International Journal of Computational Geometry
& Applications
Vol. 16, No. 4 (2006) 345–371

World Scientific
www.worldscientific.com

THE STEINER CENTRE OF A SET OF POINTS: STABILITY, ECCENTRICITY, AND APPLICATIONS TO MOBILE FACILITY LOCATION*

STEPHANE DUROCHER[†] and DAVID KIRKPATRICK[‡]

Department of Computer Science, University of British Columbia
2366 Main Mall Vancouver, British Columbia V6T 1Z4, Canada
[†] *durocher@cs.ubc.ca*
[‡] *kirk@cs.ubc.ca*

Received 18 November 2004
Revised 21 June 2005
Communicated by J.S.B. Mitchell

ABSTRACT

The Euclidean centre (centre of the smallest enclosing sphere) of a set of points P in two or more dimensions is *unstable*; small perturbations at only a few points of P can result in an arbitrarily large relative change in the position of the Euclidean centre. Any centre function more stable than the Euclidean centre is *eccentric*; that is, its associated radius exceeds the radius of the smallest enclosing circle for some point sets P. Motivated by applications in mobile facility location (in which clients move continuously with some maximum velocity) we seek alternative notions of *centrality* that are stable while maintaining low eccentricity. In general there is a trade-off; centre functions with lower eccentricity are less stable. In an attempt to balance the conflicting goals of closeness of approximation and stability, we apply the Steiner centre, traditionally defined for convex polytopes, as a centre function of a set of points in the plane. Although previously defined, the notion of a Steiner centre had not been analyzed in terms of its approximation of the Euclidean centre. Exploiting the equivalence of the two definitions of the Steiner centre established by Shephard,[27] we prove the stability of the Steiner centre is $\pi/4$ and show that the associated radius is at most 1.1153 times the Euclidean radius of any point set P. It follows that a mobile facility located at the Steiner centre of the positions of a set of mobile clients remains close to the Euclidean centre of the clients yet never moves with relative velocity that exceeds $4/\pi$.

Keywords: Centre; Steiner point; Euclidean centre; smallest enclosing circle; stability; eccentricity; approximation; facility location; mobile facility location.

*Funding for this research was made possible by the Natural Sciences and Engineering Research Council of Canada and the Mathematics of Information Technology and Complex Systems.

1. Introduction

Finding a point that is *central* to a collection P of data points drawn from some metric space is a fundamental problem of geometry and data analysis. The centre of the smallest d-sphere enclosing a set of points P in Euclidean d-space, referred to as the Euclidean centre, provides a natural, and broadly applicable, definition for a centre function.

Unfortunately, the Euclidean centre is unstable in the sense that small perturbations at only a few points of P can result in an arbitrarily large relative change in the position of the Euclidean centre.[10] Even small error, perhaps introduced by discretization in the representation of point coordinates, can result in large relative error in the position of the Euclidean centre. Clearly, any centre function, Υ_d, that is more stable than the Euclidean centre must, for some point sets P, have an associated radius (maximum distance from $\Upsilon_d(P)$ to any point in P) that exceeds the Euclidean radius of P. These two attributes, eccentricity (relative radius) and stability, are in opposition; increased stability implies increased eccentricity and vice-versa. We formalize these notions through the definitions of κ-stability and λ-eccentricity. Together, the properties of stability and eccentricity allow us to quantify and to compare the quality of various *centre functions*.

In this paper we apply and analyze the Steiner centre of a set of points in the plane with the objective of identifying a centre function that balances high stability with low eccentricity. The Steiner centre has two quite different but equivalent definitions. Maintaining dual definitions proves significant: the first formulation using weights defined by turn angles at extreme points (definition by Gaussian weights) allows for simple implementation using kinetic data structures while the second formulation using projection and integration lends itself to proving bounds on stability and eccentricity. We show the Steiner centre is more stable and less eccentric than several other natural notions of centre functions.

The utility of the Steiner centre extends beyond its definition as a robust centre of a set of static points. Indeed, our primary motivation arises from the field of facility location within which the definition and computation of centre functions is a fundamental and well-studied task. Recently, motivated in part by applications in mobile computing, there has been considerable interest in recasting a number of basic questions of facility location in a mobile context.[2,3,5,9,10,17,20] Given a set of mobile clients, modelled as points in $\mathbb{R}^d$ that move continuously and with bounded velocity, the utility of a mobile facility is determined by its eccentricity as well as the continuity and maximum relative velocity of its motion. We show the stability of a centre function is inversely related to the maximum relative velocity of a mobile facility whose location is specified by that function. We evaluate the mobile Steiner centre as a strategy for the maintenance of a bounded-velocity mobile facility with low eccentricity.

This paper is organized as follows. Section 2 formalizes the notions of eccentricity and stability. Section 3 motivates and presents two definitions for the Steiner centre of a set of points in $\mathbb{R}^2$. Sections 4 and 5 respectively derive the eccentricity and stability of the Steiner centre. Section 6 explores the concepts of mobile facility location related to the centre problem, defines the Steiner centre within the mobile setting, and establishes a tight bound on its maximum relative velocity. Section 7 discusses implementation issues for maintaining the Steiner centre using kinetic data structures. Section 8 compares the stability and eccentricity of the Steiner centre against other centre functions. Finally, Section 9 briefly discusses extensions of the Steiner centre to three dimensions.

2. The Centre of a Set of Points

2.1. *Preliminary definitions*

A $(d-1)$-dimensional hyperplane H partitions $\mathbb{R}^d$ into three regions: H itself and the two open connected components of $\mathbb{R}^d - H$, which we denote by H^+ and H^-. By convention, if A is a set, let $\overline{A}$ denote its closure, $CH(A)$ its convex hull, and $\widehat{\mathscr{P}}(A)$ the set of all nonempty bounded subsets of A.

Definition 1. A point p is an **extreme point** of the set $P \subseteq \mathbb{R}^d$ if and only if for some $(d-1)$-dimensional hyperplane H and associated half-space H^+, p satisfies $\overline{P} \cap \overline{H^+} = \{p\}$.

Note that the extreme points of P are just the vertices of $CH(P)$.

2.2. *Notions of centrality*

Given a nonempty bounded set of points $P \subseteq \mathbb{R}^d$, a fundamental problem of geometry and data analysis concerns the characterization and computation of points that are central to P. If δ is any metric on $\mathbb{R}^d$, then a **δ-centre** of P is defined as a point $c \in \mathbb{R}^d$ that minimizes $\max_{p\in\overline{P}} \delta(p, c)$. A natural, and for many applications the default, metric for measuring distance between two points p and q in $\mathbb{R}^d$ is the Euclidean metric, equivalent to the ℓ_2 norm, $||p-q||$, of the vector $p-q$. The corresponding *Euclidean centre* of P is the (unique) centre of the smallest d-sphere enclosing P.

We refer to an arbitrary function $\Upsilon_d : \widehat{\mathscr{P}}(\mathbb{R}^d) \to \mathbb{R}^d$ as a **centre function.**

Definition 2. The **Euclidean centre** is the centre function whose value $\Xi_d(P)$, for an arbitrary $P \in \widehat{\mathscr{P}}(\mathbb{R}^d)$, is the point in $\mathbb{R}^d$ that minimizes

$$\max_{p\in\overline{P}} ||p - \Xi_d(P)||. \tag{1}$$

The value $\max_{p\in\overline{P}} ||p - \Xi_d(P)||$ is referred to as the **Euclidean radius** of P.

The minimum enclosing circle (for points in $\mathbb{R}^2$) and minimum enclosing sphere (for points in $\mathbb{R}^3$) problems are well studied with both deterministic and randomized linear-time algorithmic solutions. Megiddo[23] gives a deterministic $\Theta(n)$-time linear programming solution in $\mathbb{R}^2$, where $n = |P|$. Agarwal et al.[1] extend this result to $\mathbb{R}^d$ for any fixed d in $O(d^{O(d)}n)$ time. Since every point must be examined, these results are asymptotically optimal when d is fixed. Welzl[31] gives a simpler randomized algorithm with $\Theta(n)$ expected time in $\mathbb{R}^d$ for any fixed d.

2.3. *Centre stability*

Point coordinates are commonly represented by discretization of real positions to nearby grid coordinates. That is, each point is approximated by the nearest grid point. Given a set of points $P \in \widehat{\mathscr{P}}(\mathbb{R}^d)$ and its Euclidean centre $\Xi_d(P)$, small perturbations at only a few points of P can result in a relatively large change in the corresponding position of $\Xi_d(P)$. In this sense, the Euclidean centre is *unstable*. As will be discussed in Sec. 6, this implies that the relative velocity of a mobile facility located at the Euclidean centre of a collection of mobile clients is potentially unbounded. We formalize the notion of stability by defining κ-stability for a centre function Υ_d as a measure of its maximum volatility. This requires preliminary definitions for an ε-perturbation and a continuous function.

Definition 3. Given $\varepsilon > 0$, function $f : P \to \mathbb{R}^d$ is an **ε-perturbation** on $P \in \widehat{\mathscr{P}}(\mathbb{R}^d)$ if for all $p \in P$, $||p - f(p)|| \leq \varepsilon$.

Let F_ε^P denote the set of all ε-perturbations on P. A prerequisite for stability is continuity. Specifically, if the stability of centre function Υ is bounded, then Υ must be continuous.

Definition 4. A centre function $\Upsilon_d : \widehat{\mathscr{P}}(\mathbb{R}^d) \to \mathbb{R}^d$ is **continuous** if for all $P \in \widehat{\mathscr{P}}(\mathbb{R}^d)$ and all $\delta > 0$ there exists an $\varepsilon > 0$ such that for all $f \in F_\varepsilon^P$,

$$||\Upsilon_d(P) - \Upsilon_d(f(P))|| < \delta. \tag{2}$$

Definition 5. A centre function $\Upsilon_d : \widehat{\mathscr{P}}(\mathbb{R}^d) \to \mathbb{R}^d$ is **κ-stable** if

$$\forall \varepsilon > 0,\ \forall f \in F_\varepsilon^P,\ \kappa||\Upsilon_d(P) - \Upsilon_d(f(P))|| \leq \varepsilon, \tag{3}$$

for all $P \in \widehat{\mathscr{P}}(\mathbb{R}^d)$.

Clearly, every centre function is 0-stable. It has been shown previously, in the context of mobile facility location, that the Euclidean centre Ξ_d, though continuous, is arbitrarily unstable. Specifically, Bespamyatnikh et al.[10] show, in effect, that for $d \geq 2$, Ξ_d is not κ-stable for any $\kappa > 0$. For arbitrary Υ_d, the maximum stability factor κ ranges from 0 to ∞, where a greater κ value corresponds to a more stable point. If Υ_d is constant it is ∞-stable, but of course such a function is of little benefit as a notion of centre; as we will see in Sec. 3.1, only maximum stability factors in the range $[0, 1]$ are of interest.

2.4. *Measuring centrality*

If centre function Υ_d is κ-stable for any $\kappa > 0$ then $\max_{p \in \overline{P}} ||p - \Upsilon_d(P)||$ must exceed the Euclidean radius of P for some $P \in \widehat{\mathscr{P}}(\mathbb{R}^d)$. We formalize the notion of the relative radius of a centre function Υ_d in terms of the *eccentricity* of Υ_d:

Definition 6. A centre function Υ_d is **λ-eccentric** if

$$\max_{p \in \overline{P}} ||p - \Upsilon_d(P)|| \leq \lambda \max_{q \in \overline{P}} ||q - \Xi_d(P)||, \tag{4}$$

for all $P \in \widehat{\mathscr{P}}(\mathbb{R}^d)$.

The eccentricity factor λ ranges from 1 to ∞, with a 1-eccentric centre function being the least eccentric (most central). The stability and eccentricity factors, κ and λ, allow us to compare the utility of different centre functions. In general, functions with lower eccentricity have lower stability. Subject to this trade-off we seek centre functions with stability and eccentricity close to one.

3. The Steiner Centre

Named after Steiner who first introduced this point in the late nineteenth century,[29] the original definition of the Steiner centre was phrased in terms of projection and integration, leading to the definition in Sec. 3.2. A second, fundamentally different definition, phrased in terms of Gaussian weights given by turn angles at the extreme points leads to the definition in Sec. 3.1. The equivalence of these two definitions was shown by Shephard.[27]

3.1. *Definition by Gaussian weights*

Many centre problems find their simplest non-trivial definitions in $\mathbb{R}^2$. In this section we motivate using Gaussian weights to define a centre function. The resulting definition is that of the Steiner centre of a set of points in the plane, defined first for a finite set P, and then, more generally, for any nonempty bounded set $P \in \widehat{\mathscr{P}}(\mathbb{R}^2)$. As will be discussed in Secs. 6 and 7, the simple and intuitive definition of the Steiner centre by Gaussian weights will prove effective in efficiently defining the position of a mobile facility that balances low maximum velocity and low eccentricity

Let $P \subseteq \mathbb{R}^2$ denote a nonempty finite set of points. The simplest definition of a centre function Υ_2 (that is not independent of P) simply assigns $\Upsilon_2(P) = p$, for some fixed point $p \in P$. Since $||p - f(p)|| = ||\Upsilon_2(P) - \Upsilon_2(f(P))||$ for any ε-perturbation $f : \mathbb{R}^2 \to \mathbb{R}^2$, Υ_2 is 1-stable. However, Υ_2 is 2-eccentric but not $(2 - \varepsilon)$-eccentric for any $\varepsilon > 0$ (the worst case occurs when $\Upsilon_2(P)$ lies at one end of a diameter of the minimum enclosing circle of P). In fact, any centre function that lies within the convex hull of P, regardless of its stability, is at worst 2-eccentric. This sets an upper bound for λ; any reasonable centre function should have eccentricity at most 2.

Taking the average of points in P improves neither eccentricity nor stability; Bespamyatnikh et al.[10] show the centre of mass is also 1-stable and 2-eccentric. Furthermore, they show that any centre function with stability factor at least one must have eccentricity at least two. This relationship extends easily to show that if a centre function has stability factor greater than one, then its eccentricity must be infinite. This sets an upper bound for κ-stability; any reasonable κ-stable λ-eccentric centre function will have a stability at most 1.

Reducing eccentricity decreases stability and vice-versa. The challenge lies in understanding the trade-off between eccentricity (in the range $[1,2]$) and stability (in the range $[0,1]$). The actual correlation between κ and λ is a strictly increasing bijection over all centre functions Υ_2 and all sets of points P. For a fixed $\kappa \in [0,1]$, let $\lambda^*(\kappa)$ denote the minimum eccentricity over all κ-stable centre functions. This defines a function $\lambda^* : [0,1] \to [1,2]$, where $\lambda^*(1) = 2$ and $\lambda^*(0) = 1$. Thus, the eccentricity of any κ-stable centre function Υ_2 is at least $\lambda^*(\kappa)$. While the precise value of function $\lambda^*(\kappa)$ for $\kappa \in (0,1)$ remains unknown, the asymptotic behaviour of $\lambda^*(\kappa)$ is understood and shown to be bounded from below by $\lambda^*(\kappa) \geq 1 + \kappa^2/64$ by Bereg et al.[9]

Observe that any point $p \in P$ that lies on the minimum enclosing circle of P must be an extreme point of P. A natural attempt at defining a centre function Υ_2 might be to define it to be the average of the extreme points of P. However, the instability (and in fact, discontinuity) of this centre function Υ_2 becomes evident whenever a small perturbation of P alters the composition of the set of extreme points. For the same reason, any centre function defined as a fixed weighted average of the extreme points of P is not κ-stable for any $\kappa > 0$ (nor is it continuous). Nevertheless, by choosing weights that depend on the degree of extremity of individual points it is possible to ensure not only continuity but also high stability.

For clarity, Definitions 7 and 8 assume $|P| \geq 2$. In the case when $|P| = 1$ (that is, $P = \{p\}$, for some p) the Steiner centre is simply $\Gamma_2(P) = p$.

Definition 7. Let $P \subseteq \mathbb{R}^2$ be a finite set of points with $|P| \geq 2$. Let V_P be the set of extreme points of P. For every $p \in V_P$, let α_p be the interior angle formed on the convex hull boundary at p. The **Gaussian weight** of p is

$$w_2(p) = \begin{cases} \pi - \alpha_p & \text{if } p \in V_P \\ 0 & \text{if } p \in P - V_P. \end{cases} \tag{5}$$

For $p \in V_P$, $w_2(p)$ corresponds to the turn angle at p on $CH(P)$. Consequently, $\sum_{p\in P} w_2(p) = 2\pi$. Note, $w_2(p) > 0$ if and only if p is an extreme point of P. Expressed in terms of Gaussian weight, the Steiner centre is defined as the normalized weighted centre of mass of P.

Definition 8. Let $P \subseteq \mathbb{R}^2$ be a finite set of points with $|P| \geq 2$. The **Steiner centre** of P is the normalized weighted mean of P:

$$\Gamma_2(P) = \frac{1}{2\pi} \sum_{p\in P} w_2(p) p, \tag{6}$$

where $w_2(p)$ is the Gaussian weight of point p.

For example, let $P = \{p_1, \ldots, p_6\} = \{(-2,-1), (-1,-1), (2,-1), (2,1), (0,1), (1,0)\}$, respectively. See Fig. 1A. Since $w_2(p) = \pi - \alpha_p$, points $p_1, \ldots, p_6$

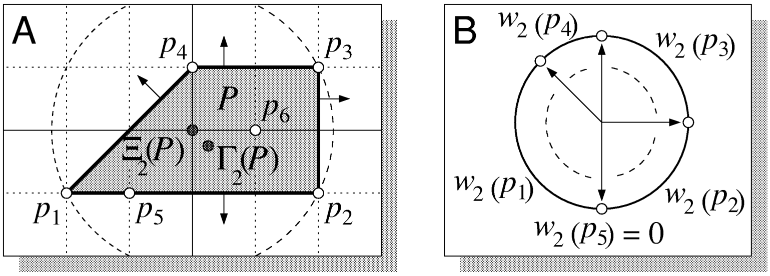

Fig. 1. The Steiner centre $\Gamma_2(P)$ defined by Gaussian weights and the Gaussian map of P.

have weights $3\pi/4$, 0, $\pi/2$, $\pi/2$, $\pi/4$, and 0, respectively. The Steiner centre of P, $\Gamma_2(P)$, lies in position $(1/4, -1/4)$. The Euclidean centre of P, $\Xi_2(P)$, lies at the origin.

The Gaussian map (normal map) provides an equivalent definition for Gaussian weights. The Gaussian map of a convex polygon $Q \subseteq \mathbb{R}^2$ is the set of normals to edges of Q projected from the origin as vertices on the unit circle (see Ref. 13 for a discussion of the Gaussian map). Given a nonempty finite set of points $P \subseteq \mathbb{R}^2$, the Gaussian map G_P of the convex hull boundary of P divides the unit circle into sectors such that the Gaussian weight of each extreme point of P is given by the length of its corresponding arc in G_P or, equivalently, the corresponding sector angle. The example in Fig. 1B displays the Gaussian map of the set of points P from Fig. 1A.

As a centre function, Gaussian weight formulation of the Steiner centre has several desirable properties. The Steiner centre is defined solely in terms of the geometry of the boundary of the convex hull of P. Small changes in the convex hull result in small changes in the weights of points. Specifically, if a point p is moved continuously, the weight of p changes continuously, even when p moves along, joins, or leaves the convex hull boundary. This continuous change in weights results in continuity in the motion of the Steiner centre by smoothly blending the contribution of each point.

3.2. *Definition by projection*

In one dimension, the Euclidean centre of a set of points $P \in \widehat{\mathscr{P}}(\mathbb{R})$ is simply

$$\frac{1}{2}\left(\min_{p\in \overline{P}} p + \max_{q\in \overline{P}} q\right). \tag{7}$$

That is, the one-dimensional Euclidean centre is the average of the two extreme points. As discussed in Sec. 3.1, while the mean of the extrema does not provide a robust centre function, Eq. (7) suggests other possible generalizations to higher dimensions.

One possibility is to project points onto a line through the origin, to find the one-dimensional centre of the projection, and to average these one dimensional

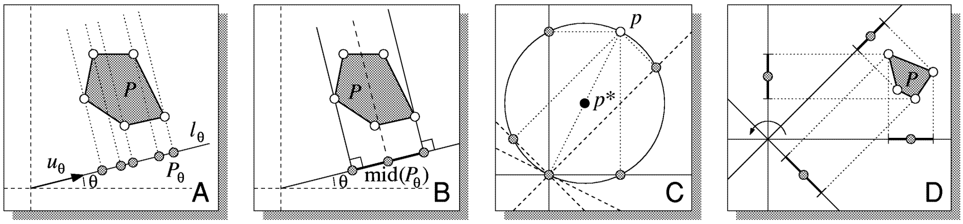

Fig. 2. Defining the Steiner centre Γ_2 by projection.

centres for all lines through the origin.

Let line l_θ be the line through the origin parallel to the unit vector $u_\theta = (\cos\theta, \sin\theta)$. Given a set of points $P \in \widehat{\mathscr{P}}(\mathbb{R}^2)$ and an angle $\theta \in [0, \pi)$, let P_θ denote the projection of P onto the line l_θ. See Fig. 2A. That is,

$$P_\theta = \{u_\theta \langle p, u_\theta \rangle \mid p \in P\}. \tag{8}$$

The midpoint of P_θ is just the Euclidean centre of P_θ,

$$\mathrm{mid}(P_\theta) = \frac{u_\theta}{2}\left(\min_{p \in \overline{P}} \langle p, u_\theta \rangle + \max_{q \in \overline{P}} \langle q, u_\theta \rangle\right) = \Xi_2(P_\theta). \tag{9}$$

See Fig. 2B. Let $p \in \mathbb{R}^2$ be any fixed point. The average over all projections of p onto lines l_θ is

$$\frac{1}{\pi}\int_0^\pi u_\theta \langle p, u_\theta \rangle \, d\theta = p/2. \tag{10}$$

See Fig. 2C. Equivalently, if $P = \{p\}$,

$$p = \frac{2}{\pi}\int_0^\pi u_\theta \langle p, u_\theta \rangle \, d\theta = \frac{2}{\pi}\int_0^\pi \mathrm{mid}(P_\theta) \, d\theta. \tag{11}$$

This suggests the following definition of a centre function (shown to be equivalent to Definition 8 by Shephard[27]):

Definition 9. The **Steiner centre** of $P \in \widehat{\mathscr{P}}(\mathbb{R}^2)$ is

$$\Gamma_2(P) = \frac{2}{\pi}\int_0^\pi \mathrm{mid}(P_\theta) \, d\theta, \tag{12}$$

where $\mathrm{mid}(P_\theta) = \Xi_2(P_\theta)$ is the midpoint of the projection of P onto line $y = x\tan\theta$.

This second definition of the Steiner centre of P can be interpreted in terms of bounding boxes of P. The **bounding box** of P with orientation θ is simply $CH(P_\theta) + CH(P_{\theta+\pi/2})$, where addition denotes the Minkowski sum. Its centre is the point $\mathrm{mid}(P_\theta) + \mathrm{mid}(P_{\theta+\pi/2})$. See Fig. 3. Hence,

Lemma 1. *The Steiner centre of $P \in \widehat{\mathscr{P}}(\mathbb{R}^2)$, $\Gamma_2(P)$, is equivalent to the average of the centres of all bounding boxes of P.*

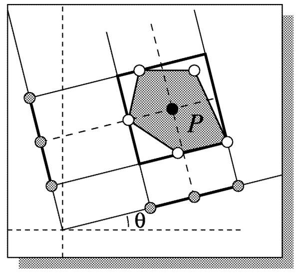

Fig. 3. Illustration supporting Lemma 1.

Proof.

$$\begin{aligned}\Gamma_2(P) &= \frac{2}{\pi}\int_0^{\pi} \operatorname{mid}(P_\theta)\ d\theta \\ &= \frac{2}{\pi}\left[\int_0^{\pi/2} \operatorname{mid}(P_\theta)\ d\theta + \int_{\pi/2}^{\pi} \operatorname{mid}(P_\theta)\ d\theta\right] \\ &= \frac{2}{\pi}\int_0^{\pi/2} \left[\operatorname{mid}(P_\theta) + \operatorname{mid}(P_{\theta+\pi/2})\right]\ d\theta. \qquad \square\end{aligned}$$

Observe that the minimum of P_θ corresponds to the maximum of $P_{\theta+\pi}$. Specifically, we can rewrite Eq. (12) as

$$\begin{aligned}\Gamma_2(P) &= \frac{2}{\pi}\int_0^{\pi} \operatorname{mid}(P_\theta)\ d\theta \\ &= \frac{2}{\pi}\int_0^{\pi} \frac{u_\theta}{2}\left(\min_{p\in\overline{P}}\langle p, u_\theta\rangle + \max_{q\in\overline{P}}\langle q, u_\theta\rangle\right)\ d\theta \\ &= \frac{1}{\pi}\int_0^{2\pi} u_\theta \cdot \max_{q\in\overline{P}}\langle q, u_\theta\rangle\ d\theta. \qquad (13)\end{aligned}$$

The latter, Eq. (13), is used in the proof of Theorem 1.

3.3. *Related work*

The Steiner centre is known under various names including Steiner curvature centroid,[12,21] Steiner point,[18,25,26,27,28] Kimberling triangle centre $X(1115)$,[22] Gaussian centre,[14,15] and projection centre.[15] Several useful properties of the Steiner centre have been established:

(1) **locality**[7] $\Gamma_2(P) \in CH(P)$.
(2) **continuity**[26,28] Γ_2 is continuous (see Definition 4).
(3) **additivity**[28] $\Gamma_2(P) + \Gamma_2(Q) = \Gamma_2(P + Q)$, where addition denotes the Minkowski sum.
(4) **invariance under similarity transformations**[28] $\Gamma_2(t(P)) = t(\Gamma_2(P))$ for any similarity transformation $t : \mathbb{R}^d \to \mathbb{R}^d$.

(5) **convex decomposition**[25] A relationship analogous to the inclusion-exclusion principle holds on a convex decomposition of P; that is, given polytopes $P_1, \ldots, P_n$ such that $P = P_1 \cup \ldots \cup P_n$ is also a polytope, then

$$\Gamma_2(P) = \sum \Gamma_2(P_i) - \sum_{i<j} \Gamma_2(P_i \cap P_j) + \ldots + (-1)^{n-1}\Gamma_2(P_1 \cap \ldots \cap P_n).$$

(6) **decomposition into j-faces**[27] $\Gamma_2(P)$ can be expressed in terms of the Steiner centres of the faces, edges, and vertices of P; that is,

$$(1 + (-1)^{d-1})\Gamma_2(P) = \sum \Gamma_2(F_i^0) - \sum \Gamma_2(F_i^1) + \ldots + (-1)^{d-1} \sum \Gamma_2(F_i^{d-1}),$$

where F_i^j are the j-faces of P.

To our knowledge, previous to our work, neither had the Steiner centre been evaluated as a stable approximation to the Euclidean centre nor had its quality in defining the position of a mobile facility been examined.

4. Eccentricity of the Steiner Centre

In this section we prove that the Steiner centre is λ-eccentric, where $\lambda \approx 1.1153$. We show that this maximum is achieved when the extreme points form an arc opposite an isolated point on the circle as displayed in Fig. 5B, where $\alpha = 0$ and $\beta = \gamma \approx 0.8105$.

Lemma 2. *Among all sets $P \in \widehat{\mathscr{P}}(\mathbb{R}^2)$ with Euclidean radius $r > 0$, the worst-case eccentricity of Γ_2 is realized when the extreme points of P consist of an arc A and an isolated point m on the circle C with radius r and centre $\Xi_2(P)$.*

Proof. Since $\Gamma_2(P) = \Gamma_2(CH(P))$ and $\max_{p \in P} ||\Gamma_2(P) - p||$ is realized at an extreme point of P, we can assume that P is a convex set. Let $m \in P$ be a furthest point from $\Gamma_2(P)$. Let a_x (respectively, a_y) denote the x-coordinate (respectively, y-coordinate) of a point $a \in \mathbb{R}^2$. Since Γ_2 is invariant under rotation and translation, without loss of generality, we can further assume that $m_y = \Gamma_2(P)_y$ and $m_x \geq \Gamma_2(P)_x$. Since $\max_{p \in P} ||\Gamma_2(P) - p|| \geq r > 0$, the line induced by m and $\Gamma_2(P)$ is well defined.

For $p \in P$, let

$$p' = \begin{cases} \text{left translation of } p \text{ to } C & \text{if } p \neq m \\ \text{right translation of } p \text{ to } C & \text{if } p = m \end{cases}. \tag{14}$$

Let set $P' = \{p' \mid p \in P\}$. Observe that every point in P' corresponds to a horizontal translation of some point in P. See Fig. 4. The x-coordinate of the Steiner centre of P' is given by

$$\Gamma_2(P')_x = \frac{2}{\pi} \int_0^{\pi} \text{mid}(P'_\theta)_x \, d\theta. \tag{15}$$

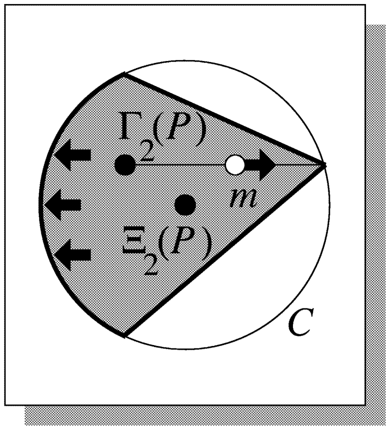

Fig. 4. Illustrations supporting Lemma 2.

Since all points of $P' - \{m'\}$ are left translations of points in P,

$$\text{mid}(P'_\theta)_x \leq \text{mid}(P_\theta)_x + \frac{m'_x - m_x}{2}, \tag{16}$$

for any $\theta \in [0, \pi]$. Therefore,

$$\Gamma_2(P')_x \leq \Gamma_2(P)_x + (m'_x - m_x), \tag{17}$$

and hence

$$m'_x - \Gamma_2(P')_x \geq m_x - \Gamma_2(P)_x. \tag{18}$$

Since $m_x \geq \Gamma_2(P)_x$ and $m'_x \geq \Gamma_2(P')_x$,

$$|m'_x - \Gamma_2(P')_x| \geq |m_x - \Gamma_2(P)_x|. \tag{19}$$

Therefore,

$$||m' - \Gamma_2(P')|| \geq |m'_x - \Gamma_2(P')_x| \geq |m_x - \Gamma_2(P)_x| = ||m - \Gamma_2(P)||. \tag{20}$$

Since all points of P' lie within the minimum enclosing circle of P, the Euclidean radius of P' is at most the Euclidean radius of P. Therefore, Eq. (20) implies that the eccentricity of P' is at least as great as the eccentricity of P. The extreme points of set P' consist of an arc of C opposite the isolated point m'. □

Theorem 1. *The eccentricity of the Steiner centre* Γ_2 *is* $\lambda \approx 1.1153$.

Proof. It follows from Lemma 2 that to understand the eccentricity of Γ_2 it suffices to study point sets P formed by an arc A of a circle C and an isolated point m on C. Since Γ_2 is preserved by translation, reflection, rotation, and uniform scaling, we can assume C is the unit circle centred at the origin such that m lies in the first quadrant and the line induced by m and $\Gamma_2(P)$ lies parallel to the

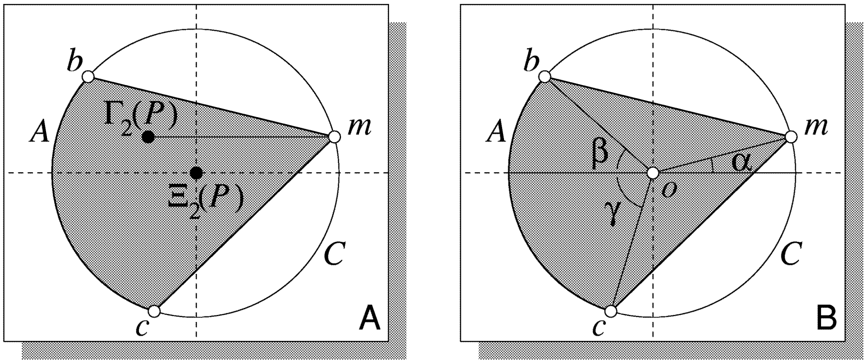

Fig. 5. Theorem 1: maximizing the eccentricity of the Steiner centre.

x-axis. See Fig. 5A. Thus, point sets of interest are completely characterized by three parameters which specify the angles α, β, and γ formed, respectively, by the position of m relative to the positive x-axis and the endpoints of A relative to the negative x-axis. See Fig. 5B. Let $P_{\alpha,\beta,\gamma}$ denote such a set of points. To find a point set that realizes the worst-case eccentricity of Γ_2 we need only maximize $||\Gamma_2(P_{\alpha,\beta,\gamma}) - m||$. Since $\Gamma_2(P_{\alpha,\beta,\gamma})_x \leq m_x$, this corresponds to identifying values of α, β, and γ that maximize $m_x - \Gamma_2(P_{\alpha,\beta,\gamma})_x$ while maintaining the property that $\Gamma_2(P_{\alpha,\beta,\gamma})_y = m_y$.

The Steiner centre of $P_{\alpha,\beta,\gamma}$ is straightforward to calculate by examination of the various cases for which specific extreme points of $P_{\alpha,\beta,\gamma}$ remain extreme in P_θ. The coordinates of the extreme points of P are $m = (\cos\alpha, \sin\alpha)$, $b = (-\cos\beta, \sin\beta)$, $c = (-\cos\gamma, -\sin\gamma)$, and $u_\theta = (\cos\theta, \sin\theta)$, for $\theta \in [\pi - \beta, \pi + \gamma]$.

Table 1 divides the range of integration, $\theta \in [0, 2\pi]$, into intervals for which each of the points m, b, c, and u_θ induce a maximum of P_θ.

Table 1. Case analysis of extreme points in $\Gamma_2(P_{\alpha,\beta,\gamma})$.

interval of θ		$\arg\max_{p\in P}\langle p, u_\theta\rangle$
$[0,$	$(\pi+\alpha-\beta)/2]$	m
$[(\pi+\alpha-\beta)/2,$	$\pi-\beta]$	b
$[\pi-\beta,$	$\pi+\gamma]$	u_θ
$[\pi+\gamma,$	$(3\pi+\alpha+\gamma)/2]$	c
$[(3\pi+\alpha+\gamma)/2,$	$2\pi]$	m

The x-coordinate of the Steiner centre of $P_{\alpha,\beta,\gamma}$ is given by

$$\Gamma_2(P_{\alpha,\beta,\gamma})_x = \frac{1}{\pi}\int_0^{2\pi} \cos\theta \cdot \max_{p\in P}\langle u_\theta, p\rangle \; d\theta$$

$$= \frac{1}{\pi}\left[\int_0^{(\pi+\alpha-\beta)/2} \cos\theta\langle u_\theta, m\rangle\, d\theta + \int_{(\pi+\alpha-\beta)/2}^{\pi-\beta} \cos\theta\langle u_\theta, b\rangle\, d\theta \right.$$
$$+ \int_{\pi-\beta}^{\pi+\gamma} \cos\theta\langle u_\theta, u_\theta\rangle\, d\theta + \int_{\pi+\gamma}^{(3\pi+\alpha+\gamma)/2} \cos\theta\langle u_\theta, c\rangle\, d\theta$$
$$\left. + \int_{(3\pi+\alpha+\gamma)/2}^{2\pi} \cos\theta\langle u_\theta, m\rangle\, d\theta \right]$$
$$= \frac{1}{4\pi}\left[-2\sin\beta - 2\sin\gamma - (\pi-\alpha-\beta)\cos\beta \right.$$
$$\left. -(\pi+\alpha-\gamma)\cos\gamma + (2\pi-\gamma-\beta)\cos\alpha\right]. \tag{21}$$

Let f denote the function $f(\alpha,\beta,\gamma) = m_x - \Gamma_2(P_{\alpha,\beta,\gamma})_x$. Values of α, β, and γ that define a local maximum of f must satisfy the following conditions:

$$\frac{\partial}{\partial\alpha}f = \frac{\partial}{\partial\beta}f = \frac{\partial}{\partial\gamma}f = 0.$$

Specifically,

$$\frac{\partial}{\partial\beta}f = \frac{1}{4\pi}[\cos\beta - (\pi-\alpha-\beta)\sin\beta + \cos\alpha] = 0, \tag{22}$$
$$\frac{\partial}{\partial\gamma}f = \frac{1}{4\pi}[\cos\gamma - (\pi+\alpha-\gamma)\sin\gamma + \cos\alpha] = 0, \tag{23}$$
$$\text{and } \frac{\partial}{\partial\alpha}f = \frac{1}{4\pi}[\cos\gamma - \cos\beta - (2\pi+\beta+\gamma)\sin\alpha] = 0. \tag{24}$$

We now show that the constraints imposed by Eqs. (22) through (24) imply that for $(\alpha,\beta,\gamma) \in [0,\pi/2]^3$, f has only one local (and hence global) maximum occurring at $\alpha = 0$ and $\beta = \gamma \approx 0.81047$.

Since α, β, and γ lie in the interval $[0,\pi/2]$, the term $-(2\pi+\beta+\gamma)\sin\alpha$ in Eq. (24) is nonpositive, meaning that $\cos\gamma - \cos\beta \geq 0$ and, consequently, $\gamma \leq \beta$. Furthermore, in order for the unit circle to define the minimum enclosing circle of $P_{\alpha,\beta,\gamma}$, line segment $\overline{cm}$ must pass below the origin, implying that $\gamma \geq \alpha$. See Fig. 5B. These constraints impose an ordering on the angles: $0 \leq \alpha \leq \gamma \leq \beta \leq \pi/2$.

We bound the value of α. Solving for $\sin\alpha$ in Eq. (24) gives

$$\sin\alpha = \frac{\cos\gamma - \cos\beta}{2\pi+\beta+\gamma} \leq \frac{1}{2\pi}.$$

Therefore,

$$0 \leq \alpha \leq \arcsin\left(\frac{1}{2\pi}\right) \approx 0.159835 < \frac{3\pi}{50}. \tag{25}$$

We derive an upper bound on β using this bound on α. By Eq. (22),

$$0 = \cos\beta - (\pi-\alpha-\beta)\sin\beta + \cos\alpha,$$
$$\leq \cos\beta - \left(\frac{47\pi}{50} - \beta\right)\sin\beta + 1, \text{ since } \alpha \in \left[0, \frac{3\pi}{50}\right]. \tag{26}$$

Let $g(\beta) = \cos\beta - (47\pi/50 - \beta)\sin\beta + 1$. Observe that $g'(\beta) \leq 0$ for $\beta \in [0, \pi/2]$. Furthermore, $g(1) < 0$. Consequently, $g(\beta) < 0$ for all $\beta \in [1, \pi/2]$. Since $g(\beta)$ must be nonnegative by Eq. (26), it follows that $\gamma \leq \beta < 1$.

We now take a linear combination of Eqs. (22), (23), and (24).

$$\begin{aligned}
4\pi\left(\frac{\partial}{\partial\gamma}f - \frac{\partial}{\partial\beta}f - \frac{\partial}{\partial\alpha}f\right) &= 0\\
\Rightarrow (2\pi + \beta + \gamma)\sin\alpha - (\pi + \alpha - \gamma)\sin\gamma + (\pi - \alpha - \beta)\sin\beta &= 0\\
\Rightarrow \qquad \underbrace{\beta\sin\alpha - \alpha\sin\beta}_{t_1} + \underbrace{\gamma\sin\alpha - \alpha\sin\gamma}_{t_2} &\\
+ \underbrace{(\pi - \beta)\sin\beta - (\pi - \gamma)\sin\gamma}_{t_3} + \underbrace{2\pi\sin\alpha}_{t_4} &= 0. \qquad (27)
\end{aligned}$$

We examine terms t_1 through t_4 from Eq. (27). Let $h(x) = x/\sin x$. Observe that $\lim_{x\to 0} h'(x) = 0$ and $h''(x) \geq 0$ for $x \in [0, \pi/2]$. Thus, $h(x)$ is nondecreasing on the interval $[0, \pi/2]$, meaning that for any $0 \leq \alpha \leq \gamma \leq \beta \leq \pi/2$,

$$\frac{\beta}{\sin\beta} \geq \frac{\alpha}{\sin\alpha} \quad \text{and} \quad \frac{\gamma}{\sin\gamma} \geq \frac{\alpha}{\sin\alpha}. \qquad (28)$$

Therefore, terms t_1 and t_2 in Eq. (27) are nonnegative.

Let $i(x) = (\pi - x)\sin x$. Observe that $i''(x) \leq 0$ for $x \in [0, \pi/2]$ and $i'(1) > 0$. Therefore, $i(x)$ is nondecreasing on the interval $[0, 1]$. Consequently, since $0 \leq \gamma \leq \beta < 1$, we get

$$(\pi - \beta)\sin\beta - (\pi - \gamma)\sin\gamma \geq 0. \qquad (29)$$

Therefore, term t_3 in Eq. (27) is nonnegative. Since terms t_1, t_2, and t_3 are nonnegative and Eq. (27) is equal to zero, term t_4 must be nonpositive. Thus,

$$2\pi\sin\alpha \leq 0 \Rightarrow \alpha = 0. \qquad (30)$$

Furthermore, by Eq. (24),

$$\cos\gamma - \cos\beta = 0 \Rightarrow \gamma = \beta, \qquad (31)$$

and by Eq. (22),

$$\cos\beta - (\pi - \beta)\sin\beta + 1 = 0. \qquad (32)$$

Since $\alpha = 0$ and $\beta = \gamma$, we get $\Gamma(P_{\alpha,\beta,\gamma})_y = m_y$ as required. Eq. (32) has a single root on $\beta \in [0, \pi/2]$. This can be seen by the fact that its derivative is nonpositive and its second derivative is strictly positive on this interval. This root occurs near $\beta = 0.81047$. These values are substituted into $f(\alpha, \beta, \gamma)$ to give

$$\sup_{(\alpha,\beta,\gamma)\in[0,\pi/2]^3} ||\Gamma(P_{\alpha,\beta,\gamma}) - m|| \approx 1.1153. \qquad (33)$$

Since the Euclidean radius of P is one, this implies the eccentricity of the Steiner centre is also approximately 1.1153. □

5. Stability of the Steiner Centre

Closely related to our definition of stability, Alt et al.[6] define the *quality* of a reference point using Hausdorff distance and show that the quality of the Steiner point is $4/\pi$. Our definition of stability lends itself better to the notion of a perturbation of a set of points (for example, our definition enables the stability of the centre of mass to be analyzed) and allows us to exploit the inverse relationship between stability and maximum velocity (see Theorem 3). For completeness, we include our proof of the stability of the Steiner centre.

Theorem 2. *The Steiner centre* Γ_2 *is* $\frac{\pi}{4}$*-stable.*

Proof. Choose any $P \in \widehat{\mathscr{P}}(\mathbb{R}^2)$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ denote any ε-perturbation. Let set $Q = f(P)$. Since Γ_2 is invariant under rotation and translation, without loss of generality assume $\Gamma_2(P)$ and $\Gamma_2(Q)$ lie on the x-axis. Since the extreme points in any projection A_θ define the midpoint of A_θ, for any θ,

$$||\mathrm{mid}(P_\theta) - \mathrm{mid}(Q_\theta)|| \leq \max_{p \in \overline{P}} ||p - f(p)||. \tag{34}$$

Thus, for any θ,

$$\begin{aligned} |\mathrm{mid}(P_\theta)_x - \mathrm{mid}(Q_\theta)_x| &= |\cos\theta| \cdot ||\mathrm{mid}(P_\theta) - \mathrm{mid}(Q_\theta)|| \\ &\leq |\cos\theta| \cdot \max_{p \in \overline{P}} ||p - f(p)|| \\ &\leq |\cos\theta| \cdot \varepsilon. \end{aligned}$$

We bound the stability of Γ_2 from below by

$$\begin{aligned} ||\Gamma_2(P) - \Gamma_2(f(P))|| &= |\Gamma_2(P)_x - \Gamma_2(Q)_x| \\ &= \left| \frac{2}{\pi} \int_0^\pi \mathrm{mid}(P_\theta)_x \, d\theta - \frac{2}{\pi} \int_0^\pi \mathrm{mid}(Q_\theta)_x \, d\theta \right| \\ &\leq \frac{2}{\pi} \int_0^\pi |\mathrm{mid}(P_\theta)_x - \mathrm{mid}(Q_\theta)_x| \, d\theta \\ &\leq \frac{2}{\pi} \int_0^\pi |\cos\theta| \cdot \varepsilon \, d\theta \\ &= \frac{4\varepsilon}{\pi}. \end{aligned}$$

Therefore,

$$\forall \varepsilon > 0, \ \forall f \in F_\varepsilon^P, \ \frac{\pi}{4} ||\Gamma_2(P) - \Gamma_2(f(P))|| \leq \varepsilon, \tag{35}$$

for all $P \in \widehat{\mathscr{P}}(\mathbb{R}^2)$. □

The following example shows that the stability bound is tight. Let $P = \{(\cos\theta, \sin\theta) \mid 0 \leq \theta < 2\pi\}$ be the set of points on the unit circle centred at the

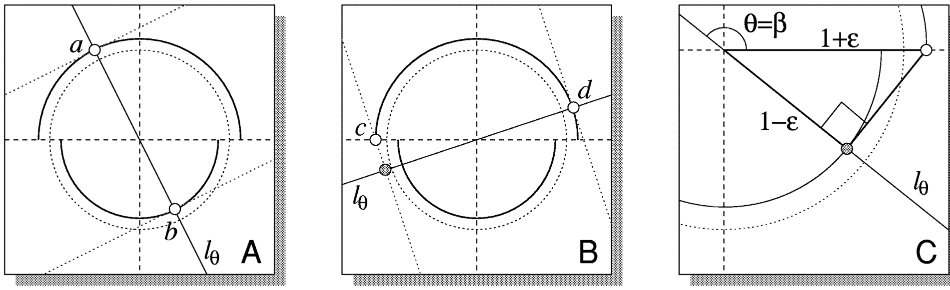

Fig. 6. Illustrations supporting Theorem 2.

origin. Let $\varepsilon \in (0,1)$ be fixed and let function $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an ε-perturbation defined by

$$f(p) = \begin{cases} (1+\varepsilon)p & \text{if } p_y \geq 0 \\ (1-\varepsilon)p & \text{if } p_y < 0 \end{cases}. \tag{36}$$

Let set $Q = f(P)$. Q corresponds to an ε-perturbation of P such that points on or above the x-axis are scaled outward by ε and points below are scaled inward by ε. See Fig. 6A.

For every $\theta \in [0, \pi]$,

$$\text{mid}(P_\theta) = \frac{1}{2}[u_\theta + (-u_\theta)] = (0,0). \tag{37}$$

Consequently, $\Gamma_2(P) = (0,0)$. The midpoint of Q_θ can be described by three cases. The simplest case occurs when one extremum of Q_θ lies on the outer semicircle and the second extremum lies on inner semicircle. For example, see points a and b in Fig. 6A. The second case occurs for angles θ near zero; in this case, one extremum of Q_θ is defined by the projection of one endpoint of the outer semicircle onto line l_θ whereas the other extremum remains on the outer outer semicircle. For example, see points c and d in Fig. 6B. The final case is analogous to the second case and occurs for angles θ near π. The angles θ for which a transition occurs from one case to the next are given by $\alpha = \arccos\left(\frac{1-\varepsilon}{1+\varepsilon}\right)$ and $\beta = \pi - \alpha = \arccos\left(\frac{\varepsilon-1}{1+\varepsilon}\right)$. See Fig. 6C.

The Steiner centre $\Gamma_2(Q)$ is defined in terms of $\text{mid}(Q_\theta)$. We examine the value $\text{mid}(Q_\theta)$ over the three intervals, $[0,\alpha]$, $[\alpha,\beta]$, and $[\beta,\pi]$. For $\theta \in [0,\alpha]$,

$$\text{mid}(Q_\theta) = \frac{1}{2}[u_\theta\langle -(1+\varepsilon,0), u_\theta\rangle + u_\theta(1+\varepsilon)] = \frac{u_\theta(1+\varepsilon)(1-\cos\theta)}{2}. \tag{38}$$

For $\theta \in [\alpha,\beta]$,

$$\text{mid}(Q_\theta) = \frac{1}{2}[(1+\varepsilon)u_\theta + (1-\varepsilon)(-u_\theta)] = \varepsilon \cdot u_\theta. \tag{39}$$

For $\theta \in [\beta,\pi]$,

$$\text{mid}(Q_\theta) = \frac{1}{2}[u_\theta(1+\varepsilon) + u_\theta\langle(1+\varepsilon,0), u_\theta\rangle] = \frac{u_\theta(1+\varepsilon)(1+\cos\theta)}{2}. \tag{40}$$

The Steiner centre of set Q is

$$\begin{aligned}
\Gamma_2(Q) &= \frac{2}{\pi}\int_0^\pi \mathrm{mid}(Q_\theta)\ d\theta \\
&= \frac{2}{\pi}\left[\int_0^\alpha \mathrm{mid}(Q_\theta)\ d\theta + \int_\alpha^\beta \mathrm{mid}(Q_\theta)\ d\theta + \int_\beta^\pi \mathrm{mid}(Q_\theta)\ d\theta\right] \\
&= \frac{2}{\pi}\left[\frac{1+\varepsilon}{2}\int_0^\alpha u_\theta(1-\cos\theta)\ d\theta + \varepsilon\int_\alpha^\beta u_\theta\ d\theta\right. \\
&\quad \left. + \frac{1+\varepsilon}{2}\int_\beta^\pi u_\theta(1+\cos\theta)\ d\theta\right] \\
&= \frac{2}{\pi}\left[\left(\frac{\sqrt{\varepsilon}(1+3\varepsilon)}{2(1+\varepsilon)} - \frac{1+\varepsilon}{4}\arccos\alpha, \frac{\varepsilon^2}{1+\varepsilon}\right) + \left(0, \frac{2\varepsilon(1-\varepsilon)}{1+\varepsilon}\right)\right. \\
&\quad \left. + \left(-\frac{\sqrt{\varepsilon}(1+3\varepsilon)}{2(1+\varepsilon)} + \frac{1+\varepsilon}{4}\arccos\alpha, \frac{\varepsilon^2}{1+\varepsilon}\right)\right] \\
&= \left(0, \frac{4\varepsilon}{\pi(1+\varepsilon)}\right).
\end{aligned}$$

For any fixed $\varepsilon > 0$, $f(P)$ defines an actual point set in $\widehat{\mathscr{P}}(\mathbb{R}^2)$. Therefore, when $\Gamma_2(P) \neq \Gamma_2(f(P))$, the κ-stability of Γ_2 satisfies

$$\begin{aligned}
\kappa &\le \frac{\varepsilon}{||\Gamma_2(P) - \Gamma_2(f(P))||} \\
&\le \lim_{\varepsilon\to 0}\frac{\varepsilon}{||\Gamma_2(P) - \Gamma_2(f(P))||} \\
&= \lim_{\varepsilon\to 0}\frac{\varepsilon}{||\Gamma_2(f(P))||} \\
&= \lim_{\varepsilon\to 0}\frac{\varepsilon}{\left|\left|\left(0, \frac{4\varepsilon}{\pi(1+\varepsilon)}\right)\right|\right|} \\
&= \frac{\pi}{4},
\end{aligned}$$

where P and f are as described above. It follows that Γ_2 is $\frac{\pi}{4}$-stable, but not $(\frac{\pi}{4} + \varepsilon)$-stable for any $\varepsilon > 0$.

6. Applications to Mobile Facility Location

The Steiner centre's success at defining a point of low eccentricity and high stability is not limited to the setting of static sets of points. Recent developments within facility location examine problems related to the mobile centre of a set of mobile points. The qualities of the Steiner centre transfer quite naturally to define the position of a mobile facility that provides a close approximation to the Euclidean centre while maintaining a low upper bound on velocity. In this section we examine properties of the mobile Steiner centre.

6.1. *Related work in mobile facility location*

The traditional problems of facility location are set in a static setting; client positions are fixed and a single location is selected for each facility. The problems of *static* facility location have been studied extensively. Within the last few years, partly motivated by the applicability of mobile computing to the wireless telecommunication industries involving cellular and radio ethernet, these questions have been posed in the *mobile* setting.[2,3,4,5,9,10,17,20] Given a set of clients whose positions change continuously over time with bounded velocity, the location of a mobile facility is specified by some given centre function Υ_d of the client positions. The fitness of the mobile facility is determined not only by the quality of its optimization of the objective function (captured by the eccentricity of Υ_d) but also by the maximum velocity and continuity of its motion (inversely related to the stability of Υ_d). These additional factors usually require the optimal location to be approximated, leading to new approximation strategies quite different from previous static approximations.

Until recently, only discrete changes to the location of clients had been considered. Such problems, termed *dynamic facility location*,[32,33] either attempt to optimize the objective function summed over a finite set of discrete time slots, $T = \{t_1, \ldots, t_f\}$, or they restrict locations for facilities to a discrete set (often the set of client positions).[11] These models do not incorporate continuity or bounded-velocity constraints in the motion of the facility. Thus, the techniques employed to solve static facility location problems do not necessarily extend to their mobile counterparts.

Within the setting of continuous motion of clients and facilities, Agarwal and Har-Peled[5] maintain the approximate mobile Euclidean centre in $\mathbb{R}^2$ under ℓ_∞ and ℓ_2. Their approximations do not require continuity or bounded velocity in the motion of the centre function; their objective, rather, is to minimize the number of events processed and the update cost per event using a kinetic data structure (KDS) to maintain a $(1+\varepsilon)$-approximation on the extent of the point set. Agarwal et al.[3] use a KDS to maintain a *kd*-tree of the points and a δ-approximate mobile median in $\mathbb{R}$. Similarly, Agarwal et al.[2] use a KDS to maintain the exact (expensive) and ε-approximate (less expensive) mobile medians in $\mathbb{R}$ and $\mathbb{R}^2$. Bespamyatnikh et al.[10] maintain approximations to the mobile Euclidean centre and mobile median in $\mathbb{R}^2$ under ℓ_∞ and ℓ_2. These include an extreme point of the convex hull, the rectilinear centre, the centre of mass, and linear combinations of these.

6.2. *Defining a mobile centre*

Given a set of mobile clients whose positions are defined by bounded continuous functions, any continuous centre function Υ_d can be used to define the position of a mobile facility.

Definition 10. Let $T = [0, t_f]$ be a time interval. Let $P = \{p_1, \ldots, p_n\}$ be a nonempty finite set of **mobile clients** such that for every i, $p_i : T \to \mathbb{R}^d$ is a bounded continuous function that defines the position of client i in $\mathbb{R}^d$ at every instant $t \in T$.

For every $t \in T$, let $P(t) = \{p_i(t) \mid p_i \in P\}$ denote the set of points corresponding to the positions of clients in P at time t. Following Bespamyatnikh et al.,[10] we define the mobile Euclidean centre as a direct extension of its static definition: $\Xi_d(P(t))$. Similarly, the definition of the mobile Steiner centre is simply $\Gamma_2(P(t))$.

Since eccentricity is defined in terms of a worst-case configuration, it is independent of motion of points. Thus, the eccentricity of a mobile facility whose position is defined by centre function Υ_d is simply the eccentricity of Υ_d.

6.3. *The velocity of a mobile facility*

In mobile facility location, velocity constraints restrict the behaviour of both clients and facilities. In addition to requiring that the motion of clients be continuous, we assume that the magnitude of velocity is bounded by a constant $\sigma > 0$. That is,

$$\forall p_i \in P,\ \forall t_1, t_2 \in T,\ ||p_i(t_1) - p_i(t_2)|| \leq \sigma \cdot |t_1 - t_2|. \tag{41}$$

When p_i is differentiable, then $\forall t \in T,\ ||p_i'(t)|| \leq \sigma$. We assume a constant upper bound $\sigma = 1$ on the velocity[a] of clients (since there is no unit of reference, we may choose any σ without loss of generality).

Since many applications impose some upper bound on the velocity of facilities, we formally define the maximum relative velocity of a mobile facility.

Definition 11. Let P be a set of mobile clients, each moving with at most unit velocity. Let $\Upsilon_d : \widehat{\mathscr{P}}(\mathbb{R}^d) \to \mathbb{R}^d$ be a centre function. The **maximum relative velocity** of a mobile facility whose location is determined by Υ_d is bounded by $v_{\max}$ if

$$\forall t_1, t_2 \in T,\ ||\Upsilon_d(P(t_1)) - \Upsilon_d(P(t_2))|| \leq v_{\max}|t_1 - t_2|. \tag{42}$$

For some mobile facilities no finite velocity bound may exist.

The similarity of Definitions 5 and 11 is perhaps not surprising; the maximum relative velocity, $v_{\max}$, and the stability, κ, of a mobile centre are inversely related. Within the context of mobile facility location, maximum velocity and eccentricity describe the fitness of a mobile facility's approximation of the mobile Euclidean centre just as stability and eccentricity described the fitness of a centre function's approximation of the static Euclidean centre.

Theorem 3. *If $\Upsilon_d : \widehat{\mathscr{P}}(\mathbb{R}^d) \to \mathbb{R}^d$ is a κ-stable centre function, then a mobile facility whose position is determined by Υ_d has maximum relative velocity at most*

$$v_{\max} = \frac{1}{\kappa}. \tag{43}$$

[a]We use the terms "bounded velocity" to mean "bounded magnitude of velocity".

Proof. Let P be a set of mobile clients defined over time interval $T = [0, t_f]$. Choose any $\varepsilon > 0$ and any $0 \leq t_1 \leq t_2 \leq t_f$ such that $\varepsilon = |t_1 - t_2|$. Let set $Q = P(t_1)$. If clients move with velocity at most one, then for every $p_i \in P$, $||p_i(t_1) - p_i(t_2)|| \leq |t_1 - t_2| = \varepsilon$. Therefore, there exists an ε-perturbation of Q, denoted f, such that $f(Q) = P(t_2)$. The equivalence follows:

$$\forall t_1, t_2 \in T,\ ||\Upsilon_d(P(t_1)) - \Upsilon_d(P(t_2))|| \leq v_{\max}|t_1 - t_2|$$
$$\Leftrightarrow \frac{1}{v_{\max}}||\Upsilon_d(Q) - \Upsilon_d(f(Q))|| \leq \varepsilon. \qquad \square$$

An immediate consequence of Theorem 3 is that the mobile Euclidean centre has no finite velocity bound.[10] The result provides further motivation for the identification of a centre function that achieves both high stability and low eccentricity.

Thus, the Steiner centre has a natural extension to the mobile setting whose behaviour makes it a good approximation to the mobile Euclidean centre, both in terms of its maximum velocity and approximation factor. Theorems 2 and 3 imply that the maximum relative velocity of the mobile Steiner centre is at most $4/\pi$.

7. Kinetic Maintenance of the Steiner Centre

We examine implementation issues involving kinetic data structures for the maintenance of both exact and approximate mobile Steiner centres of a set of mobile points. We describe a simple algorithm to maintain an arbitrarily-close approximation of the Steiner centre of a set of mobile points by using a Kinetic Data Structure to maintain the k-hull of the points (see Definition 12). We show the motion of the Steiner centre of the k-hull follows a piecewise-linear trajectory. Although the Steiner centre has two equivalent definitions, in this context maintaining the Steiner centre of the k-hull is simplified by its formulation by Gaussian weights.

7.1. *Maintaining the mobile Steiner centre with a kinetic data structure*

Kinetic data structures (KDS)[8] allow for efficient implementation and maintenance of various attributes of a finite set of mobile points under linear (or bounded-degree algebraic) motion. Those related to the mobile Steiner centre include the bounding box,[5] the convex hull,[8,19] a $(1+\varepsilon)$-approximate Euclidean centre,[5] and the extent of a set of mobile points[4,5,8] in $\mathbb{R}$. The constraint on the degree of the motion allows for the occurrence of events related to the trajectories of points to be calculated exactly.

The Gaussian weight $w_2(p(t))$ of an extreme point p at time t is defined in terms of p and its two neighbouring points, a and b, on the convex hull boundary:

$$w_2(p(t)) = \pi - \arccos\left(\frac{||p(t) - a(t)||^2 + ||p(t) - b(t)||^2 - ||b(t) - a(t)||^2}{2||p(t) - a(t)|| \cdot ||p(t) - b(t)||}\right). \qquad (44)$$

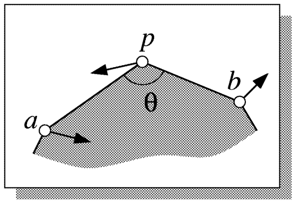

Fig. 7. The Gaussian weight of p is defined in terms of the positions of p, a, and b.

See Fig. 7. Even if the motion of points is linear, function $w_2(p(t))$ remains trigonometric. As a consequence, the position of the Steiner centre is not expressible as a polynomial and its description requires a number of terms proportional to the size of the convex hull, $\Theta(|P|)$. Similarly, even under linear motion of points, the trajectory of the Euclidean centre Ξ_2 cannot be expressed algebraically. At any given time, the position of Ξ_2 is defined by at most three points and, unlike Γ_2, the trajectory of Ξ_2 is expressible by a constant number of terms (while the same three points define Ξ_2).

Given this constraint on the complexity of a description of the Steiner centre's trajectory, the position of Γ_2 may be maintained by any KDS that maintains the convex hull of a set of mobile points. For any mobile point p, the description of its Gaussian weight $w_2(p(t))$ changes only when the neighbours of p change along the convex hull boundary or when p joins or leaves the convex hull boundary. Each such update requires only constant time. Therefore, the number of KDS events processed remains unchanged and the complexity of the new KDS is not increased. Thus, a KDS may be used to maintain the Steiner centre with responsiveness, efficiency, locality, and compactness identical to that for maintaining the convex hull. However, the expression for the position of the Steiner centre requires $\Theta(n)$ terms.

7.2. *The Steiner centre of the k-hull*

The definition of many centre functions (like the Euclidean centre and the Steiner centre) depends only on extreme points of the set P. Of course, the convex hull of any (possibly infinite) bounded set of points P can be closely approximated by some finite set of points P'. We formalize this notion by defining the k-hull of a set of points. We then show that when any set of points $P \in \widehat{\mathscr{P}}(\mathbb{R}^d)$ is approximated by its k-hull, $Q_k(P)$, the relative distance between $\Gamma_2(P)$ and $\Gamma_2(Q_k(P))$ is $O(\frac{1}{k})$.

Definition 12. Let $P \in \widehat{\mathscr{P}}(\mathbb{R}^2)$ and let $k \in \mathbb{Z}, k \geq 4$, be fixed. The ***k*-hull** of P, denoted $Q_k(P)$, is defined by the intersection of all half-planes H^+ such that $P \subseteq H^+$ and the outer normal to the boundary line of H^+ is $u_\phi = (\cos\phi, \sin\phi)$ for some $\phi = 0 \bmod \frac{2\pi}{k}$.

See the example in Fig. 8 for $k = 8$. The boundary of $Q_k(P)$ is a polygon with

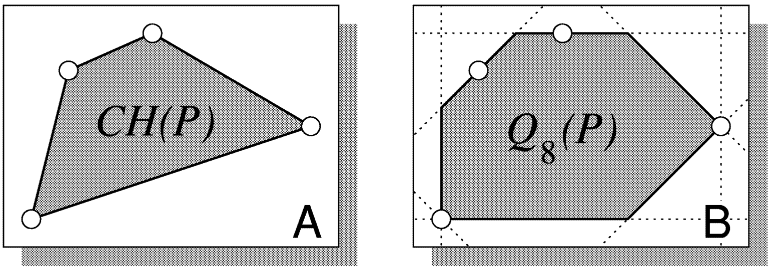

Fig. 8. The convex hull and the 8-hull of P.

at most k sides whose edges have normals parallel to $(\cos(\frac{2\pi j}{k}), \sin(\frac{2\pi j}{k}))$ for some $j \in \mathbb{Z}$. As k increases, the k-hull of P approaches the convex hull of P.

We show that when a point set P is approximated by its k-hull, $Q_k(P)$, the relative distance between $\Gamma_2(P)$ and $\Gamma_2(Q_k(P))$ is $O(\frac{1}{k})$.

Lemma 3. *Let $P \in \widehat{\mathscr{P}}(\mathbb{R}^2)$ and let $k \in \mathbb{Z}^+$ be fixed, $k \geq 4$. Let $Q_k(P)$ denote the k-hull of P and let r be the Euclidean radius of P. The distance between $\Gamma_2(P)$ and $\Gamma_2(Q_k(P))$ satisfies*

$$||\Gamma_2(Q_k(P)) - \Gamma_2(P)|| \leq \frac{16r}{\pi k}. \tag{45}$$

Proof. Since $\Gamma_2(P) = \Gamma_2(CH(P))$ and the k-hull of P is equal to the k-hull of $CH(P)$, assume without loss of generality that $P = CH(P)$. Choose any $k \in \mathbb{Z}$, $k \geq 4$. Let $Q_k(P)$ denote the k-hull of P. Let r be the Euclidean radius of P. Let f be an ε-perturbation of $Q_k(P)$ such that for every $q \in Q_k(P)$, $f(q)$ is a nearest point in P to q (the value of ε is chosen below). For every edge l of the boundary of $Q_k(P)$, there is a point $p \in P$ tangent to l. Let a and b be extreme points in P defining adjacent boundary edges l_1 and l_2 on the boundary of $Q_k(P)$. Let point $c \in Q_k(P)$ denote the intersection of l_1 and l_2. See Fig. 9B. If $c \in P$ then locally $||f(c) - c|| = 0$. Therefore assume $c \notin P$. The distance from c to line $\overline{ab}$ is maximized when $||a - c|| = ||b - c||$. Since $c \notin P$, angle $\angle acb = \pi - 2\pi/k$. Consequently, $\angle cab =$

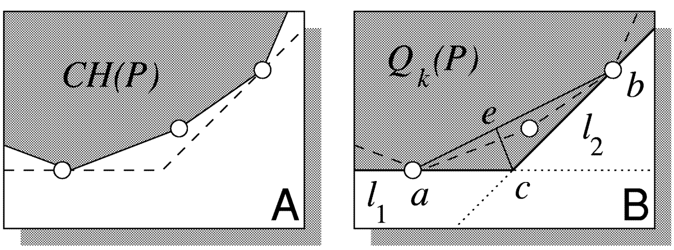

Fig. 9. Illustration in support of Lemma 3.

$\angle cba = \pi/k$. Since $a,b \in P$, $||a - b|| \leq 2r$. Let e be the midpoint of $\overline{ab}$. Therefore, $||a-e|| \leq r$ and $||e-c|| \leq r\tan(\pi/k)$. Thus, no point in $Q_k(P)$ may lie farther than $r\tan(\pi/k)$ from the convex hull of P. Therefore, the maximum distance between $Q_k(P)$ and the convex hull of P is at most $\max_{q \in \overline{Q_k(P)}} ||q - f(q)|| \leq r\tan(\pi/k)$. Thus, f is an $r\tan(\pi/k)$-perturbation of $Q_k(P)$. By Definition 5 and Theorem 2,

$$\frac{\pi}{4}||\Gamma_2(Q_k(P)) - \Gamma_2(f(Q_k(P)))|| \leq \varepsilon. \tag{46}$$

Since $P \subseteq Q_k(P)$ and by the definition of f, observe that $\Gamma_2(f(Q_k(P))) = \Gamma_2(P)$. Also note that if $\theta < \pi/4$, then $\tan\theta < 4\theta/\pi$. Therefore,

$$\begin{aligned} ||\Gamma_2(P) - \Gamma_2(Q_k(P))|| &= ||\Gamma_2(Q_k(P)) - \Gamma_2(f(Q_k(P)))|| \\ &\leq \frac{4}{\pi}\varepsilon \\ &= \frac{4}{\pi}r\tan\left(\frac{\pi}{k}\right) \\ &\leq \frac{4}{\pi}r\left[\frac{4}{\pi}\frac{\pi}{k}\right] \\ &= \frac{16r}{\pi k}. \end{aligned}$$

□

7.3. *Mobile implementation using the k-hull*

For implementation it may be desirable to define a mobile centre that carries the benefits of the Steiner centre but has a simple polynomial description. Under linear motion of points, we describe a simple discretization using the k-hull that allows the motion of the Steiner centre to be closely approximated by a piecewise-linear function.

Let $Q_k(P)$ denote the k-hull of a set $P \in \widehat{\mathscr{P}}(\mathbb{R}^2)$. See the example in Fig. 8 for $k = 8$. The boundary of $Q_k(P)$ is a polygon with at most k sides with turn angles that are multiples of $\frac{2\pi}{k}$ between 0 and π. These correspond to Gaussian weights. Therefore, the Gaussian weight of $q \in Q_k(P)$ is $w_2(q(t)) = j\frac{2\pi}{k}$ for some $j \in \{0, \ldots, \lfloor k/2 \rfloor\}$. Furthermore, while the adjacencies between edges of $Q_k(P)$ to points on the convex hull boundary of P remains unchanged, the Gaussian weight of any extreme point $q \in Q_k(P)$ remains constant. Since the weights are constant, the Steiner centre $\Gamma_2(Q_k(P))$ is simply a linear combination of the vertices of $Q_k(P)$. Therefore, under linear motion of points of P, between events along the convex hull boundary of $Q_k(P)$, the motion of $\Gamma_2(Q_k(P))$ is also linear (and continuous). In general, the motion of $\Gamma_2(Q_k(P))$ is piecewise-linear.

Maintaining the mobile k-hull of P in a KDS is simple. Associated with the k-hull are k normal vectors, $u_\phi = (\cos\phi, \sin\phi)$, where ϕ is drawn from the set of k angles $\Phi = \{j\frac{2\pi}{k} \mid 0 \leq j \leq k-1\}$. For each $\phi \in \Phi$, let $P_\phi = \{u_\phi\langle p, u_\phi\rangle \mid p \in P\}$ be the projection of P onto the line through the origin that lies parallel to u_ϕ. We maintain the maximum point in each of the k sets P_ϕ. As described by Guibas,[19] a KDS that maintains the maximum of a set of points in $\mathbb{R}$, each moving with linear motion, is responsive, efficient, compact, and local. Under linear motion the maximum point of each set P_ϕ changes at most $n = |P|$ times. We require maintaining k instances of this KDS. Therefore, the total number of times a maximum point changes is at most $k \cdot n$.

The set of k maximum points defines the k-hull, $Q_k(P)$, and ultimately the Steiner centre of $Q_k(P)$, $\Gamma_2(Q_k(P))$. Associated with each maximum point is a tangent line with normal u_ϕ. These lines are ordered and we maintain the k intersection points that define the boundary of the k-hull (intersection points may be collocated resulting in fewer than k points). Since the points of P move linearly, the motion of the intersection points is also linear. Furthermore, an intersection point only requires updating whenever the maximum point of one of its defining lines is updated. For each such event, the Gaussian weight of a point on the boundary of $Q_k(P)$ requires a constant-time update. Between events, weights of points in $Q_k(P)$ remain constant.

Although Richardson[24] provides an approximation of the convex hull of P to within $O(1/k^2)$ while requiring at most k vertices, the k-hull has the advantage that interior angles at the vertices of $Q_k(P)$ are multiples of $2\pi/k$. Consequently, maintaining the kinetic k-hull is straightforward and only requires maintaining the k supporting planes with outer normals $j \cdot 2\pi/k$, for $j = 0 \ldots k-1$.

In summary, given a set of mobile points P each moving in linear trajectories, the Steiner centre of P does not move with algebraic motion. However, the k-hull allows the maintenance of an approximation to the mobile Steiner centre of P that moves with piecewise-linear motion. Furthermore, k can be selected independently of $|P|$ to ensure the approximate Steiner centre is made arbitrarily close to the Steiner centre of P. Finally, maintaining the k-hull and approximate Steiner centre of a set of mobile points P using a KDS is responsive, efficient, compact, and local.

Remark. As suggested by an anonymous referee, the size of the KDS can be reduced from $\Theta(kn)$ to $\Theta(n \log k)$ by using a natural generalization of a kinetic tournament.[8,19] The size bound exploits the fact that the k-hull of a set P can be efficiently represented in $\Theta(\min\{k, |P|\})$ space. The total number of change events remains $O(kn)$.

8. Comparing the Steiner Centre to Other Centre Functions

We briefly discuss other common centre functions, and compare their eccentricity and stability against the Steiner centre[b].

As discussed by Bespamyatnikh et al.,[10] the centre of mass is stable ($\kappa = 1$) but very eccentric ($\lambda = 2$). The worst case eccentricity occurs when $|P| - 1$ clients are

[b] A graphical implementation of the Steiner centre of a set of mobile points is available as a Java applet on the web at `http://www.cs.ubc.ca/~durocher/gaussianDemo.html`. The demonstration provides visual intuition of the stability and eccentricity of the Steiner centre as compared with the Euclidean centre, the rectilinear centre, and the centre of mass of a set of mobile points.

collocated and one client lies elsewhere. As $n \to \infty$, the eccentricity λ approaches 2. Since the centre of mass is an average, the average magnitude of velocity is bounded by the magnitude of each component. Thus, $v_{\max} = 1$ (by Theorem 3, $\kappa = 1$).

Table 2. Comparing centre functions in $\mathbb{R}^2$.

centre function	λ-eccentricity	κ-stability $= (v_{\max})^{-1}$
Euclidean centre Ξ_d	1	0
single point	2	1
centre of mass	2	1
rectilinear centre	$\frac{1+\sqrt{2}}{2} \approx 1.2071$	$\frac{1}{\sqrt{2}} \approx 0.7071$
Steiner centre Γ_2	≈ 1.1153	$\frac{\pi}{4} \approx 0.7854$

The rectilinear centre (ℓ_∞ centre) of a set $P \in \widehat{\mathscr{P}}(\mathbb{R}^2)$ is the unique point $\Upsilon_2(P)$ that minimizes both

$$\max_{p \in P} |\Upsilon_2(P)_x - p_x| \quad \text{and} \quad \max_{p \in P} |\Upsilon_2(P)_y - p_y|. \tag{47}$$

The rectilinear centre is orientation dependent. Unlike the Euclidean and Steiner centres that are invariant under rotation, $\Upsilon_2(f(P)) \neq f(\Upsilon_2(P))$ for some rotations $f : \mathbb{R}^2 \to \mathbb{R}^2$. As shown by Bespamyatnikh et al.,[10] the rectilinear centre is is $(1+\sqrt{2})/2$-eccentric with maximum velocity $v_{\max} = \sqrt{2}$ (by Theorem 3, $\kappa = 1/\sqrt{2}$).

Finally, as discussed in Sec. 3.1, any arbitrary point $p \in P$ defines a 2-eccentric, 1-stable centre function of P. Several other common notions of centre exist that exhibit either low stability, high eccentricity, or both. Often the motion of the centre function is discontinuous or its velocity is unbounded. These include the median[16] (minimizes the sum of distances to $p \in P$) and the Lemoine point[21,30] (minimizes the sum of squared distances to edges of the convex hull). Each of these has eccentricity $\lambda = 2$ and stability $\kappa = 0$ ($v_{\max} = \infty$). Also, any two (or more) different centre functions can be combined linearly[10] to obtain some combination of the eccentricity and stability factors of each. Finally, observe that the bound $\kappa \leq 8\sqrt{\lambda - 1}$ given by Bereg et al.[9] only impacts centre functions whose eccentricity lies in the range $1 < \lambda \leq 65/64 \approx 1.0156$; for $\lambda > 65/64$, $8\sqrt{\lambda - 1} > 1$ and we bound κ from above by $\kappa \leq 1$.

As displayed is Table 2, the high stability and low eccentricity of the Steiner centre compares well with those of other centre functions. All eccentricity and stability bounds listed in Table 2 are tight.

9. Future Work

Both definitions of the Steiner centre have natural extensions into three dimensions. In three dimensions, the maximum velocity and, therefore, the stability of Γ_3 can be shown to be 3/2 and 2/3, respectively, by a method analogous to the proof of Theorem 2. Again, this is related to the work of Alt et al.[6] on the quality of the

Steiner centre in $\mathbb{R}^3$. As for eccentricity in $\mathbb{R}^3$, although Lemma 2 easily extends to $\mathbb{R}^3$, the generalization of Theorem 1 appears to be non-trivial, although examples exist to imply that the eccentricity of Γ_3 in $\mathbb{R}^3$ is greater than 1.1153. Several questions (including applications to mobile facility location) remain to be answered in three dimensions, opening the possibility for future applications of the Steiner centre in three and higher dimensions.

Furthermore, the techniques developed in this paper have applications toward defining a stable median function[16] as well as extensions to the k-centre problem.

Acknowledgments

The authors wish to thank an anonymous referee for several helpful suggestions. Special thanks to Andrea Bunt for her careful reading of numerous drafts of the manuscript.

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