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BOUNDED-VELOCITY APPROXIMATION OF MOBILE EUCLIDEAN 2-CENTRES*

STEPHANE DUROCHER
School of Computer Science, McGill University
Montréal, Québec, Canada
durocher@cs.mcgill.ca

DAVID KIRKPATRICK
Department of Computer Science, University of British Columbia
Vancouver, British Columbia, Canada
kirk@cs.ubc.ca

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ABSTRACT

Given a set P of points (clients) in the plane, a Euclidean 2-centre of P is a set of two points (facilities) in the plane such that the maximum distance from any client to its nearest facility is minimized. Geometrically, a Euclidean 2-centre of P corresponds to a cover of P by two discs of minimum radius r (the Euclidean 2-radius). Given a set of mobile clients, where each client follows a continuous trajectory in the plane with bounded velocity, the motion of the corresponding mobile Euclidean 2-centre is not necessarily continuous. Consequently, we consider strategies for defining the trajectories of a pair of mobile facilities that guarantee a fixed-degree approximation of the Euclidean 2-centre while maintaining bounded relative velocity. In an attempt to balance the conflicting goals of closeness of approximation and a low maximum relative velocity, we introduce reflection-based 2-centre functions by reflecting the position of a mobile client across the mobile Steiner centre and the mobile rectilinear 1-centre, respectively.

Keywords: 2-centre; motion; approximation; velocity; Euclidean; continuous.

1. Introduction

1.1. *Motivation*

The traditional problems of facility location are defined statically; a set of n points is given as input, corresponding to the positions of clients, and a solution consists

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of a set of k points, corresponding to the positions of facilities, that optimizes some objective function of the input set. In the k-centre problem, the objective is to select locations for k facilities such that the maximum distance from any client to its nearest facility is minimized. A common setting for these problems is to model clients and facilities as points in Euclidean space and to measure distances between these by the Euclidean (ℓ_2) distance metric; this defines the Euclidean k-centre problem.

Recently, motivated in part by applications in the field of mobile computing, there has been considerable interest in recasting a number of familiar questions of facility location in a mobile context (e.g., Refs. 1,2,4,5,11,13,22,24,25,28,32); these include results related to the mobile Euclidean 1-centre problem (see Section 3). In this paper we consider the mobile Euclidean 2-centre problem. A problem instance consists of a set of mobile clients, where each client follows a continuous trajectory through Euclidean space under bounded velocity. A mobile Euclidean 2-centre is a pair of mobile facilities defined as a function of the instantaneous positions of the clients, such that at any time, the facilities define a Euclidean 2-centre of the client set. There exist trajectories for a set mobile clients P, such that any mobile Euclidean 2-centre of P is discontinuous. Consequently, for any pair of mobile facilities whose motion has bounded velocity, the distance between some client p in P and the facility nearest to p exceeds the Euclidean 2-radius of P. Therefore, we seek strategies for defining the positions of a pair of mobile facilities that provide a good approximation to the Euclidean 2-centre while maintaining motion that is continuous and whose magnitude of velocity has a low fixed upper bound. Thus, the fitness of a mobile facility is determined not only by the quality of its optimization of the objective function but also by its maximum velocity. This additional constraint leads to a trade-off between velocity and approximation factor, requiring new approximation strategies quite different from previous static approximations.

1.2. *Main result*

We introduce reflection-based 2-centre functions, a family of approximations to the mobile Euclidean 2-centre that involves coordinating the positions of the two facilities without explicit partitioning of the client set. We obtain a solution by using reflection; the position of the first facility is set to coincide with that of a mobile client while the position of the second facility is given by the reflection of the first facility across a mobile centre of reflection. The choice of reflection centre is critical to ensuring that both mobile facilities maintain bounded velocity and a good approximation of the mobile Euclidean 2-centre. We show that the Steiner centre and the rectilinear 1-centre represent good choices for the reflection centre. Refer to Table 1 for a summary of the bounds which we derive on the approximation factor and on the maximum relative velocity of these various approximations of the mobile Euclidean 2-centre. To the authors' knowledge, no previous bounded-velocity approximations to the mobile Euclidean 2-centre have been defined.

1.3. *Overview*

This paper is organized as follows. Section 2 recalls the definition of a Euclidean 2-centre, motivates the necessity of approximation, and formalizes the measures used for comparing the quality of different approximations. Section 3 provides a brief overview of related work. In Section 4 we introduce reflection-based approximations to the Euclidean 2-centre and derive bounds on their respective maximum relative velocities and approximation factors. Section 5 compares the various approximation strategies considered and mentions directions for future research.

2. Mobile Euclidean 2-Centre: Definition and Properties

2.1. *Euclidean k-centre of a set of points*

Given a set of points P in $\mathbb{R}^2$, a fundamental problem of geometry and data analysis concerns the characterization and computation of points that are central to P. A natural, and for many applications the default, metric for measuring distance between points is the Euclidean distance metric. We now recall the definition of a Euclidean k-centre of a set P, where k denotes the number of facilities serving P.

Definition 1. Given a finite set P in $\mathbb{R}^2$, a **Euclidean k-centre** of P is a set of k points, $\{\Xi^1(P), \ldots, \Xi^k(P)\}$, that minimizes

$$\max_{p \in P} \min_{i \in \{1,\ldots,k\}} ||p - \Xi^i(P)||. \tag{1}$$

We refer to the value of (1) as the **Euclidean k-radius** of P. It is straightforward to show that a Euclidean k-centre is invariant under similarity transformations. Geometrically, a Euclidean k-centre of P corresponds to the centres of k discs whose union covers the points of P such that the radius of the largest disc is minimized. Although the Euclidean 1-centre of P is unique, when $k \geq 2$, a Euclidean k-centre of P is not unique in general. For example, four points located at the vertices of the unit square in the plane have two distinct possible 2-centres, located at the midpoints of opposite pairs of edges of the square.

The Euclidean k-centre problem is also known as planar k-centre,[15,21] min-max multicentre,[17,29] minimax radius clustering,[12] and minimax location-allocation problem.[20]

In this paper we refer to values of $k \leq 2$. We denote the Euclidean 1-centre of a set P by $\Xi(P)$ and a Euclidean 2-centre of P by $\{\Xi^1(P), \Xi^2(P)\}$.

2.2. *Mobile clients and mobile facilities*

We consider continuous motion in $\mathbb{R}^2$. Since a *point* refers to a fixed position in Euclidean space, we refer to a *client* in the context of motion. That is, each client's position traces a continuous trajectory through the Euclidean plane, defined as a function over a continuous temporal dimension.

Definition 2. Let $T = [0, t_f]$ denote a time interval. Set $P = \{p_1, \ldots, p_n\}$ is a set of **mobile clients** if for every i, $p_i : T \to \mathbb{R}^2$ is a bounded continuous function that defines the position of client i in $\mathbb{R}^2$ at every instant $t \in T$.

We consider clients whose motion is continuous and we assume that each client's velocity is bounded by a constant $\sigma > 0$. That is,

$$\forall p \in P,\ \forall t_1, t_2 \in T,\ ||p(t_1) - p(t_2)|| \leq \sigma |t_1 - t_2|. \tag{2}$$

Throughout this article we assume a constant upper bound of $\sigma = 1$ on the velocity of clients since we are interested in *relative velocity.* We make no assumption on the continuity of higher derivatives, only that the rate of change in position is bounded.

For every $t \in T$, let $P(t) = \{p(t) \mid p \in P\}$ denote the set of points corresponding to the positions of clients in P at time t. We define the mobile Euclidean 1-centre and a mobile Euclidean 2-centre as a direct extensions of their respective static definitions: $\Xi(P(t))$ and $\{\Xi^1(P(t)), \Xi^2(P(t))\}$. Since a Euclidean 2-centre is not unique, a mobile Euclidean 2-centre of P is a pair of mobile facilities that realizes *any* Euclidean 2-centre of $P(t)$ at time t.

2.3. *Discontinuity of mobile Euclidean 2-centres*

Bereg et al.[13] show that the relative velocity of the Euclidean 1-centre has no finite upper bound. Specifically, for any $\sigma \geq 0$, Bereg et al. give an example of four mobile clients in $\mathbb{R}^2$, each moving in a linear trajectory with unit velocity, such that their Euclidean 1-centre moves with average velocity at least σ over some time interval whose length depends on σ.

The unbounded velocity of the Euclidean 1-centre is easily shown to imply a similar property for the Euclidean 2-centre.[22] Unlike the Euclidean 1-centre which is continuous,[22] we show that a continuous mobile Euclidean 2-centre does not always exist.

Proposition 1. *There exists a set of mobile clients P such that any mobile Euclidean 2-centre of P is discontinuous.*

See Appendix A for an example of four mobile clients that realize such a set P. This same example can be used to show that the rectilinear 2-centre and 2-means clustering are also discontinuous. See Section 4.1 for a discussion of the rectilinear 2-centre and 2-means clustering.

2.4. *Approximating the Euclidean 2-centre: comparison measures*

Continuity of motion and, more specifically, a finite upper bound on velocity impose natural constraints on any physical moving object. Scenarios involving vehicles, mobile robots, or people with wireless communication devices suggest that bounds on velocity are necessary in many applications (e.g., Refs. 9,16,18,19,34,37).

Given its discontinuity, a Euclidean 2-centre may be unfit for certain applications and impossible to maintain exactly within specific mobile contexts. A pair of mobile facilities that approximate the Euclidean 2-centre while maintaining some fixed upper bound on maximum velocity may be better suited. We refer to a pair $\Upsilon = \{\Upsilon^1, \Upsilon^2\}$, where $\Upsilon^i : \mathrm{P}(\mathbb{R}^2) \rightarrow \mathbb{R}^2$, as a **2-centre function**.

In analogy to (2), we say 2-centre function $\Upsilon = \{\Upsilon^1, \Upsilon^2\}$ has maximum (relative) velocity $v_{\max}$ if

$$\forall t_1, t_2 \in T, \max_{i \in \{1,2\}} ||\Upsilon^i(P(t_1)) - \Upsilon^i(P(t_2))|| \leq v_{\max}|t_1 - t_2|, \tag{3}$$

for all time intervals T and all sets of mobile clients P defined on T. If Υ has bounded relative velocity then $\{\Upsilon^1(P(t)), \Upsilon^2(P(t))\}$ cannot always coincide with a Euclidean 2-centre of $P(t)$; that is, $\max_{p \in P} \min_{i \in \{1,2\}} ||p(t) - \Upsilon^i(P(t))||$ must exceed the Euclidean 2-radius of P for some sets of mobile clients P. We formalize this notion and quantify it in terms of the *approximation factor* of Υ.

Definition 3. A 2-centre function $\Upsilon = \{\Upsilon^1, \Upsilon^2\}$ is a **λ-approximation** of the Euclidean 2-centre if

$$\forall P \in \mathrm{P}(\mathbb{R}^2), \max_{p \in P} \min_{i \in \{1,2\}} ||p - \Upsilon^i(P)|| \leq \lambda \max_{q \in P} \min_{j \in \{1,2\}} ||q - \Xi^j(P)||. \tag{4}$$

2.5. *General lower bounds on comparison measures*

We establish lower bounds on the maximum relative velocity and approximation factor of any mobile 2-centre function.

Proposition 2. *A mobile 2-centre function with maximum relative velocity less than $1 + \sqrt{3}/2$ cannot guarantee any bounded approximation of the Euclidean 2-centre.*

See Appendix A for a proof of Proposition 2. This property highlights a significant difference between approximations of the mobile Euclidean 1-centre and approximations of the mobile Euclidean 2-centre; in particular, several approximations of the mobile Euclidean 1-centre have been considered that guarantee an approximation factor of 2 while requiring at most unit velocity[13,22,25] in $\mathbb{R}^2$.

Bereg et al.[11] show that for any $\lambda > 1$ there exists a bounded-velocity approximation of the Euclidean 1-centre with approximation factor λ. Again, the situation differs when approximating the Euclidean 2-centre:

Proposition 3. *A continuous mobile 2-centre function in $\mathbb{R}^2$ cannot guarantee a λ-approximation of the Euclidean 2-centre for any $\lambda < \sqrt{2}$.*

See Appendix A for a proof of Proposition 3.

3. Related Work

A Euclidean 2-centre of a set of n points is straightforward to find in linear time in $\mathbb{R}$ (e.g., Ref. 22). A considerable number of results have appeared related to the static Euclidean 2-centre in $\mathbb{R}^2$, including both deterministic and randomized algorithms for finding exact or approximate solutions (e.g., Refs. 6,7,15,21,26,27,33,35,39) and algorithms for the corresponding decision problem (e.g., Ref. 31). Currently, the algorithm with the best running time is by Chan[15] who gives a deterministic solution to the exact problem in $\mathbb{R}^2$, requiring $O(n \log^2 n \log^2 \log n)$ time. The current best lower bound is $\Omega(n \log n)$ time, shown by Segal.[38] Agarwal and Sharir[8] mention a generalization of Drezner's algorithm[21] from $\mathbb{R}^2$ to $\mathbb{R}^d$ to give an algorithm requiring $O(n^{d+1})$ time.

With respect to motion, most of the previous work in this area is related to the mobile Euclidean 1-centre. These results serve as a benchmark against which we can contrast our results on the mobile Euclidean 2-centre.

Bereg et al.[11,13,14] show that the velocity of the mobile Euclidean 1-centre is unbounded in $\mathbb{R}^2$. They consider the centre of mass and the rectilinear 1-centre as bounded-velocity approximations and derive tight bounds of $(2 - 2/n)$ and $[(1 + \sqrt{2})/2]$ on their respective approximation factors and tight bounds of 1 and $\sqrt{2}$ on their respective relative velocities, where n is the number of mobile clients. Durocher and Kirkpatrick[25] examine the Steiner centre as a bounded-velocity approximation of the mobile Euclidean 1-centre and show tight bounds of approximately 1.1153 on its approximation factor and $4/\pi$ on its relative velocity. Durocher[22] shows that no bounded-velocity approximation is possible for the Euclidean 3-centre.

Agarwal and Har-Peled[4] maintain the approximate mobile 1-centre in $\mathbb{R}^2$ under ℓ_∞ and ℓ_2 distance metrics. Their approximations do not require continuity or bounded velocity in the motion of the mobile facility; their objective, rather, is to minimize the number of events processed and the update cost per event using a kinetic data structure (KDS) to maintain a $(1 + \epsilon)$-approximation on the extent of the client positions. As shown by Bereg et al.,[11] the maximum relative velocity of an approximation to the mobile Euclidean 1-centre is inversely proportional to its approximation factor. That is, a $(1 + \epsilon)$-approximation of the Euclidean 1-centre defined in terms of a variable ϵ cannot guarantee any fixed upper bound on velocity.

4. Reflection-Based 2-Centre Functions

4.1. *Motivation and definition*

Generalizations of strategies that provide bounded-velocity approximations to the mobile Euclidean 1-centre suggest themselves as natural candidates for defining a 2-centre function. In particular, as discussed by Bereg et al.,[11] the rectilinear 1-centre and the centre of mass both provide bounded-velocity approximations of the Euclidean 1-centre (see Sections 4.5 and 4.6 for definitions). Both have natural generalizations to two facilities, namely, the rectilinear 2-centre and the 2-means centre,

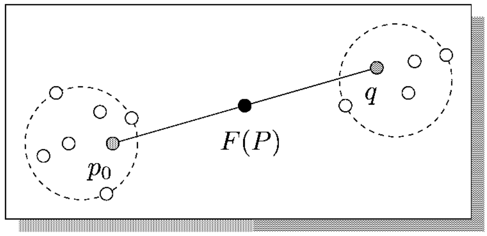

Fig. 1. Given a properly-selected reflection centre F, when the clients of P form two clusters, client p_0 and its reflection q across $F(P)$ define an approximation to the Euclidean 2-centre.

neither of which is continuous (see Section 2.3). As a result, neither the rectilinear 2-centre nor the 2-means centre define a bounded-velocity 2-centre function.

A natural strategy for finding a static approximation to the Euclidean 2-centre problem involves partitioning the clients into two sets and subsequently identifying an approximation to the Euclidean 1-centre of each partition. Such strategies generalize poorly to the mobile setting because discontinuities in the position of a mobile 2-centre function can result from changes in the partition of the client set. Thus, it is not clear that a continuous mobile 2-centre function can be defined in terms of partitions of the client set.

Nevertheless, if the clients of P form two obvious clusters, then a 2-centre function Υ should position one facility close to each cluster. In particular, when the clients of P coincide at two points a and b, the Euclidean 2-radius of P is zero, and $\Upsilon^1(P)$ and $\Upsilon^2(P)$ must coincide with a and b in order to guarantee any fixed upper bound on approximation factor. When this occurs, observe that any client p_0 in P and its reflection across the midpoint of P coincide with $\{a, b\}$.

As described by Durocher,[22] a natural definition for a bounded-velocity Euclidean 2-centre of P in one dimension is provided by partitioning P across $\Xi(P)$ and identifying the Euclidean 1-centre of each partition; the position of the first facility is specified by the Euclidean 1-centre of the partition with greater diameter while the position of the second facility can be viewed as the reflection of the first facility across the Euclidean 1-centre of P. This strategy does not generalize to higher dimensions because the Euclidean 1-centre does not induce a partition of the clients in two or more dimensions. Furthermore, the unbounded velocity of the Euclidean 1-centre in two or more dimensions precludes it from being used to define a bounded-velocity facility. Instead, we identify a mobile centre of reflection, denoted F, that remains central to P while moving under bounded velocity. A client of P, say p_0, is selected arbitrarily and the position of the first facility is set to coincide with that of p_0. The position of the second facility, q, is found by reflecting p_0 across F. See Figure 1.

Definition 4. Given a finite set of mobile clients P in $\mathbb{R}^2$, an arbitrarily-selected client p_0 in P, and a function $F : \mathrm{P}(\mathbb{R}^2) \to \mathbb{R}^2$, a **reflection-based 2-centre**

function consists of two facility functions, Υ^1 and Υ^2, whose positions are given by the position of client $p_0(t)$ and its reflection across $F(P(t))$.

We refer to F as the **reflection centre**. We select bounded-velocity approximations of the mobile Euclidean 1-centre as natural candidates for F. These include the mobile centre of mass, the mobile rectilinear 1-centre, and the mobile Steiner centre. For comparison, we also examine the case when F is the mobile Euclidean 1-centre. Sections 4.5 through 4.7 respectively begin with definitions for the centre of mass, the rectilinear 1-centre, and the Steiner centre and follow with a derivation of bounds on the maximum relative velocity and approximation factor of the corresponding reflection-based 2-centre function.

Given a fixed choice for client p_0, invariance under similarity transformations of a reflection-based 2-centre function follows if the corresponding property holds for the reflection centre. The rectilinear 1-centre is invariant under translation and uniform scaling, but not under reflection or rotation whereas the Euclidean 1-centre, the Steiner centre, and the centre of mass are invariant under all similarity transformations.[22]

4.2. *Maximum velocity*

As we now show, if an upper bound is known on the relative velocity of the reflection centre, F, then an upper bound on the relative velocity of the corresponding reflection-based 2-centre function is straightforward to establish. The worst case is achieved when the reflection centre F and the client p_0 being reflected move toward or away from each other at their respective maximum velocities.

Proposition 4. *Let a and b denote mobile clients or mobile facility functions with respective maximum velocities v_a and v_b. The maximum velocity of the reflection of a across b is $2v_b + v_a$. Furthermore, this bound is tight if the maximum velocities of a and b are simultaneously realizable in opposite directions.*

Proof. Choose any time interval T and any $v_a, v_b > 0$. Choose any functions $a : T \to \mathbb{R}^2$ and $b : T \to \mathbb{R}^2$ such that (2) holds for a and for b. The reflection of $a(t)$ across $b(t)$ corresponds to the function $c(t) = 2b(t) - a(t)$. We bound the velocity of c by

$$\begin{aligned}\forall t_1, t_2 \in T,\ ||c(t_1) - c(t_2)|| &= ||2[b(t_1) - b(t_2)] - [a(t_1) - a(t_2)]|| \\ &\leq 2||b(t_1) - b(t_2)|| + ||a(t_1) - a(t_2)|| \\ &\leq (2v_b + v_a)|t_1 - t_2|.\end{aligned}$$

Therefore, the velocity of c is at most $2v_b + v_a$. It is straightforward to see that this bound is realized when a and b move in opposite directions. □

4.3. *Lower bounds for reflection-based 2-centre functions*

We derive lower bounds on the approximation factor and maximum relative velocity of any reflection-based 2-centre function.

Proposition 5. *A bounded-approximation reflection-based 2-centre function cannot guarantee relative velocity less than three.*

Proof. A reflection-based 2-centre function Υ is defined in terms of a reflection centre F. If the velocity of F is less than the velocity of clients in P, then the clients can coincide at a point away from F and no approximation factor can be guaranteed. Therefore, the relative velocity of F must be at least one. By Proposition 4 it follows that no reflection-based 2-centre function can guarantee relative velocity less than three. □

Proposition 6. *Any mobile 2-centre function for which the position of one facility is set to coincide with the position of a mobile client cannot guarantee an approximation factor less than two.*

Proof. Choose any function $F : \mathrm{P}(\mathbb{R}^2) \to \mathbb{R}^2$. Let $P = \{(-4,0),(-2,0),(4,0)\}$. Let $p_0 = (-4,0)$ and choose any $q \in \mathbb{R}^2$. Let $\Upsilon(P) = \{p_0, q\}$. Assume there exists a $\lambda < 2$ such that Υ is a λ-approximation of the Euclidean 2-centre. Let

$$d = \max_{p \in P} \min\{||p - p_0||,\ ||p - q||\}.$$

Observe that the Euclidean 2-radius of P is one. Therefore, $\lambda \geq d$ by (4).

Since d must be less than two, point q must lie in the right half-plane; otherwise the distance between client $(4,0)$ and either facility would be at least four. Since $||(-2,0)-p_0|| = 2$ and $||(-2,0)-q|| \geq 2$, it follows that $d \geq 2$. Therefore, $\lambda \geq d \geq 2$, deriving a contradiction. □

Consequently, all reflection-based 2-centre functions have maximum relative velocity at least three and an approximation factor of at least two.

4.4. *Reflection across the Euclidean 1-centre*

Intuitively, an important criterion in the selection of a reflection centre F is the degree to which F remains "central" to a set of clients P. When the reflection centre is the Euclidean 1-centre, we refer to the corresponding reflection-based 2-centre function as the **Euclidean reflection 2-centre**. One might expect the Euclidean reflection 2-centre to have a low approximation factor. Surprisingly, in Sections 4.4 through 4.7 we show that the Euclidean 1-centre is not the optimal choice for F, both in terms of approximation factor and maximum velocity.

Theorem 1. *The Euclidean reflection 2-centre provides a 4-approximation of the Euclidean 2-centre. This bound is tight. Furthermore, the Euclidean reflection 2-centre cannot guarantee bounded relative velocity.*

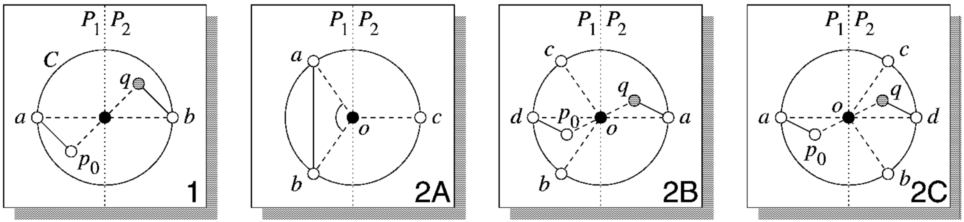

Fig. 2. Illustration in support of Theorem 1: upper bound.

Proof. The result on velocity follows from the unbounded velocity of the Euclidean 1-centre shown by Bereg et al.[13] and Proposition 4.

We first show that the Euclidean reflection 2-centre provides a 4-approximation of the Euclidean 2-centre and then demonstrate the bound is tight.

Let P denote any finite set of clients in $\mathbb{R}^2$. Let p_0 denote a client of P whose position corresponds to the first facility, $\Upsilon^1(P)$. Let q denote the reflection of p_0 across $\Xi(P)$. The position of the second facility, $\Upsilon^2(P)$ is given by q. Let C denote the minimum enclosing circle of P and let s denote the radius of C.

Let $CH(A)$ denote the convex hull of a set A in $\mathbb{R}^2$. Let $\Xi^1(P)$ and $\Xi^2(P)$ denote a Euclidean 2-centre of P. Let r denote the Euclidean 2-radius of P. Let P_1 and P_2 denote the partition of P induced by $\Xi^1(P)$ and $\Xi^2(P)$ such that $\Xi^1(P)$ is the facility closest to any client in P_1 and $\Xi^2(P)$ is the facility closest to any client in P_2. If any client p in P is equidistant from $\Xi^1(P)$ and $\Xi^2(P)$, then assume p is assigned to either partition arbitrarily. Without loss of generality assume $p_0 \in P_1$. Therefore,

$$\forall p \in P_1, \ ||p_0 - p|| \leq 2r, \tag{5}$$

since p and p_0 are both contained within the minimum enclosing circle of P_1. Therefore, we need only to verify that $||q - p|| \leq 4r$ for all clients $p \in P_2$.

Case 1. Assume C is supported by two clients $a, b \in P$ that lie opposite each other on C. Clients a and b must lie in opposite partitions, otherwise the Euclidean 2-radius equals the Euclidean 1-radius. Without loss of generality assume $a \in P_1$. See Figure 2(1). Thus, for all clients $p \in P_2$, $||p-b|| \leq 2r$ since b and p are both contained within the minimum enclosing circle of P_2. Observe that $||b - q|| = ||a - p_0||$. Therefore, by (5),

$$\forall p \in P_2, \ ||p - q|| \leq ||p - b|| + ||b - q|| \leq 2r + ||b - q|| = 2r + ||a - p_0|| \leq 4r. \tag{6}$$

Case 2. Assume no two clients in P lie opposite each other on circle C. Let o denote the centre of C. At least three clients $a, b, c \in P$ must support C such that the angles $\angle aob$, $\angle aoc$, and $\angle boc$ are all less than π. Without loss of generality, assume $\angle boc$ corresponds to the minimum of the three angles. Since the angles sum to 2π, we get $\angle boc \leq 2\pi/3$. Since all angles are less than π, we get $\angle aob \geq \pi/3$ and $\angle aoc \geq \pi/3$. Furthermore, at least one of a, b, or c must lie in each partition.

Case 2a. Assume a and b lie in the same partition. Therefore, c must lie in the opposite partition. See Figure 2(2A). Since $\pi/3 \leq \angle aob < \pi$, it follows that $||a - b|| \geq s$. Consequently, $2r \geq s$. The distance between any two points contained within C is at most $2s$. Therefore,

$$\forall p \in P_2, \ ||p - q|| \leq 2s \leq 4r. \tag{7}$$

Case 2b. Assume $c, b \in P_1$ and $a \in P_2$. See Figure 2(2B). Let d denote the point opposite a on circle C. Since $\angle aob$, $\angle aoc$, and $\angle boc$ are all less than π, one of c and b must lie above d and the other must lie below d on circle C. Consequently, d is in the minimum enclosing circle of P_1. Therefore, for all clients $p \in P_1$, $||p - d|| \leq 2r$ since p is contained within the minimum enclosing circle of P_1. Similarly, since $a \in CH(P_2)$, for all clients $p \in P_2$, $||p - a|| \leq 2r$ since p is contained within the minimum enclosing circle of P_2. Since $p_0 \in CH(P_1)$, for all points p in the minimum enclosing circle of P_1, $||p - p_0|| \leq 2r$. In particular, $||d - p_0|| \leq 2r$. Observe that $||d - p_0|| = ||a - q||$. Thus,

$$\forall p \in P_2, \ ||p - q|| \leq ||p - a|| + ||a - q|| = ||p - a|| + ||d - p_0|| \leq 4r. \tag{8}$$

Case 2c. Assume $c, b \in P_2$ and $a \in P_1$. See Figure 2(2C). Let d denote the point opposite a on circle C. Since $\angle aob$, $\angle aoc$, and $\angle boc$ are all less than π, one of c and b must lie above d and the other must lie below d on circle C. Consequently, d is in the minimum enclosing circle of P_2. Therefore, for all clients $p \in P_2$, $||p-d|| \leq 2r$ since p is contained within the minimum enclosing circle of P_2. Similarly, since $a \in CH(P_1)$ for all clients $p \in P_1$, $||p-a|| \leq 2r$ since p is contained within the minimum enclosing circle of P_1. By (5), $||a - p_0|| \leq 2r$. Observe that $||a - p_0|| = ||d - q||$. Thus,

$$\forall p \in P_2, \ ||p - q|| \leq ||p - d|| + ||d - q|| = ||p - d|| + ||a - p_0|| \leq 4r. \tag{9}$$

Case 2d. Assume $c, a \in P_1$ and $b \in P_2$. This case is analogous to Case 2b since we have not made any assumptions to differentiate a from b.

Case 2e. Assume $c, a \in P_2$ and $b \in P_1$. This case is analogous to Case 2c since we have not made any assumptions to differentiate a from b.

The result follows from (5) through (9). We show this bound is tight with the following example.

Let $\theta \in (0, \pi/4)$. Let $P = \{p_0, p_1, p_2, p_3\}$ where $p_0 = (-\cos\theta, -\sin\theta)$, $p_1 = (-1, 0)$, $p_2 = (1, 0)$, and $p_3 = (\cos\theta, -\sin\theta)$. The Euclidean 1-centre of P lies at the origin. The unique Euclidean 2-centre of P lies at $(p_0 + p_1)/2$ and $(p_2 + p_3)/2$. Let the first facility, $\Upsilon^1(P)$, coincide with p_0. Let q denote the reflection of p_0 across $\Xi(P)$. The position of the second facility, $\Upsilon^2(P)$ is given by q. See Figure 3.

The Euclidean 2-radius is $\frac{1}{2}\sqrt{(1 - \cos\theta)^2 + \sin^2\theta} = \frac{1}{2}\sqrt{2(1 - \cos\theta)}$. The furthest client from q is p_2, separated by a distance of $2\sin\theta$. It follows that

$$\lambda \geq \lim_{\theta \to 0^+} \frac{2\sin\theta}{\frac{1}{2}\sqrt{2(1 - \cos\theta)}} = 4\sqrt{\lim_{\theta \to 0^+} \frac{\sin^2\theta}{2(1 - \cos\theta)}} = 4 \text{ (by L'Hôpital's rule).} \quad \square$$

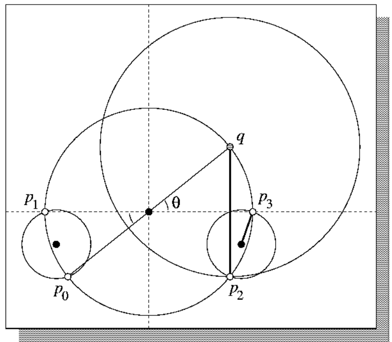

Fig. 3. Illustration in support of Theorem 1: lower bound.

Given the unbounded velocity of the Euclidean reflection 2-centre, bounded-velocity approximations of the Euclidean 1-centre provide natural candidates for defining the reflection centre F. The properties of low maximum relative velocity and low approximation factor exhibited by the rectilinear 1-centre, the centre of mass, and the Steiner centre[11,13,22,25] suggest these functions as a natural candidates for defining the reflection centre in a reflection-based 2-centre function. As we demonstrate in Sections 4.5 through 4.7, selecting either the rectilinear 1-centre or the Steiner centre for the reflection centre results in a 2-centre function with a lower approximation factor than that guaranteed when the reflection centre is the Euclidean 1-centre; selecting the centre of mass for the reflection centre results in a 2-centre function that cannot guarantee any bounded approximation factor.

4.5. *Reflection across the centre of mass*

Given a finite set P in $\mathbb{R}^2$, the **centre of mass** of P is the point

$$C(P) = \frac{1}{|P|} \sum_{p \in P} p. \tag{10}$$

The centre of mass is the unique point that minimizes the sum of the square distances from each clients to its nearest facility.[36,40]

When the reflection centre is the centre of mass, we refer to the corresponding reflection-based 2-centre function as the **mean reflection 2-centre**. We now examine its maximum relative velocity and approximation factor.

Theorem 2. *The mean reflection 2-centre has maximum relative velocity 3. This bound is tight. Furthermore, the mean reflection 2-centre cannot guarantee any bounded approximation of the Euclidean 2-centre.*

Proof. As shown by Bereg et al.,[11] the centre of mass has at most unit relative velocity. Furthermore, this velocity is realizable. Although the velocity of the centre of mass is not independent of the velocity of p_0, the contribution of p_0 to the velocity of the centre of mass approaches zero as $|P|$ increases. The result on velocity follows from Proposition 4.

Let set P be defined by two clients be located at the origin and a single client a located at $(1, 0)$. The centre of mass of P lies at $C(P) = (1/3, 0)$. The reflected facility has position either $(1/3, 0)$ or $(2/3, 0)$, depending on the position of p_0. The Euclidean 2-radius of P is zero. The distance from client a to the nearest facility is at least $1/3$. Consequently, no λ satisfies (4). □

4.6. *Reflection across the rectilinear 1-centre*

We now consider the rectilinear 1-centre as the reflection centre.

Given a finite set P in $\mathbb{R}^2$, the **rectilinear 1-centre** of P is the function whose value, $R(P)$, is the point located at the centre of the bounding box of P. That is, $R(P)$ is a point that minimizes

$$\max_{p \in P} ||R(P) - p||_\infty, \tag{11}$$

where $||x||_\infty$ denotes the ℓ_∞ norm of $x \in \mathbb{R}^2$. The rectilinear 1-centre of P is given by finding the one-dimensional 1-centre of P in each dimension. That is, $R(P)_i = R(P_i) = \Xi(P_i)$, where $P_i = \{p_i \mid p \in P\}$ and p_i denotes the i-coordinate of a point p in $\mathbb{R}^2$ for $i \in \{x, y\}$.

When the reflection centre is the rectilinear 1-centre, we refer to the corresponding reflection-based 2-centre function as the **rectilinear reflection 2-centre**. We now examine its maximum relative velocity and approximation factor.

We first bound the approximation factor of the one-dimensional rectilinear reflection 2-centre in Lemma 1. This result allows us to derive a bound in $\mathbb{R}^2$ in Theorem 3.

Lemma 1. *Given a set of collinear clients P in $\mathbb{R}^2$, the rectilinear reflection 2-centre of P provides a 2-approximation of the Euclidean 2-centre of P.*

Proof. It is straightforward to show that the definitions of the rectilinear 1-centre, the rectilinear reflection 2-centre, the Euclidean 1-centre, and the Euclidean 2-centre are consistent across dimensions. Since we only consider collinear sets, we may assume without loss of generality that P is a set of points in $\mathbb{R}$. Let p_0 denote any client of P. Let the position of the first facility, $\Upsilon^1(P)$, coincide with p_0. Let q denote the reflection of p_0 across $R(P)$. Let the position of the second facility, $\Upsilon^2(P)$ be given by q.

Let P_1 and P_2 denote the partition of P induced by clients positioned respectively to the left and right of $\Xi(P)$. If any client p in P coincides with $\Xi(P)$, then assume p is assigned to partition P_1. There exists a Euclidean 2-centre of P, $\Xi^1(P)$ and $\Xi^2(P)$, such that $\Xi^1(P)$ is the facility closest to any client in P_1 and $\Xi^2(P)$ is the facility closest to any client in P_2. Let d denote the maximum of the diameters of P_1 and P_2. It follows that $d = 2r$, where r denotes the Euclidean 2-radius of P. Without loss of generality, assume $p_0 \in P_1$. Therefore,

$$\max_{p \in P_1} ||p - p_0|| \leq d \quad \text{and} \quad \max_{p \in P_2} ||p - q|| \leq d,$$
$$\Rightarrow \max_{p \in P} \min_{i \in \{1,2\}} ||p - \Upsilon^i(P)|| \leq 2r. \qquad \square$$

Theorem 3. *The rectilinear reflection 2-centre provides a* $2\sqrt{2}$*-approximation of the Euclidean 2-centre and has maximum relative velocity* $2\sqrt{2}+1$*. Both bounds are tight.*

Proof. Bereg et al.[13] show a tight bound of $\sqrt{2}$ on the velocity of the rectilinear 1-centre. The result on velocity follows Proposition 4 because the velocity of the rectilinear 1-centre is independent of the velocity of p_0 whenever p_0 not an extreme point of P.

We first show that the rectilinear reflection 2-centre provides a $2\sqrt{2}$-approximation of the Euclidean 2-centre and then demonstrate the bound is tight.

Let P denote any finite set of clients in $\mathbb{R}^2$. Let p_0 denote a client of P whose position corresponds to first facility, $\Upsilon^1(P)$. Let q denote the reflection of p_0 across $R(P)$. The position of the second facility, $\Upsilon^2(P)$ is given by q.

Since $q = 2R(P) - p_0$, therefore $q_i = 2R(P_i) - [p_0]_i$. Consequently,

$$\begin{aligned}
\max_{p \in P} \min_{j \in \{1,2\}} ||p - \Upsilon^j(P)|| &= \max_{p \in P} \min_{j \in \{1,2\}} \sqrt{\sum_{i=1}^{2} |p_i - \Upsilon^j(P)_i|^2} \\
&\leq \sqrt{\sum_{i=1}^{2} \left[\max_{p \in P} \min_{j \in \{1,2\}} |p_i - \Upsilon^j(P)_i|\right]^2} \\
&\leq \sqrt{\sum_{i=1}^{2} (2r_i)^2},
\end{aligned}$$

by Lemma 1, where r_i denotes the Euclidean 2-radius of P_i,

$$\begin{aligned}
&\leq \max_{i \in \{1,2\}} \sqrt{2(2r_i)^2} \\
&= \max_{i \in \{1,2\}} 2r_i\sqrt{2} \\
&\leq 2r\sqrt{2}, \qquad (12)
\end{aligned}$$

where r denotes the Euclidean 2-radius of P.

We show this bound is tight with the following example. Let $P = \{(-2,0), (-2,-2), (0,2), (2,2)\}$. Let $p_0 = (-2,0)$. The unique Euclidean 2-centre of P has positions $(-2,-1)$ and $(1,2)$. The Euclidean 2-radius of P is 1. The rectilinear 1-centre of P, $R(P)$, is located at the origin. The reflection of p_0 across $R(P)$,

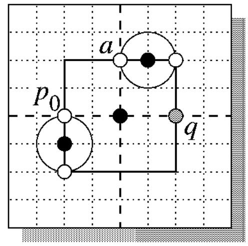

Fig. 4. Illustration in support of Theorem 3.

denoted q, is located at $(2,0)$. Client $a = (0,2)$ lies a distance $2\sqrt{2}$ from both q and p_0. See Figure 4. □

4.7. *Reflection across the Steiner centre*

Durocher and Kirkpatrick[25] examine the Steiner centre of P, denoted $\Gamma(P)$, as a bounded-velocity approximation of the mobile Euclidean 1-centre.

Given a finite set P in $\mathbb{R}^2$, the **Steiner centre** of P is the point

$$\Gamma(P) = \frac{2}{\pi} \int_0^{\pi} \Xi(P_\theta)\, d\theta, \tag{13}$$

where $P_\theta = \{(\cos\theta, \sin\theta)\langle p, (\cos\theta, \sin\theta)\rangle \mid p \in P\}$ is the projection of P onto the line passing through the origin, parallel to the vector $(\cos\theta, \sin\theta)$. See Refs. 22 and 25 for motivation, discussion, and alternate representations of the Steiner centre within the context of continuous motion.

When the reflection centre is the Steiner centre, we refer to the corresponding reflection-based 2-centre function as the **Steiner reflection 2-centre**. We now examine its maximum relative velocity and approximation factor.

Theorem 4. *The Steiner reflection 2-centre has maximum relative velocity* $8/\pi + 1$. *This bound is tight. The Steiner reflection 2-centre provides a* $8/\pi$*-approximation but cannot guarantee a* λ*-approximation of the Euclidean 2-centre for any* λ *less than* $2\sqrt{1 + 1/\pi^2}$.

Proof. As shown by Durocher and Kirkpatrick,[25] the Steiner centre has maximum relative velocity $4/\pi$ and this velocity is realizable. Therefore, the result on velocity follows from Proposition 4 because the velocity of the Steiner centre is independent of the velocity of p_0 whenever p_0 is not an extreme point of P.

We first show that the Steiner reflection 2-centre provides a $8/\pi$-approximation of the Euclidean 2-centre and then demonstrate a lower bound of $2\sqrt{1 + 1/\pi^2}$.

Let d_ϕ denote the ℓ_∞ norm relative to a rotation by ϕ of the reference axis. That is, $d_\phi(x) = ||f_\phi(x)||_\infty$, where f_ϕ is a clockwise rotation about the origin by ϕ. Let $R_\phi(P) = f_{-\phi}(R(f_\phi(P)))$ denote the rectilinear 1-centre with respect to norm d_ϕ.

As shown by Durocher and Kirkpatrick,[25] the Steiner centre of a set of clients P in $\mathbb{R}^2$ can be defined as the limit of the convex combinations of the rotated rectilinear 1-centres of P. That is,

$$\Gamma(P) = \frac{2}{\pi} \int_0^{\pi/2} R_\theta(P)\ d\theta. \tag{14}$$

Let P denote any finite set of clients in $\mathbb{R}^2$. Let p_0 denote a client of P whose position corresponds to the first facility, $\Upsilon^1(P)$. Let q denote the reflection of p_0 across $\Gamma(P)$. The position of the second facility, $\Upsilon^2(P)$ is given by q.

Let $\Xi^1(P)$ and $\Xi^2(P)$ denote a Euclidean 2-centre of P. Let r denote the Euclidean 2-radius of P. Let P_1 and P_2 denote the partition of P induced by $\Xi^1(P)$ and $\Xi^2(P)$ such that $\Xi^1(P)$ is the facility closest to any client in P_1 and $\Xi^2(P)$ is the facility closest to any client in P_2. If any client p in P is equidistant from $\Xi^1(P)$ and $\Xi^2(P)$, then assume p is assigned to either partition arbitrarily. Without loss of generality assume $p_0 \in P_1$. Therefore,

$$\forall p \in P_1,\ ||p_0 - p|| \leq 2r, \tag{15}$$

since p and p_0 are both contained within the minimum enclosing circle of P_1. Therefore, we need only to verify that $||q - p|| \leq (8/\pi)r$ for all clients $p \in P_2$.

As shown in the proof of Theorem 3 with respect to the rectilinear reflection 2-centre, if $q_R = 2R(P) - p_0$, then

$$\max_{p \in P_2} |p_x - [q_R]_x| \leq 2r \quad \text{and} \quad \max_{p \in P_2} |p_y - [q_R]_y| \leq 2r.$$

That is, every client in P_2 is contained within a box of width and height $4r$ centred at q_R. Let $q_\theta = 2R_\theta(P) - p_0$. It follows that every client in P_2 is contained within a box of width and height $4r$, whose axis is rotated by θ relative to the x-axis, and whose centre is q_θ. Consequently,

$$\max_{p \in P_2} |p_x - [q_\theta]_x| \leq 2\sqrt{2}\cos(\pi/4 - \theta)r. \tag{16}$$

We now bound the maximum distance in the x-coordinates from any client in P_2 to q.

$$\begin{aligned}
\max_{p \in P_2} |p_x - q_x| &= \max_{p \in P_2} |p_x - (2\Gamma(P)_x - [p_0]_x)| \\
&= \max_{p \in P_2} \left| p_x - \left(2 \left[\frac{2}{\pi} \int_0^{\pi/2} R_\theta(P)_x\ d\theta \right] - [p_0]_x \right) \right|, \qquad \text{by (14)}, \\
&= \max_{p \in P_2} \left| \frac{2}{\pi} \int_0^{\pi/2} p_x - (2R_\theta(P)_x - [p_0]_x)\ d\theta \right| \\
&= \max_{p \in P_2} \left| \frac{2}{\pi} \int_0^{\pi/2} p_x - [q_\theta]_x\ d\theta \right|
\end{aligned}$$

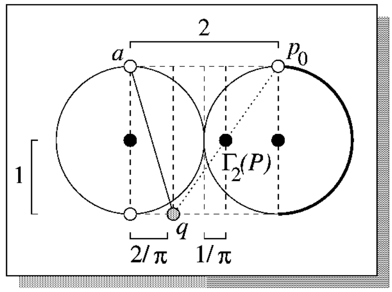

Fig. 5. Illustration in support of Theorem 4.

$$\begin{aligned}
&\leq \max_{p\in P_2} \frac{2}{\pi}\int_0^{\pi/2} |p_x - [q_\theta]_x|\; d\theta \\
&\leq \frac{2}{\pi}\int_0^{\pi/2} \max_{p\in P_2} |p_x - [q_\theta]_x|\; d\theta \\
&\leq \frac{2}{\pi}\int_0^{\pi/2} 2\sqrt{2}\cos(\pi/4-\theta) r\; d\theta, \qquad \text{by (16)}, \\
&= \frac{8r}{\pi}.
\end{aligned}$$

The Steiner reflection 2-centre is invariant under rotation. Consequently,

$$\max_{p\in P_2} |p_x - q_x| \leq \frac{8r}{\pi} \Rightarrow \max_{p\in P_2} ||p-q|| \leq \frac{8r}{\pi}.$$

We now show that the Steiner reflection 2-centre cannot guarantee a λ-approximation of the Euclidean 2-centre for any λ less than $2\sqrt{1+1/\pi^2}$.

Let a continuous arc of clients lie on a unit semicircle centred at the origin on the positive x-axis. Let two clients lie opposite the arc at $a = (-1, 1)$ and $b = (1, 1)$. The unique Euclidean 2-centre of P lies at $(0, 0)$ and $(-1, 0)$. The corresponding Euclidean 2-radius, r, is one. The Steiner centre of P lies at $(1/\pi - 1/2, 0)$. Let $p_0 = (0, 1)$ define the position of the first facility, $\Upsilon^1(P)$. Let q denote the reflection of p_0 across $\Gamma(P)$. Observe that $q = (2/\pi - 1, -1)$. The position of the second facility, $\Upsilon^2(P)$ is given by q. See Figure 5.

The reflection of q across $\Gamma(P)$ lies at a distance $2\sqrt{1+1/\pi^2}$ from client a. $\square$

4.8. *Kinetic algorithms for reflection-based 2-centres*

The kinetic maintenance of a reflection-based 2-centre reduces to the problem of maintaining the corresponding mobile reflection centre. Kinetic data structures (KDS), introduced by Basch et al.,[10] allow for efficient maintenance of various attributes of a set of mobile objects under linear (or bounded-degree algebraic) motion. Those related to our discussion of mobile 2-centre functions include the maximum

Table 1. Comparing reflection-based 2-centre functions in $\mathbb{R}^2$.

reflection centre	**approximation factor**	**maximum relative velocity**
Euclidean 1-centre	$\lambda = 4$	$v_{\max} = \infty$
centre of mass	$\lambda = \infty$	$v_{\max} = 3$
rectilinear 1-centre	$\lambda = 2\sqrt{2} \approx 2.8284$	$v_{\max} = 2\sqrt{2} + 1 \approx 3.8284$
Steiner centre	$2\sqrt{1 + 1/\pi^2} \leq \lambda \leq 8/\pi$ $\Rightarrow 2.0989 \leq \lambda \leq 2.5465$	$v_{\max} = 8/\pi + 1 \approx 3.5465$
bounded approx. factor		$3 \leq v_{\max}$
any reflection centre	$2 \leq \lambda$	

(in $\mathbb{R}$),[10,30] the rectilinear 1-centre,[4] the Steiner centre,[25] the convex hull,[10,30] a $(1+\epsilon)$-approximate Euclidean 1-centre,[4] and the extent of a set of mobile clients.[3,4]

5. Conclusion

5.1. *Summary of results*

Since continuity and a fixed upper bound on velocity are natural constraints on mobile problems, the discontinuity of mobile Euclidean 2-centres leads us to consider bounded-velocity approximations. The goal of this paper is to identify 2-centre functions that guarantee a fixed-degree approximation of a mobile Euclidean 2-centre while maintaining motion whose magnitude of velocity does not exceed a fixed upper bound.

We established general lower bounds of $\sqrt{2}$ on the approximation factor and $1 + \sqrt{3}/2$ on the maximum relative velocity of any bounded-velocity 2-centre function. We introduced reflection-based 2-centre functions for which we established stronger lower bounds of 2 on the approximation factor and 3 on the maximum relative velocity. When the reflection centre is the Euclidean 1-centre we showed unbounded velocity and a tight bound of 4 on the approximation factor. When the reflection centre is the rectilinear 1-centre we showed a tight bound of $2\sqrt{2}+1$ on the relative velocity and a tight bound of $2\sqrt{2}$ on the approximation factor. When the reflection centre is the Steiner centre we showed a tight bound of $8/\pi + 1$ on the relative velocity, an upper bound of $8/\pi$ on the approximation factor, and a lower bound of $2\sqrt{1 + 1/\pi^2}$ on the approximation factor. Finally, when the reflection centre is the centre of mass we showed a tight bound of 3 on the relative velocity and that no fixed approximation factor can be guaranteed. These properties are summarized in Table 1.

In conclusion, we have identified two strategies for defining a bounded-velocity approximation of the mobile Euclidean 2-centre: the Steiner reflection 2-centre and the rectilinear reflection 2-centre.

5.2. *Directions for future research*

The questions explored in this paper have natural extensions into higher dimensions. Preliminary results in three dimensions are discussed in Ref. 22. Another possible continuation of this work is to attempt to define a 2-centre function without using reflection. Finally, some of the bounds established here can be improved or shown to be tight, such as the general lower bounds on approximation factors and velocity and the bounds on the approximation factor of the Steiner reflection 2-centre. As mentioned earlier, even in one dimension no bounded-velocity approximation is possible for the mobile Euclidean 3-centre.[22]

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Appendix A. Proofs for Propositions 1 through 3

Proposition 1. *There exists a set of mobile clients P such that any mobile Euclidean 2-centre of P is discontinuous.*

Proof. Let $P = \{a, b, c, d\}$ denote a set of four mobile clients such that

$$a(t) = \begin{cases} (2-t, 1) & t \leq 1 \\ (1, t) & t > 1 \end{cases}, \qquad b(t) = \begin{cases} (2-t, -1) & t \leq 1 \\ (1, -t) & t > 1 \end{cases},$$

$$c(t) = \begin{cases} (t-2, -1) & t \leq 1 \\ (-1, -t) & t > 1 \end{cases}, \qquad \text{and } d(t) = \begin{cases} (t-2, 1) & t \leq 1 \\ (-1, t) & t > 1 \end{cases}.$$

Observe that each client moves with unit velocity. When $t < 1$, the unique Euclidean 2-centre of $P(t)$ is $\{(2-t, 0), (t-2, 0)\}$. See Figure 6A. Similarly, when $t > 1$, the unique Euclidean 2-centre of $P(t)$ is $\{(0, t), (0, -t)\}$. See Figure 6B. The

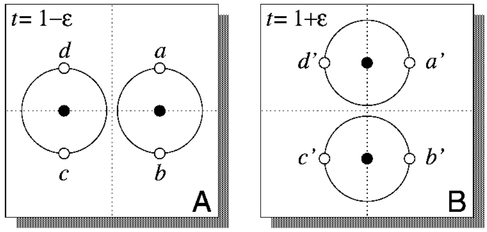

Fig. 6. Illustration in support of Proposition 1.

corresponding Euclidean 2-radius is one in both instances. It follows that

$$\forall t_1 < 1,\ \forall t_2 > 1,\ \min_{\substack{i\in\{1,2\}\\ j\in\{1,2\}}} ||\Xi^i(P(t_1)) - \Xi^j(P(t_2))|| \geq \sqrt{2}.$$

Therefore, any Euclidean 2-centre of P is discontinuous at $t = 1$. □

Proposition 2. *A mobile 2-centre function with maximum relative velocity less than* $1 + \sqrt{3}/2$ *cannot guarantee any bounded approximation of the Euclidean 2-centre.*

Proof. Let $P = \{a, b, c\}$ denote a set of three mobile clients with initial positions (at time $t = 0$) at the vertices of an equilateral triangle in $\mathbb{R}^2$ such that any two clients in P lie a distance two from each other. Let R_a, R_b, and R_c denote the Voronoi regions induced by $a(0)$, $b(0)$, and $c(0)$, respectively. See Figure 7A. Choose any $\Upsilon^1(P(t))$ and $\Upsilon^2(P(t))$ in $\mathbb{R}^2$ for the positions of the 2-centre function. The interior of at least one of R_a, R_b, or R_3 must be empty of $\Upsilon^1(P(0))$ and $\Upsilon^2(P(0))$. Without loss of generality assume R_a is empty. Let b and c move toward each other at unit velocity until they meet at their midpoint after one time unit. Let a move away from their midpoint with unit velocity. See Figure 7B. Thus, the Euclidean 2-radius of $P(1)$ is zero. If Υ has any fixed approximation factor, then $\Upsilon^1(P(1))$ and $\Upsilon^2(P(1))$ must coincide with $a(1)$ and $b(1) = c(1)$. Two points lie nearest to $a(1)$ along the boundary of R_a, which we denote d and e. Since these two cases are symmetric, we examine the left point, d. Let $f = [a(0)+b(0)]/2$. Either $\Upsilon^1(P(1))$ or $\Upsilon^2(P(1))$ must travel from the boundary of R_a to $a(1)$ during the time interval $T = [0, 1]$. This distance is at least as great as the length of the longer edge of the right trapezoid induced by f, $a(0)$, $a(1)$, and d. Angle $\angle da(1)b(1) = \angle fa(0)b(1) = \pi/6$. Since $||f - a(0)|| = ||a(1) - a(0)|| = 1$, it follows that $||d - a(1)|| = 1 + \sqrt{3}/2 \approx 1.8660$. □

Proposition 3. *A continuous mobile 2-centre function in* $\mathbb{R}^2$ *cannot guarantee a* λ*-approximation of the Euclidean 2-centre for any* $\lambda < \sqrt{2}$.

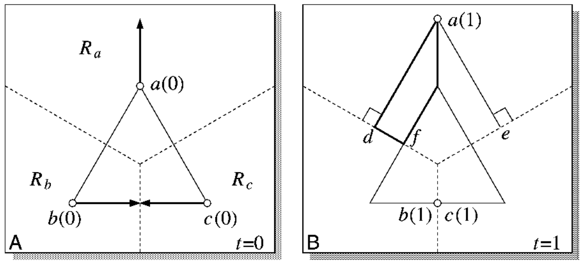

Fig. 7. Illustration in support of Proposition 2.

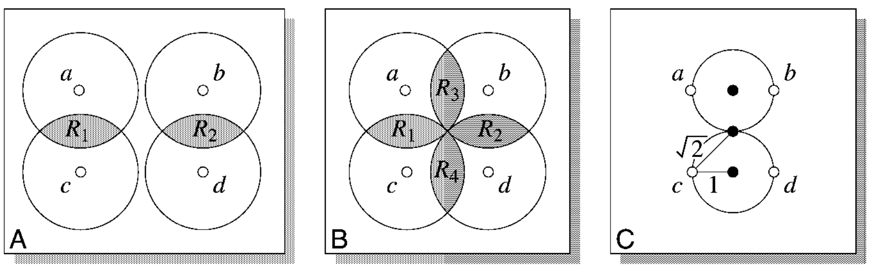

Fig. 8. Illustration in support of Proposition 3.

Proof. The result follows from the example described in the proof of Proposition 1. If Υ guarantees an approximation factor $\lambda = \sqrt{2}$, then for any t there exists a partition of $P(t)$ into two sets P_1 and P_2 such that $\Upsilon^1(P(t))$ is contained within the intersection of circles of radius $\sqrt{2}$ centred at each of the clients in P_1 and the same holds for $\Upsilon^2(P(t))$ and P_2. These circles have a fixed radius of $\sqrt{2}$ because the Euclidean 2-radius remains one throughout the motion of the clients. When $t < 1 - \sqrt{2}$, a unique partition of $P(t)$ exists such that this intersection is nonempty. We denote the corresponding regions R_1 and R_2. See Figure 8A. The same holds for $P(t)$ when $t > 1 + \sqrt{2}$, for which we denote the corresponding regions R_3 and R_4. At some point t_0, $\Upsilon^1(P(t_0))$ and $\Upsilon^2((t_0))$ must make a transition from regions R_1 and R_2 over to R_3 and R_4. Since the motion of Υ must be continuous, the transition must occur when the regions overlap. The regions have a unique point of intersection occurring at $t_0 = 1$ at the origin. See Figure 8B. Therefore, $\Upsilon^1(P(1)) = \Upsilon^2((1)) = (0, 0)$. Thus, the lower bound on the approximation factor is realized at time $t = 1$. If the radius of the circles is decreased to less than $\sqrt{2}$, then no such intersection exists. □