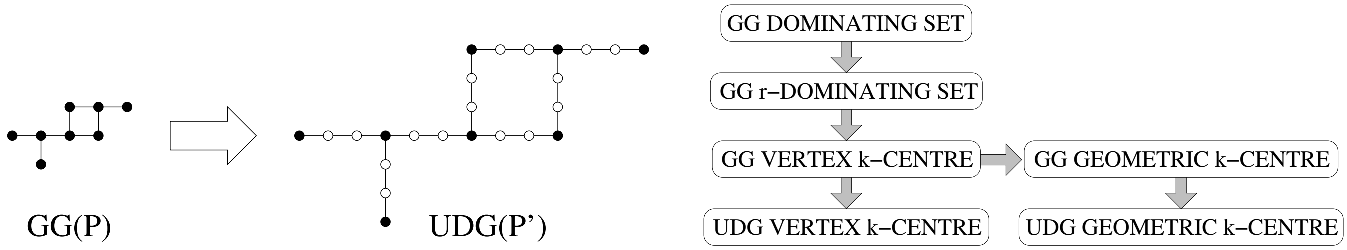

Figure 5: (Left) $\mathrm{GG}(P)$ has a dominating set of cardinality k if and only if $\mathrm{UDG}(P')$ has a geometric k-centre of radius r. In this example $s = 3$ and $r = 4$. (Right) In Theorem 9 we describe a reduction from DOMINATING SET on grid graphs to GEOMETRIC k-CENTRE on unit disc graphs. The hardness of other problems in this hierarchy can be derived by a reduction corresponding to a subset of the steps described in our proof of Theorem 9.

at most k if and only if $\mathrm{UDG}(P')$ has a geometric k-centre of radius r.

Case 1. ($\Rightarrow$) Suppose $\mathrm{GG}(P)$ has a dominating set, denoted by D, of cardinality at most k. Observe that $f(D) \subseteq P'$. Furthermore,

$$\forall q \in f(P)\ \exists t \in f(D) \text{ such that } d_{\mathrm{UDG}(P')}(q,t) \leq s. \tag{4}$$

By the triangle inequality, (3), and (4),

$$\forall p \in P'\ \exists t \in f(D) \text{ such that } d_{\mathrm{UDG}(P')}(p,t) \leq s + \lfloor s/2 \rfloor = r.$$

Therefore, $f(D)$ is a geometric k-centre of P' with radius r.

Case 2. ($\Leftarrow$) Suppose $\mathrm{UDG}(P')$ has a geometric k-centre of radius r. By Definition 2, there exists a set $F \subseteq \mathbb{R}^2$ such that $|F| \leq k$ and

$$\forall p \in P'\ \exists q \in F \text{ such that } d_{\mathrm{UDG}(P' \cup F)}(p,q) \leq r.$$

For any $t \in \mathbb{R}^2$, there exists some $q \in P'$ such that $N(t) \subseteq N(q)$ in $\mathrm{UDG}(P' \cup \{t\})$. By Observation 7, no two facilities need to be adjacent in $\mathrm{UDG}(P' \cup F)$. Consequently, there exists a set $F' \subseteq P'$ such that $|F'| \leq k$ and

$$\forall p \in P'\ \exists q \in F' \text{ such that } d_{\mathrm{UDG}(P')}(p,q) \leq r. \tag{5}$$

By the triangle inequality, (5), and (3),

$$\forall p \in P'\ \exists q \in F' \text{ such that } d_{\mathrm{UDG}(P')}(p, g(q)) \leq r + \lfloor s/2 \rfloor < 2s. \tag{6}$$

Observe that

$$\forall \{p_1, p_2\} \subseteq f(P)\ d_{\mathrm{UDG}(P')}(p_1, p_2) \bmod s = 0. \tag{7}$$

Therefore, by (6), (7), and since $g(q) \in f(P)$,

$$\forall p \in f(P)\ \exists q \in F' \text{ such that } d_{\mathrm{UDG}(P')}(p, g(q)) \leq s.$$

Consequently,

$$\forall p \in P\ \exists q \in F' \text{ such that } d_{\mathrm{GG}(P)}(p, f^{-1}(g(q))) \leq 1.$$

Let $D = f^{-1}(g(F'))$. Since $|F'| \leq k$, therefore $|D| \leq k$ and D is a dominating set of $\mathrm{GG}(P)$ whose cardinality is at most k. $\square$

5. DIRECTIONS FOR FUTURE RESEARCH

Motivated by gateway placement in wireless networks, we have examined the problem of finding a geometric k-centre in unit disc graphs. Of course, unit disc graphs are not the only model for representing wireless networks. In addition to the results described in this paper, we have considered the geometric k-centre problem in one-dimension (i.e., on interval graphs), resulting in linear-time algorithms for finding a geometric 1-centre and 2-centre and an $O(n \log n)$-time algorithm for finding a geometric k-centre. Details are omitted due to space constraints. As well, we have partial results for generalizations to the setting of disc graphs (intersection graphs of discs of differing radii), to three dimensions, and to rectangle intersection graphs.

Another possible direction is to model obstacles and interference in wireless networks by applying the geometric k-centre problem to the setting of visibility graphs. Given a set of points P (clients) in a polygonal region R, the objective is to select a set F of k points (facilities) in R such that the maximum graph distance between any client and its nearest facility is minimized in the visibility graph of $P \cup F$ in R; a pair of nodes is connected in the visibility graph if and only if the line segment between them is unobstructed by polygon R. By applying observations similar to those made in Section 4.1, a solution can be found discretely and, furthermore, the corresponding partition of the plane into visibility regions is a partial order relation for which it suffices to consider the sinks. Thus, visibility graphs seems like a natural setting to which to apply some of the ideas developed in this paper. See [2] and [3] for results on properties of visibility regions and the corresponding partial order.

One might consider generalizations of the optimization function that is minimized in selecting positions for gateways. In particular, two fundamental problems of facility location are the k-centre and k-median problems. In this paper we restrict attention to the first of these. The two problems are defined analogously, with the exception that the maximum over all $R_i \in R$ in (1) is replaced by a summation over all $R_i \in R$. Whereas a geometric k-centre minimizes the maximum node-to-gateway distance, a geometric k-median minimizes the *average* node-to-gateway distance. The algorithms for finding a geometric 1-centre and geometric k-centre for a fixed k presented in this paper are straightforward to adapt to identify a geometric 1-median or geometric k-median, respectively. In this case, each call to BFS-DEPTH is replaced by a call to BFS-SUM, which returns the corresponding sum of the distances from every node to the nearest gateway. The resulting run times remain $O(mn^2)$ and $O(mn^{2k})$, respectively.

Finally, can a geometric 1-centre of a unit disc graph be found in $O(n^3)$ worst-case time? That is, can our $O(n^2m)$-time algorithm be improved? Cubic time is a natural goal for solving this problem since the fastest known algorithms for finding a vertex 1-centre run in nearly $O(n^3)$ time.

Acknowledgements

The authors would like to thank the following individuals whose ideas and suggestions have helped shape various aspects of this work: Ian Munro and Matthew Skala with whom we discussed convexity in continuous metric spaces, Csaba Tóth who suggested the example in Figure 2C, Rolf Klein with whom we discussed this example, and Christian Knauer with whom we discussed Theorem 4.

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