2-Coloring Point Guards in a k-Guarded Polygon*

Stephane Durocher[1], Myroslav Kryven[1], Fengyi Liu[1], Amirhossein Mashghdoust[1], and Ikaro Penha Costa[1]

1 **Department of Computer Science, University of Manitoba**
`{stephane.durocher,myroslav.kryven,fengyi.liu,amirhossein.mashghdoust,`
`ikaro.penhacosta}@umanitoba.ca`

Abstract

For every $k \geq 2$, we describe how to construct a polygon P and a set G of points in P, such that P is *k-guarded* by G (i.e., every point in P is visible to at least k points in G) and for every *2-coloring* of G (i.e., for every bipartition of G) at least one of the colors does not guard P. This answers an open question posed by Morin [10].

1 Introduction

The *art gallery problem*, introduced by Klee [11] in 1973, is a well-known and extensively studied classical problem in the field of Computational Geometry. Given a simple polygon P (without holes) in the plane, the objective is to find a set G of points in P, called *guards*, such that every point $p \in P$ is *visible* to at least one guard $g \in G$; that is, the line segment $\overline{pg}$ does not pass outside P. Chvátal [6] showed that $\lfloor n/3 \rfloor$ guards suffice to guard any n-vertex simple polygon P, and that there exist polygons that require $\lfloor n/3 \rfloor$ guards. Fisk [7] later gave a simplified proof (one of the *Proofs from THE BOOK* [2]) using a 3-coloring argument. The optimization problem of finding a set G of points of minimum cardinality that guards a given simple polygon P is NP-hard [8], and was recently shown to be $\exists\mathbb{R}$-complete [1].

To introduce robustness and redundancy to the model, the art gallery problem generalizes to the *k-guarding problem*, in which each point in the input polygon P must be visible to at least k guards. Belleville et al. [3] examined a variant of k-guarding, in which guards are placed at the interior of the edges of P. Salleh [12] studied k-guarding with the constraint that guards are placed on the vertices of P, called *k-vertex guarding*. Salleh showed that $\lfloor 2n/3 \rfloor$ guards are sometimes necessary when $k = 2$, and $\lfloor 3n/4 \rfloor$ guards are sometimes necessary when $k = 3$ (see also [9]). Bereg [4] showed that Fisk's coloring argument can be used to prove these bounds. The k-guarding problem has also been studied from an algorithmic perspective; Busto et al. [5] gave a polynomial-time $O(k \log \log OPT_k(P))$-approximation algorithm for the k-guarding problem, where $OPT_k(P)$ is the optimal number of guards. As observed by Busto et al., if guards must be placed at different vertices of P, then there exist simple polygons that cannot be k-vertex guarded for $k \geq 4$ because some points in P are seen by fewer than k vertices. In k-guarding, this problem is naturally resolved by placing multiple guards arbitrarily close to each other.

During the open problem session at WADS 2023, Morin [10] asked whether there exists a positive integer k such that for all polygons P and all sets G of points that k-guard P, there exists a bipartition of G (equivalently, a 2-coloring of G) that gives two sets that each guard (1-guard) P. Morin presented counter-examples for $k = 2$ and $k = 3$ for which no such bipartition exists (see Figure 1) and asked whether this property generalizes to higher

* This work was funded in part by the Natural Sciences and Engineering Council of Canada (NSERC).

40th European Workshop on Computational Geometry, Ioannina, Greece, March 13–15, 2024.
This is an extended abstract of a presentation given at EuroCG'24. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

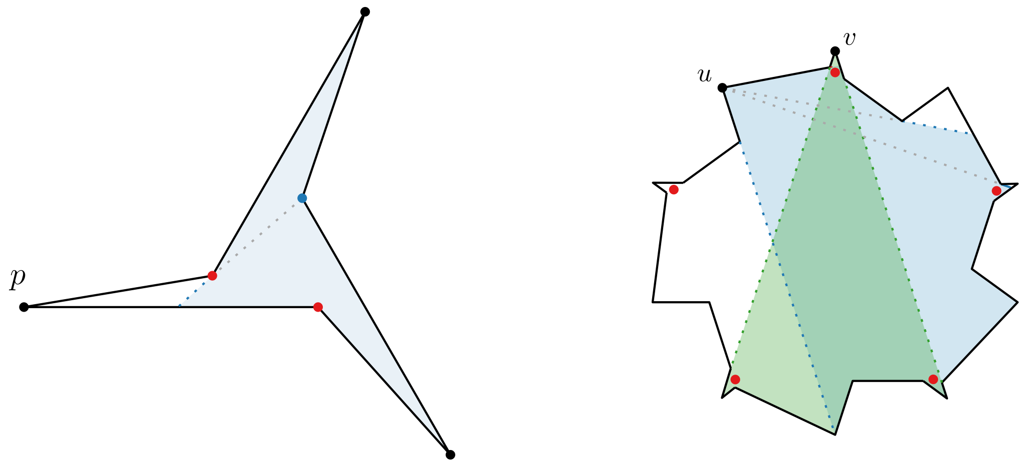


Figure 1 Examples for $k = 2$ and $k = 3$ [10]. The polygon P on the left is 2-guarded by the set G of three guards (red and blue points). Any 2-coloring of G partitions G either into 3 and 0, or 1 and 2. In both cases, at least one of the three convex vertices of P is not seen by any guard of the color with fewer guards. In this example, the blue guard, whose visibility region is shaded blue, cannot see the vertex p [10]. The polygon P' on the right is 3-guarded by the set G' of five guards (red points). There are $\binom{5}{3} = 10$ subsets of G' of cardinality three. Observe that each of these 10 subsets uniquely 3-guards exactly one of the 10 convex vertices of P'. E.g., the vertex v is 3-guarded by the three guards that are not consecutive on the boundary of P' in the visibility region shaded green, whereas the vertex u is 3-guarded by the three guards that are consecutive on the boundary of P' in the visibility region shaded blue. Any 2-coloring of G will result in one color class containing at most two guards. Consequently, some convex vertex of P is visible only by guards of the same color [10].

values of k. We answer this question in this paper. Observe that for any set G_1 that guards P, k copies of G_1 k-guard P and can be partitioned into k sets (and, therefore, into two sets) that each guard P. Consequently, Morin's question asks whether *every* set G that k-guards P can be partitioned into two sets that each guard P.

We formally define k-guarding as considered in this paper.

▶ **Definition 1.1** (k-guarding)**.** Given a simple polygon P, an integer $k \geq 1$, and a set G of points (guards) in P, P is *k-guarded* by G if for all $p \in P$, there exists $G' \subseteq G$, such that $|G'| = k$ and for all $g \in G'$, the line segment $\overline{gp}$ does not pass outside P. That is, every point in P is visible to at least k guards in G.

We say that the set of guards G is *2-colorable* if there exists a bipartition of G that partitions G into two sets such that each 1-guards P. The notions of k-guardability and 2-colorability characterize the degree to which a set G of guards sees the polygon P. Intuitively, a larger value of k should increase the probability that a set G of guards that k-guards a polygon is 2-colorable. We show that it is not the case in general. For every $k \geq 2$, we describe (see Section 2) how to construct a polygon P and a set G of guards, such that G k-guards P, but G is not 2-colorable.

Before presenting details of our construction, we first introduce some helpful definitions.

▶ **Definition 1.2.** A *k-ary tree* is a tree in which every non-leaf vertex has exactly k children.

▶ **Definition 1.3.** Given a simple polygon P and a set G of guards in P, a region $R \subseteq P$ is *uniquely guarded* by $G' \subseteq G$ if every point in R is visible (relative to P) to every guard in G', and there exists a point in R that is not visible (relative to P) to any guard in $G \setminus G'$.

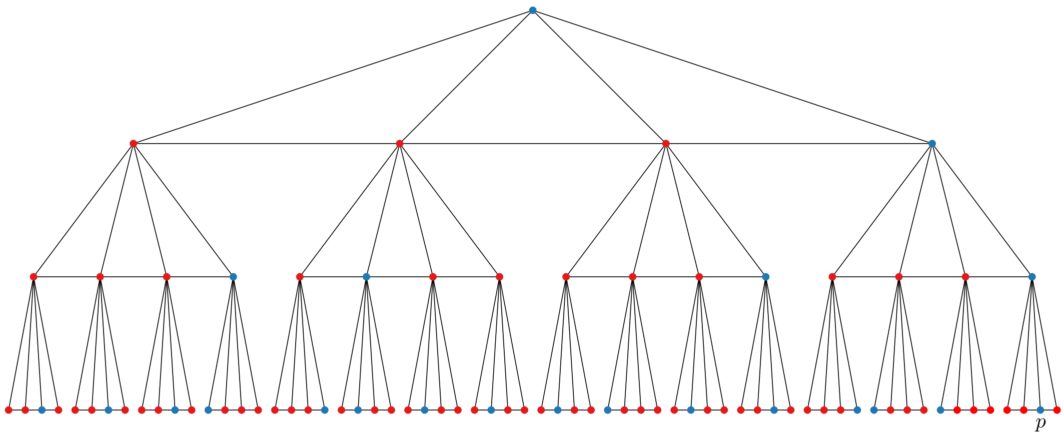


Figure 2 Proof idea. Our construction embeds a set G of guards in a polygon P_k, where G forms a perfect k-ary tree T_k of height $k-1$. Every path from root to leaf in T_k is a set of k guards in G that has an associated uniquely guarded region in P_k. Similarly, every set of siblings in T_k is a set of k guards in G that has an associated uniquely guarded region in P_k. Consequently, every set of siblings must include at least one node of each color. Therefore, there exists a monochromatic path from the root node to some leaf node. In this example, $k = 4$ and the path from the root to node p is monochromatic.

▶ **Definition 1.4.** For a simple polygon P, a set G of guards in P, and a region $R \subseteq P$ uniquely guarded by $G' \subseteq G$, we call a point in R that is only visible to G' a *witness point*, and a region composed of witness points a *witness region*.

2 Guards of a k-Guarded Polygon Are Not Always 2-Colorable

In this section, we prove our main result. The key idea is sketched in Figure 2, then proved formally in Lemma 2.1 and Theorem 2.2.

▶ **Lemma 2.1.** *For any $k \geq 1$, there exists a polygon P_k and a set G of guards in P_k that form a perfect k-ary tree T_k of height $k-1$ such that:*

1. *The polygon P_k is k-guarded by G.*
2. *For every root-to-leaf path in T_k, the points of G on that path uniquely guard some region of P_k.*
3. *For each internal node of T_k, its children uniquely guard some region of P_k.*

Proof. We will prove existence of a polygon, Π_k, defined below, that satisfies Properties 1–3 above. Consider a polygon Π_h with a set of guards G arranged in Π_h as a perfect k-ary tree of height $h-1$ (guards in each level are aligned horizontally, see Figure 4a) such that:

A The polygon Π_h is h-guarded by G.

B For every root-to-leaf path $g_{v_r}g_v$ (i.e. the path from the root v_r with the guard g_{v_r} to the leaf v with the guard g_v in the tree), the points (guards) of G on that path uniquely guard a convex region Q_v of Π_h with a witness triangle $\Delta_v = A_vB_vC_v$ of Q_v such that:

(1) Δ_v does not contain any of the guards;
(2) B_v is the bottommost point of Q_v;
(3) $A_v \in K_lB_v$, where K_l is the first vertex on Q_v after B_v clockwise;
(4) $C_v \in B_vK_r$, where K_r is the first vertex on Q_v after B_v counter clockwise.

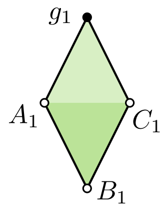
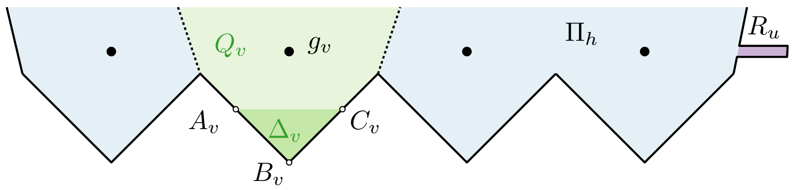
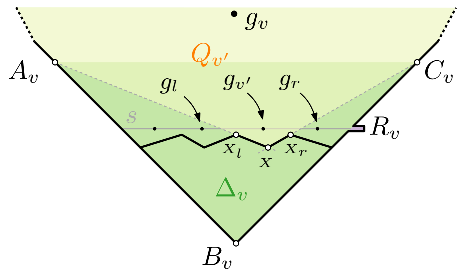


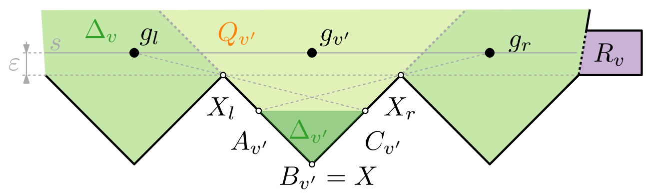
Figure 3 Base case, Π_1



(a) the polygon Π_h



(b) the region Q_v with the witness triangle Δ_v



(c) extending Q_v to $Q_{v'}$ with the witness triangle $\Delta_{v'}$

Figure 4 Illustration in support of the proof of Lemma 2.1

C For each internal node g_u in the tree, the children of g_u uniquely guard a trapezoidal region R_u.

Observe that any polygon P and set G of guards that satisfy Properties A–C also satisfy Properties 1–3. In what follows, we show how to construct Π_k by induction.

Base case. Let Π_1 be a diamond polygon with a single guard g_1 at its topmost vertex; see Figure 3. The entire polygon Π_1 is uniquely guarded by a single guard g_1, that defines a perfect k-ary tree of height 0. Therefore, Properties A–C are trivially satisfied.

Induction step. Now we show how to extend the polygon Π_h to Π_{h+1} so that Properties A–C hold. Place k guards on a horizontal segment s strictly contained in Δ_v; see Figure 4b. For a new guard $g_{v'}$, we reshape Q_v by drawing rays from A_v and C_v that cross at some point X in Δ_v below s. We ensure that A_vX crosses s between $g_{v'}$ and the guard g_l immediately to the left, and that C_vX crosses s between $g_{v'}$ and the guard g_r immediately to the right. Let $Q_{v'}$ denote the convex polygon obtained from Q_v by adding the edges A_vX and C_vX and removing away from Q_v the parts that are below these two edges. Let $X_l X$ and XX_r be the new edges forming Π_{h+1} by placing $X_l \in A_v X$ and $X_r \in C_v X$ right below s (a sufficiently small distance $\varepsilon > 0$); see Figure 4c.

We let $\Delta_{v'} = A_{v'} B_{v'} C_{v'}$, where $B_{v'} = X$, $A_{v'}$ is the point where the ray from g_r through X_r hits $X_l B_{v'}$, and $C_{v'}$ is the point where the ray from g_l through X_l hits $X_r B_{v'}$; see Figure 4c.

Let us show that Property B is satisfied. First, observe that all the guards on the root-to-leaf path $g_{v_r} g_{v'}$ are contained in the convex region $Q_{v'}$ (this holds, because by induction the guards on the root-to-leaf path $g_{v_r} g_v$ are inside Q_v, $Q_{v'} \subset Q_v$, and the guard $g_{v'}$ is inside $Q_{v'}$); therefore, all the guards on the root-to-leaf path $g_{v_r} g_{v'}$ see $Q_{v'}$. Second, notice that $\Delta_{v'} \subset \Delta_v$; therefore, no guards from the previous levels (guarding Π_h), except the root-to-leaf path $g_{v_r} g_v$ and $g_{v'}$ can see $\Delta_{v'}$. Let us show that out of the new guards (added at level $h+1$) only $g_{v'}$ can see $\Delta_{v'}$. Observe that all these guards are arranged horizontally and $\Delta_{v'}$ is contained below the line through g_l (that is, a guard immediately to the left of $g_{v'}$) and $C_{v'}$, that is, an endpoint of $\Delta_{v'}$. Therefore, $\Delta_{v'}$ is not seen by g_l, nor by any guard left of g_l. By an analogous argument, $\Delta_{v'}$ is not seen by g_r, nor by any guard right of g_r. Therefore, $\Delta_{v'}$ is a witness triangle of $Q_{v'}$ guarded by the root-to-leaf path $g_{v_r} g_{v'}$. Properties B.(1)–B.(4) are satisfied by construction with the vertices $B_{v'}$, $A_{v'}$, and $C_{v'}$ respectively; see Figure 4c.

To satisfy Property C, we make a trapezoidal pocket R_v of height 2ε and width $\delta(\varepsilon)$ aligned with s (so that every point of the pocket is visible to the children of g_v) on the right side of $B_v C_v$; see Figures 4b and 4c. For sufficiently small ε, the width $\delta(\varepsilon)$ of R_v can be made arbitrarily small, so that it does not interfere with the rest of the polygon Π_h and the right end of R_v is only seen to the guards that are children of g_v.

Finally, to see that Property A is satisfied (that is, that Π_h is h-guarded) observe that every point of the polygon is either contained in at least one convex region Q_v that contains h guards or it is contained in some trapezoidal pocket R_v that is seen by $k \geq h$ children of g_v. ◀

▶ **Theorem 2.2.** *There exists a polygon P and a set of guards G such that P is k-guarded by G but there is no 2-coloring of G.*

Proof. Consider a k-guarded polygon P_k from Lemma 2.1 with a set of guards G embedded in P_k as a perfect k-ary tree T_k of height $k-1$. Suppose there exists a 2-coloring of G. For each internal node g_u in T_k, the children of g_u uniquely guard some region of P_k. Since G is 2-colored, this set of siblings must include at least one blue guard and at least one red guard. Suppose, without loss of generality, that the root is colored blue. Therefore, there is a root-to-leaf path $g_{v_r} g_v$ that follows only the blue guards. According to Property 2, that path is uniquely guarding some region of P_k, and, therefore, there is a point in P_k that is only seen by blue guards, contradicting our assumption that there exists a 2-coloring of G. ◀

3 Directions for Future Research

We conclude with some open questions.

In the construction of polygon P_k in the proof of Lemma 2.1, the ratio of the lengths of the longest edge and the shortest edge is exponential in k. Consequently, we ask the following questions.

▶ Question 1. Is there a polygon P that is k-guarded by a set of guards G that is not 2-colorable for which the ratio of the lengths of the longest edge and the shortest edge is polynomial in k?

Is there a simpler construction than P_k? For example, does there exist a *weakly visible* polygon P (that is, every point of P is visible from some point on a given line segment in P) such that P is k-guarded by some set G of guards, but no bipartition of G exists such that each part guards P?

▶ Question 2. Is there a weakly visible polygon P that is k-guarded by a set of guards G that is not 2-colorable?

We can also examine the complexity (number of vertices) of P_k in terms of k. Our construction for P_k has $\Theta(k^k)$ vertices.

▶ Question 3. Can we show that P_k always needs $\omega(k)$ vertices?

References

1. Mikkel Abrahamsen, Anna Adamaszek, and Tillmann Miltzow. The art gallery problem is $\exists\mathbb{R}$-complete. *Journal of the ACM*, 69(1), 2021.
2. Martin Aigner and Günter M. Ziegler. *Proofs from THE BOOK*. Springer, 4th edition, 2009.
3. Patrice Belleville, Prosenjit Bose, Jurek Czyzowicz, Jorge Urrutia, and Joseph Zaks. K-guarding polygons on the plane. In *Proc. 6th Canadian Conference on Computational Geometry (CCCG)*, pages 381–386, 1994.
4. Sergey Bereg. On k-vertex guarding simple polygons. Technical report, Kyoto University, 2009. Computational Geometry and Discrete Mathematics.
5. Daniel Busto, William S. Evans, and David G. Kirkpatrick. On k-guarding polygons. In *Proc. 25th Canadian Conference on Computational Geometry (CCCG)*, pages 283–288, 2013.
6. Václav Chvátal. A combinatorial theorem in plane geometry. *Journal of Combinatorial Theory, Series B*, 18(1):39–41, 1975.
7. Steve Fisk. A short proof of Chvátal's Watchman Theorem. *Journal of Combinatorial Theory, Series B*, 24(3):374, 1978.
8. D. T. Lee and Arthur K. Lin. Computational complexity of art gallery problems. *IEEE Transactions on Information Theory*, 32(2):276–282, 1986.
9. Kurt Mehlhorn, Jörg Sack, and Joseph Zaks. Note on the paper "K-vertex guarding simple polygons" [Computational Geometry 42 (4) (May 2009) 352–361]". *Computational Geometry: Theory and Applications*, 42:722, 2009.
10. Pat Morin, Prosenjit Bose, and Paz Carmi. Open problem session. 18th Algorithms and Data Structures Symposium (WADS), 2023.
11. J. O'Rourke. *Art Gallery Theorems and Algorithms*. International series of monographs on computer science. Oxford University Press, 1987.
12. Ihsan Salleh. K-vertex guarding simple polygons. *Computational Geometry: Theory and Applications*, 42(4):352–361, 2009.