On Geometric Duality for 2-Admissible Geometric Set Cover and Hitting Set Problems

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The boundaries of any two translations of a convex object intersect in at most two points. This property is generalized beyond convexity by *2-admissibility*, which applies to a broad class of sets of geometric objects: a set S of objects, each of which is bounded by a closed Jordan curve in the plane, is 2-admissible if the boundaries of every pair of objects in S intersect at most twice. Examples include well-studied objects such as disks or squares, sets of translations of convex objects, or the shadows of 2-intersecting functions (for which the hardness of set cover remains open [1]). Mustafa and Ray [2] described a PTAS to identify a hitting set for any given set of 2-admissible regions. In specific cases, an instance of a hitting set problem can be dualized to a corresponding set cover problem (e.g., for sets of unit disks or unit squares), but whether such a dualization is possible for all sets of 2-admissible objects remained unknown. We investigate geometric duality with respect to 2-admissible regions to address this gap, as establishing complete duality between 2-admissible hitting set problems and 2-admissible set cover problems would yield PTAS solutions for the latter.

Rather surprisingly, the problems cannot always be dualized, which we prove by counterexample. However, the counterexample characterizes a property that prevents dualization, and we conjecture that instances of problems lacking this property may always be dualized.

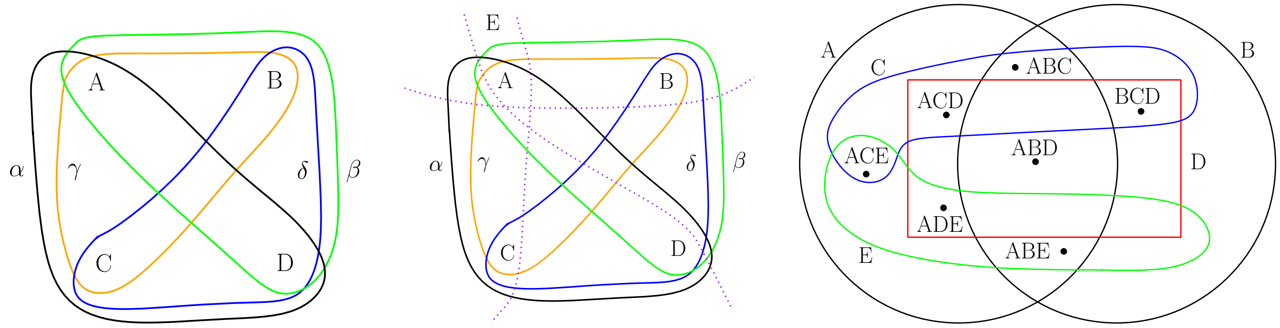
Definitions. Given a geometric range space $\mathcal{S} = (X, \mathcal{R})$, the *geometric set cover problem* is to find a subset $\mathcal{R}^\star \subseteq \mathcal{R}$ of minimum cardinality so that all elements of X are covered by $\mathcal{R}^\star$, i.e., $X = (\cup_{R \in \mathcal{R}^\star} R) \cap X$. Given a range space $\mathcal{H} = (P, \mathcal{Q})$, the *hitting set problem* is to find a subset $P^\star \subseteq P$ of minimum cardinality so that all sets of $\mathcal{Q}$ contain at least one element of $P^\star$, i.e., $P^\star \cap Q \neq \emptyset, \forall Q \in \mathcal{Q}$. Given a geometric set $R \in \mathcal{R}$ or $Q \in \mathcal{Q}$, we call $R \cap \mathbb{R}^2$ and $Q \cap \mathbb{R}^2$ *objects*.

Given an instance of the geometric set cover problem $\mathcal{S} = (X, \mathcal{R})$ (the *primal* setting), an instance of the hitting set problem $\mathcal{H} = (P, \mathcal{Q})$ is a *geometric dual* of $\mathcal{S}$ (in the *dual* setting) if there are bijections between X and $\mathcal{Q}$ as well as $\mathcal{R}$ and P so that $|X| = |\mathcal{Q}|$, $|\mathcal{R}| = |P|$, and any point $p_i \in P$ hits a range $Q_j \in \mathcal{Q}$ if and only if the corresponding point $x_j \in X$ is covered by the range $R_i \in \mathcal{R}$ in the primal setting. An optimal solution $P^\star$ for the dual setting corresponds exactly to an optimal solution $\mathcal{R}^\star$ for the primal setting.

A Counterexample for Complete Duality

In Figure 1, we present a counterexample for complete duality. We construct the counterexample by first describing a hitting set instance that cannot be made 2-admissible, and then we present a set cover instance that is 2-admissible and is the dual of the impossible configuration.

Consider four points $P_1 = \{A, B, C, D\}$ in the plane and all four combinatorially distinct 2-admissible objects $\mathcal{Q}_1 = \{\alpha, \beta, \gamma, \delta\}$ that are hit by exactly three of the points, as shown in Figure 1(a). Consider a fifth point E and the set $\mathcal{Q}_E$ containing all sets of cardinality three hit by E and two points in P_1 while remaining 2-admissible. Any pair of points in P_1 hits all four sets in $\mathcal{Q}_1$. Furthermore, any set in $\mathcal{Q}_E$ must both include and exclude at least one point from each set in $\mathcal{Q}_1$, and so the boundary



(a) (b) (c)

Figure 1: The counterexample for complete duality. (a) Four points and all combinatorially distinct 2-admissible objects hit by exactly three points. (b) The dotted curves are some cuts of cardinality two on the original four points. No cutting curve exists including points B and C while excluding A and D while intersecting the boundaries of the four objects at most eight times. (c) A set cover instance requiring the impossible configuration described in (b).

of any object in $\mathcal{Q}_E$ must intersect the boundary of each object in $\mathcal{Q}_1$ exactly twice (at least twice to impose the requisite partition and at most twice for 2-admissibility). Therefore, the boundary of an object in $\mathcal{Q}_E$ can be regarded as a cutting curve that will intersect the boundaries of the objects in $\mathcal{Q}_1$ exactly 8 times (and then is closed to create a 2-admissible region containing E). Consider the sequence of entry and exit points of the cutting curve with the objects of $\mathcal{Q}_1$. The sequence must begin by entering three objects, since each point of P_1 hits three sets. Exiting an object, say α, before entering three objects would partition α so that all points hitting α are in one of the partitions, so the cutting curve sequence begins by entering three objects and concludes by leaving three objects. The remaining options for the sequence are equivalent: if a sequence begins by entering $\{\alpha, \beta, \gamma\}$ and ends by leaving $\{\beta, \gamma, \delta\}$, it makes no difference in terms of partitioning whether δ is entered before leaving α. Therefore, a cutting curve having two points of P_1 on each side cannot separate all possible pairs of P_1. In Figure 1(b), $\{A, B\}$ may be separated from $\{C, D\}$ or $\{A, C\}$ from $\{B, D\}$, but $\{A, D\}$ cannot be separated from $\{B, C\}$.

Figure 1(c) illustrates an instance of set cover which is the dual of this impossible configuration. There are points covered by every subset of the objects $\{A, B, C, D\}$ of cardinality three, as well as all three objects covering the points E, A and one other. This counterexample leads us to the following problem definition: The *Pairwise-Cover-Free 2-Admissible Set Cover (P2SC) Problem* is a geometric set cover problem $\mathcal{S} = (X, \mathcal{R})$ where the objects are 2-admissible and the object $R_i \not\subset (R_j \cup R_k)$ for any $\{R_i, R_j, R_k\} \subseteq \mathcal{R}$. The latter condition is critical, as allowing an object to be covered by a pair of other objects is the property that was exploited to create the counterexample to the general problem.

Conjecture 1. *Any instance $\mathcal{S} = \{X, \mathcal{R}\}$ of P2SC may be reduced to an instance of a hitting set problem $\mathcal{H} = \{P, \mathcal{Q}\}$ in polynomial time, so that $\mathcal{H}$ is a geometric dual of $\mathcal{S}$.*

References

[1] Timothy M Chan and Elyot Grant. Exact algorithms and APX-hardness results for geometric packing and covering problems. *Computational Geometry*, 2012, `http://dx.doi.org/10.1016/j.comgeo.2012.04.001`.

[2] Nabil H Mustafa and Saurabh Ray. Improved results on geometric hitting set problems. *Discrete & Computational Geometry*, 44(4):883–895, 2010.