

# Bounded-velocity Approximations of the Mobile Euclidean 2-centre

extended abstract

Stephane Durocher<sup>\*†</sup>

David Kirkpatrick<sup>\*‡</sup>

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## 1 The Euclidean 2-centre

A Euclidean 2-centre of a finite and nonempty set of points  $P$  in  $\mathbb{R}^2$  is a set of two points, denoted  $\Xi(P) = \{\xi_1(P), \xi_2(P)\}$ , that minimizes the maximum Euclidean ( $\ell_2$ ) distance from any point  $p$  in  $P$  to the point  $\xi_i(P) \in \Xi(P)$  nearest to  $p$ . Equivalently,  $\xi_1(P)$  and  $\xi_2(P)$  correspond to the centres of two circles whose union contains the points of  $P$ , such that the radius of the larger circle is minimized. Traditionally, the points of  $P$  represent positions of clients while the points of  $\Xi(P)$  represent positions of facilities serving these clients. Given client positions, the problem is to select facility positions that minimize a given objective function representing cost as a function of distances between clients and facilities. For the  $k$ -centre, the optimization function is the maximum Euclidean distance from any client to the nearest facility.

Several efficient algorithms exist for solving the Euclidean 2-centre problem in  $\mathbb{R}^2$ . Eppstein [Epp92] gives algorithms that run in  $O(n^2 \log^2 n \log^2 \log n)$  expected time and  $O(n^2 \log^4 n)$  worst-case time. The worst-case time is improved to  $O(n^2 \log n)$  using the algorithm of Jaromczyk and Kowaluk [JK94]. Sharir [Sha97] reduces the time to  $O(n \log^9 n)$ . Eppstein [Epp97] gives a simpler randomized algorithm in  $O(n \log^2 n)$  expected time. Finally, Chan [Cha99] gives a deterministic algorithm in  $O(n \log^2 n \log^2 \log n)$  time. The general Euclidean  $k$ -centre problem in  $\mathbb{R}^2$  is NP-hard when  $k$  is an input parameter [MS84].

## 2 Mobile Clients and Facilities

Recently, motivated in part by applications in mobile computing, there has been considerable interest in recasting a number of basic questions of facility location in a mobile context [AdBG<sup>+</sup>05, AGG02, AGHV01, AH01, BBKS00, BBKS05, DK05a, DK05b, GGH<sup>+</sup>03, Her05]. Given a set of mobile clients, modelled as points in  $\mathbb{R}^2$  that change continuously and with bounded velocity, the utility of a mobile facility is determined by its approximation of the optimization function as well as the continuity and maximum relative velocity of its motion. In many cases, the optimal location for a facility exhibits unbounded velocity or discontinuous motion; thus, we seek to identify functions that define the positions of mobile facilities under the dual objectives of requiring that their motion be continuous and have bounded velocity while also maintaining a good approximation of the optimization function. Closely related to the mobile Euclidean 2-centre is recent work of Bereg et al. [BBKS00, BBKS05] and Durocher and Kirkpatrick [DK05b, DK05a] that examines bounded-velocity (hence, continuous) approximations to the mobile Euclidean 1-centre and to the mobile Euclidean 1-median, both in  $\mathbb{R}^2$ .

## 3 The Mobile Euclidean 2-centre in $\mathbb{R}$ and $\mathbb{R}^2$

We show that the Euclidean 2-centre in  $\mathbb{R}$  moves continuously and with relative velocity at most one. We give an algorithm for efficient maintenance of the mobile 2-centre in  $\mathbb{R}$  using the kinetic data structures of Agarwal et al. [AH01, AGHV01] to maintain the extent of point sets.

By a four-client example, we demonstrate that there exist sets of mobile clients  $P$  such that every mobile Euclidean 2-centre of  $P$  moves discontinuously. Any pair of mobile facilities that move continuously must,

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<sup>\*</sup>Department of Computer Science, University of British Columbia, Vancouver, British Columbia, Canada. Funding for this research was made possible by NSERC and the MITACS project on Facility Location Optimization.

<sup>†</sup>email: durocher@cs.ubc.ca

<sup>‡</sup>email: kirk@cs.ubc.ca

Table 1: comparing approximations of the mobile Euclidean 2-centre

reflection across	guaranteed $\lambda$ -approximation	lower bound on worst-case $\lambda$	relative velocity
Euclidean 1-centre	4	4	$\infty$
rectilinear 1-centre	$2\sqrt{2} \approx 2.8284$	$2\sqrt{2} \approx 2.8284$	$2\sqrt{2} + 1 \approx 3.8284$
Steiner centre	$\sqrt{10(2 - \sqrt{2})} \approx 2.4203$	$2\sqrt{1 + 1/\pi^2} \approx 2.0989$	$8/\pi + 1 \approx 3.5465$
any mobile point		2	$\geq 3$

therefore, differ from the Euclidean 2-centre for some client configurations. When this occurs, the distance from some client to the nearest facility must exceed the optimal value. Let  $\Upsilon(P) = \{v_1(P), v_2(P)\}$  denote a mobile facility pair, where  $v_i : \mathcal{P}(\mathbb{R}^2) \rightarrow \mathbb{R}^2$ . We say that  $\Upsilon$  is a  **$\lambda$ -approximation** of the Euclidean 2-centre if

$$\forall P \forall t, \max_{p \in P} \min_{i \in \{1,2\}} \|p(t) - v_i(P(t))\| \leq \lambda \cdot \max_{p \in P} \min_{i \in \{1,2\}} \|p(t) - \xi_i(P(t))\|.$$

We show that no mobile facility pair with maximum relative velocity less than two can guarantee a  $\lambda$ -approximation for any fixed  $\lambda > 0$ .

#### 4 Defining Mobile Facilities by Reflection

Typically, a 2-centre problem involves partitioning the clients into two sets and subsequently identifying a center for each partition. Discontinuities in the position of a mobile 2-centre can occur when the partitions change discontinuously. To prevent this from occurring, we identify a mobile point, denoted  $r$ , that remains “central” to  $P$  while moving under bounded velocity. A client of  $P$ ,  $p_0$ , is selected arbitrarily and the position of the first facility is set to coincide with that of  $p_0$ . The position of the second facility is found by reflecting  $p_0$  across  $r$ . As natural candidates for  $r$ , we select bounded-velocity approximations of the mobile Euclidean 1-centre. These include the mobile rectilinear 1-centre [BBKS00, BBKS05] and the mobile Steiner centre [DK05b]. For comparison, we also examine the case when  $r$  is the mobile Euclidean 1-centre [BBKS00, BBKS05].

If  $r$  moves with relative velocity at most  $v$ , then the reflection of  $p_0$  across  $r$  moves with relative velocity at most  $2v + 1$ . As shown by Bepamyatnikh and Kirkpatrick [BBKS00], the rectilinear 1-centre moves with relative velocity at most  $\sqrt{2}$ , whereas the velocity of the Euclidean 1-centre is unbounded. As shown by Durocher and Kirkpatrick [DK05b], the relative velocity of the Steiner centre is at most  $4/\pi$ . All three of these velocity bounds are tight, inducing the relative velocities in Tab. 1.

For facilities defined by reflection across the Euclidean 1-centre and across the rectilinear 1-centre, we show tight bounds on the  $\lambda$ -approximation of 4 and  $2\sqrt{2}$ , respectively. For facilities defined by reflection across the Steiner centre, we show  $2\sqrt{1 + 1/\pi^2} \leq \lambda \leq \sqrt{10(2 - \sqrt{2})}$ . See Tab. 1.

Finally, we show that no bounded-velocity  $\lambda$ -approximation of the Euclidean 3-centre exists in  $\mathbb{R}$ .

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