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Competitive Online Routing on Delaunay Triangulations

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Abstract Let G be a graph, $s \in G$ be a source node and $t \in G$ be a target node. The sequence of adjacent nodes (graph walk) visited by a routing algorithm is a c-*competitive* route if its length in G is at most c times the length of the shortest path from s to t in G. We present 21.766-, 17.982- and 15.479-competitive online routing algorithms on the Delaunay triangulation of an arbitrary given set of points in the plane. This improves the competitive ratio on Delaunay triangulations from the previous best of 45.749. We present 20.926- and 7.621-competitive online routing algorithms for Delaunay triangulations of point sets in convex position.

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1 Introduction

We study the fundamental problem of finding a route in a geometric graph from a given source vertex s to a given target vertex t. In our context, a geometric graph G is a weighted graph whose vertex set is a set P of n points in the plane, and whose edges are line segments joining pairs of points in P, where each edge is weighted by its length (the Euclidean distance between its endpoints). When full knowledge of the graph is provided, numerous algorithms exist for finding shortest paths in a weighted graph (e.g., Dijkstra's algorithm [10, 12]). The problem is more challenging in the *online* setting, where a route is constructed incrementally and a partial route from s to an intermediate node u is extended by selecting one of u's neighbours as a function of limited information available locally at u. Without knowledge of the full graph, an online routing algorithm cannot identify a shortest path in general; the goal is to follow a path whose length is as short as possible. A path between two vertices s and t in G is a *c-spanning path* if its length is at most c times the length of the shortest path from s to t in G. An online routing algorithm is *c-competitive* on a class $\mathcal{G}$ of geometric graphs if for any graph $G \in \mathcal{G}$ and any pair of vertices $\{s, t\}$ in G, the algorithm constructs a c-spanning path from s to t in G. When c is a constant, we say the online routing algorithm is *competitive*. In this paper we examine the problem of designing an online routing algorithm that is c-competitive on the Delaunay triangulation for the smallest value c possible.

The Delaunay triangulation, denoted $DT(P)$, of a point set P in the plane is a triangulation of P with the property that the triangle $\triangle abc$ is a face in $DT(P)$ if and only if $\{a, b, c\} \subseteq P$ and $\bigcirc abc \cap P = \{a, b, c\}$, where $\bigcirc abc$ denotes the unique disk that has a, b, and c on its boundary[1]. The Delaunay triangulation and its dual, the Voronoi diagram, are well studied; see [1, 22] for comprehensive surveys of these structures. To simplify the presentation we assume that points in P are in general position.

An online routing algorithm sends a message m together with a header h from a source vertex s to a target vertex t in a graph G. Both the header and the message can be considered to be bit strings. Initially the algorithm only has knowledge of s, t and $N(s)$, where for each vertex v, $N(v)$ denotes the set of vertices directly adjacent to v in G (and their respective coordinates). Upon reception of a message m and its header h, a node u must select one of its neighbours to which to forward the message as a function of h and $N(u)$. This procedure repeats until the message reaches the target node t. Different routing algorithms are possible

[1] This property holds for the closed disk $\bigcirc abc$ when P is any set of points in general position; it holds regardless of general position on any point set P for the open disk $\bigcirc abc$.

depending on the size of h and the fraction of G that is known to each node. In the setting considered in this paper, the header h stores the coordinates of the node s from which the message originated, the coordinates of the node t which is the final destination of the message, the coordinates of the neighbour of u that last forwarded the message, and possibly one additional value that is computed from distances between vertices visited by the message and may be modified by the algorithm during computation.

Online routing is also known as *local geometric routing* on geometric graphs, or simply as *local routing* when geometric information is not provided (or does not exist). Previous work in online routing includes results on triangulations [6, 9, 19, 23], on more general planar or near-planar geometric graphs [7, 9, 14–17, 19, 21], and on arbitrary (non-geometric) graphs [3, 8]. When h stores only the coordinates of the destination node t, we say an online routing algorithm is *oblivious*. That is, the forwarding decision at each node u is made as a function of only u, $N(u)$, and t. No competitive oblivious online routing algorithm exists [20], even on Delaunay triangulations [2]. In this paper we focus on *competitive* online routing algorithms. Allowing the header h to store slightly more information (some of which can be modified dynamically during routing) enables an online routing algorithm to guarantee not only that each route reaches its destination, but that it does so along a c-competitive path.

The *spanning ratio* of a graph G is the maximum ratio κ between the length of a shortest path σ on G joining any pair of nodes s and t and the Euclidean distance between s and t. That is, for any two vertices s and t in G there exists a path σ from s to t in G such that $|\sigma| \leq \kappa|st|$, where $|\sigma|$ denotes the sum of the lengths of the edges in σ and $|st|$ denotes the Euclidean distance from s to t in G. Several previous results examine upper bounds on the spanning ratio κ of the Delaunay triangulation [4, 11, 13, 18, 24]. Dobkin et al. [13] proved that $\kappa \leq (1+\sqrt{5})\pi/2$ in $DT(P)$. Using this bound, Bose and Morin [6] found a $(9(1+\sqrt{5})\pi/2)$-competitive online routing algorithm for Delaunay triangulations (where $9(1+\sqrt{5})\pi/2 \approx 45.749$). To the authors' knowledge, this was the smallest known competitive ratio for an online routing algorithm on Delaunay triangulations prior to our results.

We show that for each known upper bound κ on the spanning ratio of the Delaunay triangulation for every set of points P and every $\{s, t\} \subseteq P$, there exists a path σ from s to t that is contained on the edges of the sequence of Delaunay triangles that intersects the line segment from s to t such that $|\sigma| \leq \kappa|st|$. We show that this property of the location of the path is true both for points in general position and for points in convex position. This allows us to apply a hybrid of searching techniques developed in

[5] with new techniques to define a corresponding online routing algorithm whose competitive ratio is at most 9κ for each previous upper bound on κ. The current best upper bound is $\kappa \leq 1.998$, resulting in a corresponding competitive ratio of $9 \cdot 1.998 \approx 17.982$. Although this technique yields two new online routing algorithms for Delaunay triangulations, both of which improve on the previous best competitive ratio, we apply a new strategy to define a third online routing algorithm that reduces the competitive ratio further still to $\pi(5\pi + 4)/4 \approx 15.479$. Therefore, we improve the previous best competitive ratio for online routing on Delaunay triangulations by describing $(4\pi\sqrt{3})$-competitive, 17.982-competitive, and $(\pi(5\pi+4)/4)$-competitive online routing algorithms in Sections 2.1, 2.2, and 2.3, respectively, where $4\pi\sqrt{3} \approx 21.766$ and $\pi(5\pi + 4)/4 \approx 15.479$. In Section 3.2 we examine Delaunay triangulations of sets of points in convex position for which we present 20.926-competitive and $(11 + 3\sqrt{2})/2$-competitive online routing algorithms using new techniques, where $(11 + 3\sqrt{2})/2 \approx 7.621$.

2 Routing on Delaunay Triangulations of Points in General Position

The problem of designing a competitive online routing algorithm on $DT(P)$ is challenging, in large part, because it seems difficult to compute a shortest path between two points in $DT(P)$ when complete knowledge of the graph is unavailable. This difficulty is related to the fact that a small perturbation in P can cause the shortest path from s to t to change drastically. By focusing on specific local triangles in $DT(P)$ to the reduce the search space of candidate vertices to which to forward the message, and by exploiting geometric properties of the Delaunay triangulation, we can design online routing algorithms with good competitive ratios.

The search space is restricted by focusing on two specific paths that lie respectively above and below the line segment from s to t, where s and t denote the respective source and target nodes in $DT(P)$. Consider the ordered sequence of triangles that intersect the line segment st. Each triangle in this sequence has at least one edge whose interior is either completely above or completely below the line segment st. Define two ordered subsequences of triangles with one subsequence containing the triangles with an edge that lies above st, and the other containing the triangles with an edge that lies below st. The subsequence of edges lying above st determines a path from s to t in $DT(P)$. As is done by Bose and Morin [5], we refer to this path as the *upper chain* from s to t and denote it by U. Similarly, the subsequence of edges lying below st forms the *lower chain* from s to t and is denoted by L. Refer to Figure 1(a).

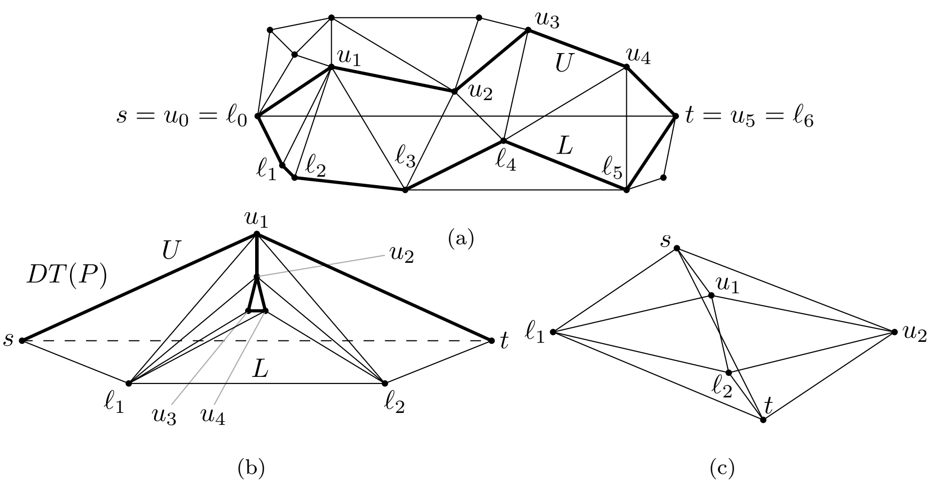


Fig. 1 (a) A Delaunay triangulation with the upper and lower chains (in bold) with respect to s and t. (b) The upper chain U (in bold) follows the sequence s, u_1, u_2, u_3, u_4, u_2, u_1, t. (c) The vertices ℓ_1 and u_2 can be moved arbitrarily far from st, implying that neither U nor L is a constant spanning path.

The upper chain is not necessarily a simple path since it may contain repeated edges or vertices (refer to Figure 1(b)). Moreover, neither the upper chain nor the lower chain is necessarily a constant spanning path (refer to Figure 1(c)). However, the subgraph of $DT(P)$ induced by $U \cup L$ contains a path whose length is at most $(1+\sqrt{5})|st|\pi/2$, which is the property used to provide the only competitive online routing algorithm [6] with competitive ratio at most $9(1+\sqrt{5})|st|\pi/2$.

Bose and Morin [5] generalized this approach slightly to triangulated weakly simple polygons. A polygon is *weakly simple* provided that the graph defined by its vertices and edges is plane, the outer face is a cycle, and one bounded face is adjacent to all vertices and edges. The weakly simple polygon is triangulated when the bounded face is triangulated.

Theorem 1 (Bose and Morin [5]) *Given a plane geometric graph G that is a triangulated weakly simple polygon, and two vertices s, t in G, there exists an online competitive routing strategy that computes a path from s to t in G whose competitive ratio is at most 9.*

Notice that the subgraph of $DT(P)$ induced by $U \cup L$ is a triangulated weakly simple polygon since it is the ordered sequence of triangles intersecting st in $DT(P)$. Therefore, showing the existence of a short path in this subgraph immediately gives a competitive online routing algorithm whose ratio is at most 9 times the length of this short path. This approach was used in [6], where the proof of the constant spanning ratio of the Delaunay triangulation by Dobkin et al. [13] was shown to construct a path of length at most $(1+\sqrt{5})|st|\pi/2 \approx 5.083$ in the subgraph induced by $U \cup L$. On the other hand, Xia [24] proves that there exists a path in the subgraph induced by $U \cup L$ whose length is at most $1.998|st|$, which implies an online routing algorithm whose ratio is at most $9 \cdot 1.998 = 17.982$.

In Section 2.1, we will use the proof by Keil and Gutwin [18] (showing an upper bound on the spanning ratio of the Delaunay triangulation) to give a new online routing algorithm with competitive ratio at most $4\pi\sqrt{3} \approx 21.766$. Note that Keil and Gutwin's [18] inductive proof does not necessarily construct a path in the subgraph induced by $U \cup L$; however, we show that whenever their proof satisfies the inductive hypothesis by including a vertex in a shortest path that lies outside the induced subgraph, there always exists an alternate vertex in the induced subgraph that also satisfies the requirements of the inductive hypothesis.

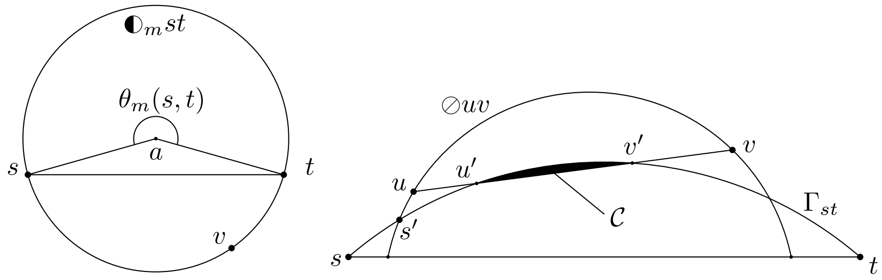
In Section 2.3 we introduce a different strategy to define an online routing algorithm with competitive ratio at most $\pi(5\pi + 4)/4 \approx 15.479$, drawing inspiration from Dobkin et al. [13] and Bose and Morin [6].

2.1 $(4\pi\sqrt{3}) \approx 21.766$-Competitive Online Routing

Keil and Gutwin [18] proved that for any two vertices s and t in $DT(P)$, there exists a path σ from s to t in $DT(P)$ such that $|\sigma| \leq \frac{4\pi\sqrt{3}}{9}|st| \leq 2.419|st|$. Although the path in the original proof may fall outside $U \cup L$, we show that the proof also implies the existence of a path of the same length among the vertices in $U \cup L$. We follow the construction given by Bose and Keil [4] (who proved the same result, but for the more general constrained Delaunay triangulations).

Our proof that there exists a path from s to t on $U \cup L$ having length at most $2.419|st|$ has two main parts. The first highlights a geometric property of Delaunay triangulations. The second part uses this geometric property to prove the result by induction. We begin with the former.

Consider the directed line segment st from s to t. Let $◐st$ be a circle through s and t such that the part of $◐st$ below st does not contain any points of P. We say that $◐st$ is a *right-empty circle* with respect to s and t. Let r denote the radius of $◐st$ and let $\theta(s, t)$ denote its *spanning angle*, corresponding to the reflex angle $\angle sat$, where a denotes the centre of $◐st$ (refer to Figure 2(a)). Let $◐_m st$ denote the right-empty circle with respect to s and t that has the minimum spanning angle and let $\theta_m(s, t)$ denote its spanning angle. Bose and Keil [4, Lemma 2.1] proved the following lemma by induction on the rank of the minimum-spanning angles (with ties being broken arbitrarily).



(a) (b)

Fig. 2 (a) The smallest spanning angle with respect to s and t. (b) Illustration of the proof of Lemma 4.

Lemma 1 (Bose and Keil [4]) *For any set of points P in the plane and any $\{s,t\} \subseteq P$, if there is a right-empty circle* ◐st *with radius r and spanning angle $\theta(s,t)$, then there exists a path τ in $DT(P)$ from s to t whose length is at most $r \cdot \theta(s,t)$ such that every edge in τ has length at most $|st|$.*

The path τ of Lemma 1 satisfies the following property.

Lemma 2 *All the vertices of the path τ are in $U \cup L$.*

To prove Lemma 2, we need the following technical lemmas.

Lemma 3 (Bose and Keil [4]) *Suppose that st is not an edge of $DT(P)$. Let $v \in P$ be a vertex in* ◐$_m st$ *to the left of st with the property that the circle through s, v and t is* ◐$_m ts$. *If v is in the circle with respect to st as diameter, then $\theta_m(s,v) \le \theta(s,t)$, $\theta_m(v,t) \le \theta(s,t)$ and $v \in \tau$. Otherwise, we have $\theta_m(t,s) \le \theta(s,t)$.*

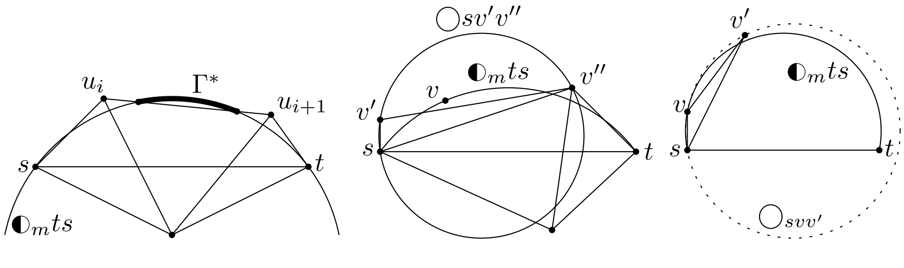
Proof This lemma is a direct consequence of Cases 1, 2, and 3, respectively, in the proof of Lemma 2.1 in [4]. □

Lemma 4 *Let st and uv be two line segments such that st and uv do not intersect and the projection of uv onto st is contained in st (refer to Figure 2(b)). Let $\oslash uv$ be a circle such that u and v are on the boundary of $\oslash uv$. Let Γ_{st} be a circular arc such that*

- *s and t are the endpoints of Γ_{st},*
- *Γ_{st} and uv intersect at u' and v',*
- *Γ_{st} and $\oslash uv$ intersect at a point $s' \notin \{s,u'\}$ that is between s and u' along Γ_{st}.*

Let $\mathcal{C}$ be the circular cap defined by the intersection of Γ_{st} with uv. Then $\oslash uv$ contains $\mathcal{C}$.

Proof Suppose that $\oslash uv$ does not contain $\mathcal{C}$. Therefore, $\mathcal{C}$ and $\oslash uv$ intersect at 2 points. But then, Γ_{st} and $\oslash uv$ intersect at 3 points. This is impossible since in general, two different circles can intersect in at most 2 points. □



(a) (b) (c)

Fig. 3 Illustration of the proof of Lemma 2. (a) If v is not in $U \cup L$, then $\LEFTcircle_m ts$ intersects an edge of $U \cup L$. (b) If v is not in $U \cup L$, then v is adjacent to s. (c) If v is not in $U \cup L$, then t is inside the circle defining $\triangle svw \in DT(P)$, which is a contradiction.

We can now show the proof of Lemma 2.

Proof (Lemma 2) If st is an edge of the convex hull of P, then it is also an edge of $DT(P)$, thus the lemma holds. For the remainder of the proof, we assume that st is not an edge of the convex hull of P. We proceed by induction on the rank of the minimum spanning angles (ties are broken arbitrarily).

For the base case, $\theta_m(s,t)$ has lowest rank. Bose and Keil showed that in this case, st is an edge of $DT(P)$. Therefore, the lemma holds.

We make the following induction hypothesis: *for any pair of vertices s and t whose minimum spanning angle has rank at most $k \geq 1$, all the vertices of the path τ from s to t belong to the union of the upper and lower chains with respect to s and t.*

Consider two vertices s and t whose minimum spanning angle has rank $k+1$. If st is an edge of $DT(P)$, then we are done. Otherwise, there must be at least one vertex of $DT(P)$ in $\LEFTcircle st$ to the left of st. Let $v \in P$ be a vertex in $\LEFTcircle st$ to the left of st with the property that the circle through s, v and t is $\LEFTcircle_m ts$.

We consider two cases: either (1) v is inside the circle $\ominus st$ having st as diameter or (2) v is outside of $\ominus st$.

1. If $v \in \ominus st$, then, by Lemma 3, we can apply induction on sv and on vt. Moreover, $v \in \tau$. This way, we get a path τ_{sv} (respectively τ_{vt}) from s to v (respectively from v to t) that satisfies the induction hypothesis.

   The heart of the proof is to argue that $v \in U \cup L$. We prove this by contradiction. Suppose $v \notin U \cup L$. Therefore, $\LEFTcircle_m ts$ intersects an edge of $U \cup L$ and contains v on its boundary. More precisely, there is an arc $\Gamma^* \subset \LEFTcircle_m ts$ such that $v \in \Gamma^*$, Γ^* intersects an edge e of $U \cup L$ and no part of Γ^* is inside $U \cup L$ (refer to Figure 3(a)). Without loss of generality, suppose $e \in U$, so that $e = u_i u_{i+1}$, where $u_i, u_{i+1} \in U$.

If $u_i \neq s$ and $u_{i+1} \neq t$, then we can apply Lemma 4 to st and $u_i u_{i+1}$. We get an empty circular sector that contains v, which is impossible. Therefore, either $u_i = s$ or $u_{i+1} = t$. Without loss of generality, suppose that $u_i = s$. Therefore, $e = su_1 \in DT(P)$.

We now prove that v is adjacent to s. Suppose that v is not. Since $\varGamma^*$ intersects su_1, then $\varGamma^*$ intersects an edge $v'v'' \in DT(P)$ in $\triangle sv'v'' \in DT(P)$ where $\angle tsv'' < \angle tsv'$ (refer to Figure 3(b)). In such a case, $\varGamma^*$ intersects sv'' in its interior. Therefore, we can apply Lemma 4 to ts and $v''v'$. We get that v is inside the empty circle $\bigcirc sv'v''$ that defines $\triangle sv'v''$. This is a contradiction. Therefore, v is adjacent to s.

Since v is adjacent to s, v is on the boundary of $\LEFTcircle_m ts$ and $v \notin U \cup L$, there exists a triangle $\triangle svw \in DT(P)$, where $\angle tsw < \angle tsv$ and $w \notin \LEFTcircle_m ts$. Let $\bigcirc svw$ be the empty circle defining $\triangle svw$. Since $w \notin \LEFTcircle_m ts$, then $t \in \bigcirc svw$, which is a contradiction (refer to Figure 3(c)). Therefore, $v \in U \cup L$.

We now have that $v \in U \cup L$, and the vertices of τ_{sv} (respectively of τ_{vt}) belong to the union of the lower and upper chains with respect to s and v (respectively to v and t). We need to prove that the vertices of $\tau_{sv} \cup \tau_{vt}$ belong to $U \cup L$. Notice that if v is not adjacent to s and v is not adjacent to t, then the union of the upper and lower chains with respect to s and v is a subset of $U \cup L$. Moreover, the union of the upper and lower chains with respect to v and t is also a subset of $U \cup L$. Therefore, if v is not adjacent to s and v is not adjacent to t, then $\tau_{sv} \in U \cup L$ and $\tau_{vt} \in U \cup L$ by the induction hypothesis. Consequently, $\tau_{sv} \cup \tau_{vt} \in U \cup L$.

If $v = u_1$, then $\tau_{sv} = su_1 \in U$ and by the induction hypothesis, $\tau_{vt} \in U \cup L$. Hence, $\tau_{sv} \cup \tau_{vt} \in U \cup L$. If $v = u_{k-1}$, then $\tau_{sv} \in U$ by the induction hypothesis and $\tau_{vt} = u_{k-1}t \in U$. Hence, $\tau_{sv} \cup \tau_{vt} \in U \cup L$.

2. If $v \notin \ominus st$, then we can apply induction by Lemma 3. Notice that the upper chain with respect to t and s is L and the lower chain with respect to t and s is U. Therefore, we get our result by induction. $\square$

We now outline the construction of the 2.419-path σ. Before doing this, we need to define a *lune*. Let p be a point on st and $\varGamma_{sp}$ be the circular arc from s to p such that $\varGamma_{sp}$ is above sp and the tangent to $\varGamma_{sp}$ at s makes an angle of $\pi/3$ with st (refer to Figure 4(a)). Let $\varGamma'_{sp}$ be the circular arc that is the reflection of $\varGamma_{sp}$ across sp. The *lune* $\mathcal{L}_{sp}$ with respect to s and p is defined to be $\varGamma_{sp} \cup \varGamma'_{sp}$.

To construct the 2.419-path σ from s to t, we consider the largest empty lune $\mathcal{L}_{sp}$ that has a vertex $v \in P$ on its boundary. If there is more than one vertex on the boundary of $\mathcal{L}_{sp}$, we consider the one closest to s. We can see this as the process of growing a lune from s until it hits a vertex $v \in P$. To construct σ, we first travel from s to v using the path of Lemma 2 (by considering a specific right-empty circle $\LEFTcircle sv$; refer

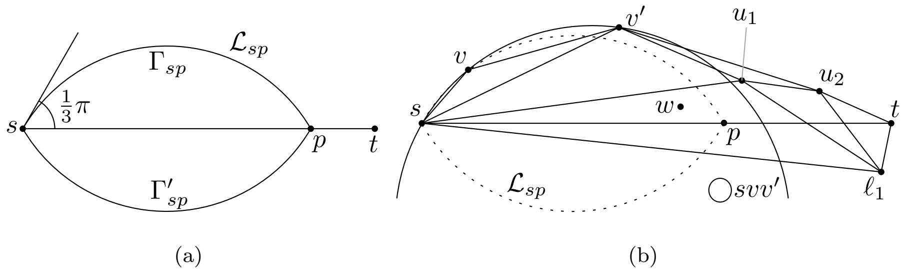


Fig. 4 (a) The lune $\mathcal{L}_{sp}$ with respect to s and p. (b) An example where the first vertex we hit by growing a lune from s is not in $U \cup L$.

to the proof of Theorem 1.1 in [4]). Then, we apply induction from v to t. When we apply Lemma 2 from s to v, we need to consider a *good* right-empty circle. A right empty circle $◐sv$ is *good with respect to* $\mathcal{L}_{sp}$ if it is centered on so, where o is the center of Γ'_{sp}.

It is possible that the first vertex v of P we encounter by growing a lune from s is not in $U \cup L$ (refer to Figure 4(b)). In the original proof by Keil and Gutwin as well as the proof in Bose and Keil, it was not necessary for v to be in $U \cup L$ to prove the spanning ratio. However, to be able to route, we need this property to apply Theorem 1. Fortunately, we are able to show that there exists a point v' in $U \cup L$ that satisfies the same properties as v and allows the inductive argument to go through. We outline this below.

Lemma 5 *Suppose that the first vertex $v \in DT(P)$ we hit by growing a lune from s is not in $U \cup L$. Let $u_1 \in U$ and $\ell_1 \in L$ be such that $su_1 \in DT(P)$ and $s\ell_1 \in DT(P)$. If we keep growing the lune until it hits a vertex $v' \in U \cup L$, then $v' = u_1$ or $v' = \ell_1$. Moreover, there exists a good right-empty circle with respect to the lune that has v on its boundary.*

Proof Without loss of generality, suppose that v is above the line through st. Denote by $\mathcal{L}_{sp}$ the empty lune that has v on its boundary. Denote by $\mathcal{L}_{sp'}$ the (not necessarily empty) lune that has u_1 on its boundary. We have that v is outside of $\bigcirc su_1\ell_1$, where $\bigcirc su_1\ell_1$ defines $\triangle su_1\ell_1 \in DT(P)$. Therefore, the part of $\mathcal{L}_{sp'}$ that is below su_1 is inside the empty circle $\bigcirc su_1\ell_1$. Consequently, if we keep growing $\mathcal{L}_{sp}$ until it hits a vertex $v' \in U \cup L$, then $v' = u_1$. Moreover, since the part of $\mathcal{L}_{sp'}$ that is below su_1 is empty, there exists a good right-empty circle $◐su_1$ with respect to $\mathcal{L}_{sp'}$. $\square$

The proof of Theorem 1.1 in [4] is based on finding a good right-empty circle before applying induction. In our case, we can use Lemma 5 within Theorem 1.1 to find such a circle; this will guarantee that there exists a 2.419-path $\sigma \in U \cup L$. Therefore, we can apply Theorem 1 to find the shortest path on $U \cup L$. The length of our routing path is at most $9\,\frac{4\pi\sqrt{3}}{9}|st| = 4\pi\sqrt{3}|st| \approx 21.766|st|$. This gives the following theorem.

Theorem 2 *There is a* $(4\pi\sqrt{3})$*-competitive online routing algorithm for Delaunay triangulations.*

2.2 17.982-Competitive Online Routing

Xia [24] showed that the stretch factor of a Delaunay triangulation of a set of points in the plane is less than 1.998. His proof restricts the search space to the set of triangles intersecting st as outlined in the proof of Corollary 1 in [24]. Therefore, by applying Theorem 1, we obtain a competitive online routing strategy whose competitive ratio is at most 17.982.

Theorem 3 *There is a* 17.892*-competitive online routing algorithm for Delaunay triangulations.*

2.3 $(\pi(5\pi + 4)/4) \approx 15.479$-Competitive Online Routing

We propose an online competitive routing algorithm inspired by the work of Dobkin et al. [13] and Bose and Morin [6]. Let P denote any set of n points in general position and let s and t denote any two vertices in P. Without loss of generality, assume s and t lie on the x-axis, with s having a smaller x-coordinate than t. Let $V_0, \ldots, V_{m-1}$ be the cells of the Voronoi diagram intersected by the line segment st, with V_0 being the Voronoi cell of s and V_{m-1} being the cell of t. The path from s to t in $DT(P)$ obtained by following the sites generating the cells $V_0, \ldots, V_{m-1}$, in order, shall be referred to as the *Voronoi path* and denoted $VP(s,t)$. Label the vertices on this path $s = v_0, \ldots, v_{m-1} = t$. The Voronoi path is x-monotone and it is not necessarily a constant spanning path [13] (see Figure 5). Dobkin et al. [13] proved the following lemma.

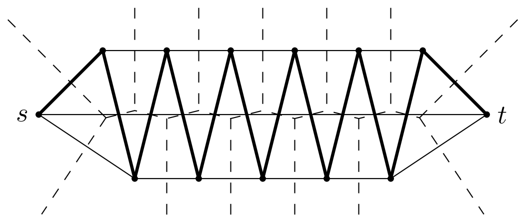


Fig. 5 This example shows that the number of times the Voronoi path (in bold) crosses st is unbounded in general. Consequently, the Voronoi path is not a constant spanning path.

Lemma 6 (Dobkin et al. [13]) *Let* N *be the set of edges of* $VP(s,t)$ *that do not cross the segment* st*. The sum of the lengths of the edges in* N *is at most* $|st|\pi/2$*.*

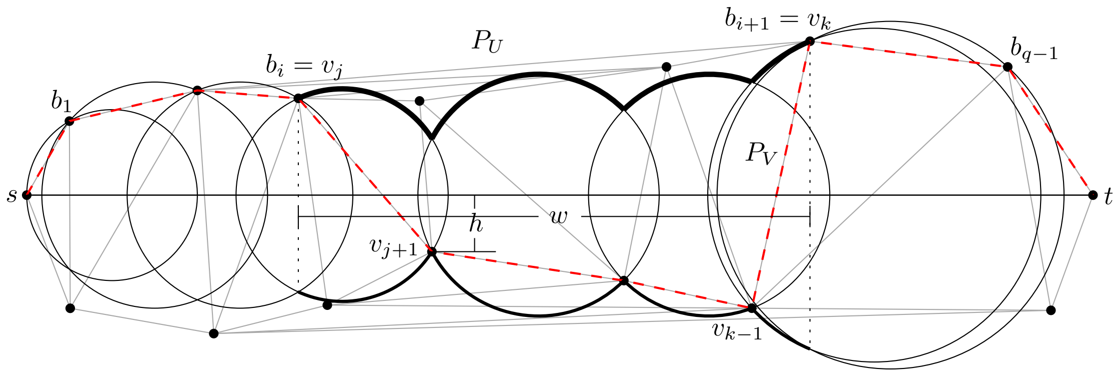


Fig. 6 The red and dashed line represents the Voronoi path P_V from $b_0 = s$ to $b_q = t$. The circles are centered on st. They are the ones that define the Voronoi path. This is an example where we would follow the Voronoi path since $h \leq w/4$.

If the vertices on $VP(s,t)$ all lie above the line through s and t, the Voronoi path is called *one-sided*. The above lemma implies that if $VP(s,t)$ is one-sided, then $|VP(s,t)| \leq |st|\pi/2$. Therefore, $VP(s,t)$ is a $\pi/2$-spanning path when it is one-sided. Note that $VP(s,t)$ is not necessarily a constant spanning path when it crosses st. Consider a Voronoi path from s to t that is not one-sided. Let $s = b_0, b_1, \ldots, b_q = t$ be the subsequence of vertices of the Voronoi path that lie above the x-axis. Consider two consecutive vertices in this subsequence $b_i = v_j$ and $b_{i+1} = v_k$ that are not consecutive on the Voronoi path, i.e. $k \neq j+1$. This means that the edge $v_j v_{j+1}$ and $v_{k-1} v_k$ both cross st. (refer to Figure 6). Let P_V be the Voronoi path $v_j, v_{j+1}, \ldots, v_k$ and let P_U be the path from v_j to v_k on the upper chain. For a point $p \in P$, let $x(p)$ and $y(p)$ be the x-coordinate and y-coordinate of p, respectively. Define $h = \min_{j<z<k} |y(v_z)|$ and $w = x(v_j) - x(v_k)$. Dobkin et al. [13] proved the following lemma:

Lemma 7 (Dobkin et al. [13]) *If $h \leq w/4$, $|P_V|$ is at most $(1+\sqrt{5})w\pi/2$ and the path from v_{j+1} to v_{k-1} has length at most $w\pi/2$.*

Using the construction given by Dobkin et al. [13], Bose and Morin [6] proved:

Lemma 8 (Bose and Morin [6]) *If $h > w/4$, $|P_U|$ is at most $w\pi^2/4$.*

Intuitively, the two lemmas state that when the Voronoi path from v_j to v_k comes "close" to the x-axis, then the length of the Voronoi path is at most a constant times w, otherwise, the length of the upper chain from v_j to v_k is at most a constant times w. These two lemmas taken together imply that the Delaunay triangulation is a $((1+\sqrt{5})\pi/2)$-spanner. Notice that given a vertex v on the upper (respectively lower) chain from s to t, one can locally determine if v is on $VP(s,t)$ simply by examining $N(v)$. Consider all the empty circles defined by the Delaunay triangles in $N(v)$ that intersect st. If any one of these circles has its center below (respectively above) the x-axis, then v is on the Voronoi path from s to t since its Voronoi cell intersects st. Armed with this observation, Lemmas 7 and 8 seem to suggest the following competitive online routing algorithm:

When at a vertex b_i, if b_{i+1} is adjacent to b_i on the Voronoi path from s to t, follow the edge. If b_i and b_{i+1} are not adjacent on the Voronoi path, follow P_V from b_i to b_{i+1} when $h \leq w/4$ and P_U when $h > w/4$. Unfortunately, the main caveat to this approach is that we do not know how to compute h or w locally from vertex b_i. It seems that knowledge of P_V is required to compute h and w, which is not necessarily available locally at b_i.

To overcome this obstacle, we slightly modify the above approach. When b_i and b_{i+1} are adjacent on the Voronoi path, we still follow the edge. However, when they are not adjacent, we take the following approach. Let $d = |v_j v_{j+1}|$. From v_j, follow P_U until either v_k is reached or a distance of at most d has been travelled on P_U. Should the latter occur at a vertex u on the upper chain, let v be the vertex furthest along the lower chain adjacent to u. Note that v must be on P_V. Move to v and continue on P_V. Proceed in this manner until t is reached. We refer to this online routing strategy as *OnlineDelaunayRoute* and outline it as follows: We assume that s is on the upper chain. The algorithm is initiated at node s with a call to *OnlineDelaunayRoute*$(s, t, 0, m)$.

Theorem 4 *OnlineDelaunayRoute is an online routing strategy that is* $(\pi(5\pi + 4)/4)$*-competitive on Delaunay triangulations.*

Proof When b_i and b_{i+1} are consecutive on the Voronoi path from s to t, the message follows the edge. By Lemma 6, the sum of all the edges of the Voronoi path that do not cross st is at most $|st|\pi/2$.

When b_i and b_{i+1} are not consecutive, the message follows two different paths depending on the length of P_U. If P_U has length at most d, then the messages travels on P_U. Otherwise, it travels on P_U for a distance of d, crosses over onto P_V and then continues travelling on P_V. Notice that by the triangle inequality, this is shorter than travelling on P_U for distance of at most d, returning to b_i and travelling on P_V. Therefore, the total distance travelled is at most $2d + |P_V|$. We bound this distance in terms of w. There are 4 cases to consider.

Algorithm 1 OnlineDelaunayRoute(s, t, d, m).

Input: c is the current node. The header contains s the source node, t the destination node, and d the distance remaining on the shortcut path. Parameter m is the message.

Output: Forward the header and message from c to one of the vertices in $N(c)$ until t is reached.

Let u be the vertex furthest along the upper chain adjacent to c, ℓ be the vertex furthest along the lower chain adjacent to c, and v be the vertex furthest along the Voronoi path adjacent to c.
if $c = t$ **then**
    Destination reached. EXIT
end if
if c is on lower chain **then**
    Forward $s, t, 0, m$ to v. EXIT
else {c is on upper chain}
    if c is on the Voronoi path **then**
        if v is on the upper chain **then**
            Forward $s, t, 0, m$ to v. EXIT
        else {v is on lower chain}
            $d \leftarrow |cv|$
        end if
    end if
end if
if $d - |cu| \geq 0$ **then**
    $d \leftarrow d - |cu|$. Forward s, t, d, m to u.
else
    Forward $s, t, 0, m$ to ℓ.
end if

Case 1: $h \leq w/4$ and the message travels $|P_U|$.

By Lemma 7, we have $|P_V| \leq (1 + \sqrt{5})w\pi/2$. Since the edge $v_j v_{j+1} \in P_V$, we have that $d \leq |P_V|$. Since the message remains on P_U, we have that $|P_U| \leq d$. Therefore, $|P_U| \leq (1 + \sqrt{5})w\pi/2 \leq 5.09w$.

Case 2: $h \leq w/4$ and the message travels $2d + |P_V|$.

By Lemma 7, we have $|P_V| \leq (1+\sqrt{5})w\pi/2$. Since the edge $v_j v_{j+1} \in P_V$, we have that $d \leq |P_V|$. Therefore, $2d + |P_V| \leq 3|P_V| \leq 3(1 + \sqrt{5})w\pi/2 \leq 15.25w$.

Case 3: $h > w/4$ and the message travels $|P_U|$.

By Lemma 8, $|P_U| \leq w\pi^2/4 \leq 2.47w$.

Case 4: $h > w/4$ and the message travels $2d + |P_V|$.

By Lemma 8, $|P_U| \leq w\pi^2/4$. By construction, $d \leq |P_U|$. Since the portion of P_V that lies below the x-axis is a one-sided Voronoi path, its length is at most $w\pi/2$ by Lemma 7. By the triangle inequality, $|P_V| \leq 2d + \pi w + |P_U|$. Therefore, putting it all together, we have $2d + |P_V| \leq \pi(5\pi + 4)w/4 \leq 15.479w$.

Since the cost of the path is dominated by the value obtained in Case 4, the result follows. □

3 Routing on Delaunay Triangulations of Points in Convex Position

When the points in P are in convex position, we can achieve a better competitive ratio in $DT(P)$. In this section, we first show that as a consequence of a result by Cui et al. [11], there exists a path from s to t on $U \cup L$ having length at most ρ, where $\rho \approx 2.326$ satisfies (1) (see below). This immediately implies that we can apply Theorem 1 to get a competitive online routing algorithm with a competitive ratio of at most $9\rho \approx 20.926$.

Using an entirely different online routing strategy, in Section 3.2 we show how to achieve a better competitive ratio of at most $(11 + 3\sqrt{2})/2 \approx 7.621$.

3.1 Lower and Upper Chains for Points in Convex Position

Cui et al. [11, Theorem 1] prove the following result about Delaunay Triangulations for points in convex position.

Lemma 9 *[Cui et al. [11]] Given any set P of points in general position and any pair of vertices $\{s, t\}$ in $DT(P)$, there is a path τ from s to t in $DT(P)$ of length at most $\rho|st| \approx 2.326|st|$, where ρ is the root of*

$$\rho^3 - \rho - \left(\pi + \arctan\left(\frac{1-\rho^2}{\rho}\right)\right)\sqrt{\rho^4 - \rho^2 + 1} = 0. \tag{1}$$

Cui et al. prove Lemma 9 by induction on the rank of $|st|$ (with ties being broken arbitrarily).

Since P is in convex position, there exists a supporting line L_s through s such that all other points of P lie on one side of L_s. Let L_t be a line through t that is parallel to L_s . Then on one side of the line through st — either above or below it — all points of P lie between L_s and L_t . Without loss of generality, assume that all points of P above the line st lie between L_s and L_t . Note that if this set of points is empty, then s and t are connected in $DT(P)$ by a horizontal path (possibly a single edge) of weight $|st|$, and the statement follows.

Let $v \in P$ be a point above st that maximizes the angle $\gamma = \angle svt$. The following three lemmas are a direct consequence of the proof of Theorem 1 in [11].

Lemma 10 *If $\gamma \leq \xi \approx 2.057$ (where $\xi / \sin \xi = \rho$), then τ is the path we obtain from Lemma 1.*

Lemma 11 *If $\gamma > \xi \approx 2.057$ (where $\xi / \sin \xi = \rho$) and sv is an edge of $DT(P)$, then the first edge of τ is sv and we can apply induction from v to t.*

If $\gamma > \xi$ and vt is an edge of $DT(P)$, then the last edge of τ is vt and we can apply induction from s to v.

Lemma 12 *Suppose that $\gamma > \xi \approx 2.057$ (where $\xi / \sin \xi = \rho$), sv is not an edge of $DT(P)$ and vt is not an edge of $DT(P)$. Let L_v be a supporting line passing through v such that all other points in P are below L_v. Let s' and t' be the intersections of L_v with L_s and L_t, respectively. Then the part of τ from s to v is inside the triangle $\triangle ss'v$ and the part of τ from v to t is inside the triangle $\triangle vt't$.*

Lemma 13 *For points in convex position, all vertices of the path τ are in $U \cup L$.*

Proof We proceed by induction on the rank of $|st|$ (ties are broken arbitrarily).

For the base case, st has lowest rank. Therefore, the circle with respect to st as diameter is empty and st is an edge. Thus, the lemma holds.

We make the following induction hypothesis: *for any pair of vertices s and t such that $|st|$ has rank at most $k \geq 1$, all the vertices of the path τ from s to t belong to the union of the upper and lower chains with respect to s and t.*

Consider two vertices s and t such that $|st|$ has rank $k+1$. If st is an edge of $DT(P)$, then we are done. Otherwise, since P is in convex position, there exists a supporting line L_s through s such that all other points of P lie on one side of L_s. Let L_t be a line through t that is parallel to L_s . Then on one side of the line through st — either above or below it — all points of P lie between L_s and L_t . Without loss of generality, assume that all points of P above the line st lie between L_s and L_t . Note that if this set of points is empty, then s and t are connected in $DT(P)$ by a horizontal path (possibly a single edge) of weight $|st|$, and the statement follows. Let $v \in P$ be a point above st that maximizes the angle $\gamma = \angle svt$.

If $\gamma \leq \xi \approx 2.057$ (where $\xi / \sin \xi = \rho$), then by Lemma 10, τ is the path we obtain from Lemma 1. Therefore, by Lemma 2, all the vertices of the path τ are in $U \cup L$.

Suppose $\gamma > \xi$. If sv is an edge of $DT(P)$ and vt is an edge of $DT(P)$, then we are done. If sv is an edge of $DT(P)$ and vt is not an edge of $DT(P)$, then by Lemma 11, the first edge of τ is sv and we can apply induction from v to t. Since v is not adjacent to t, the union of the upper and lower chains with respect to v and t is a subset of $U \cup L$. Therefore, all the vertices of τ are in $U \cup L$. If sv is not an edge of $DT(P)$ and vt is an edge of $DT(P)$, a symmetric argument applies.

Suppose that $\gamma > \xi$, sv is not an edge of $DT(P)$ and vt is not an edge of $DT(P)$. Let L_v be a supporting line passing through v such that all other points in P are below L_v. Let s' and t' be the intersections of L_v with L_s and L_t, respectively. Then by Lemma 12, the part of τ from s to v is inside the triangle $\triangle ss'v$ and the part of τ from v to t is inside the triangle $\triangle vt't$. Since the part of U from s to v belongs to $\triangle ss'v$ and the part U from v to t belongs to $\triangle vt't$, the proof is complete. □

From Lemma 13 and Theorem 1, we get the following theorem.

Theorem 5 *There is a 9ρ-competitive online routing algorithm for Delaunay triangulations of points in convex position, where $\rho \approx 2.326$ is the root of*

$$\rho^3 - \rho - \left(\pi + \arctan\left(\frac{1-\rho^2}{\rho}\right)\right)\sqrt{\rho^4 - \rho^2 + 1} = 0$$

and $9\rho \approx 20.926$.

3.2 $(11 + 3\sqrt{2})/2 \approx 7.621$-Competitive Online Routing for Points in Convex Position

We present an online routing algorithm with a competitive ratio of at most $(11 + 3\sqrt{2})/2$ for Delaunay triangulations of sets of points in convex position, where $(11+3\sqrt{2})/2 \approx 7.621$. Throughout this section we assume that P is a set of points in convex position in the plane. For ease of exposition, we assume without loss of generality that the line segment st is horizontal, with s to the left of t. Let $\ominus st$ be the circle whose diameter is the line segment st. Let $S(s,t)$ be the axis-parallel square whose bisector is the line segment st. Again, let U and L denote the respective upper and lower chains of s and t in $DT(P)$. Before proving Theorem 17, we begin with a few geometric lemmas and observations used to prove the correctness of the algorithm and to bound its competitive ratio.

Lemma 14 *If a line ℓ is not parallel to any side of a convex polygon Q, then ℓ intersects the boundary of Q in at most two points.*

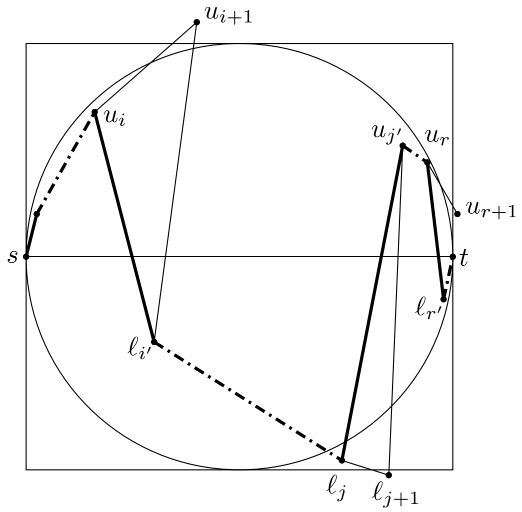


Fig. 7 The general shape of a routing path (in bold) that crosses st three times when P is in convex position.

Lemma 15 *If vertex $v \in U$ (respectively $v \in L$) is outside of $\ominus st$ then v is adjacent to at least one vertex $v' \in L$ (respectively $v' \in U$) that is in $\ominus st$.*

Proof Suppose that both v and v' are outside $\ominus st$. By definition, every edge between a vertex in U and a vertex in V must intersect st. Since vv' intersect st and st is the diameter of $\ominus st$, every circle with v and v' on its boundary will either contain s in its interior or t in its interior. This contradicts the fact that vv' is an edge of the Delaunay triangulation. □

We now describe the routing algorithm. The message starts at a node s with destination t. The algorithm first forwards the message from s to one of its neighbours on $U \cup L$ that is in $S(s,t)$. Such a vertex must exist by Lemma 15. The algorithm makes a forwarding decision at each vertex v along the route, which we now describe. Without loss of generality, suppose that v is on the upper chain (an analogous symmetric case applies if v is on the lower chain). Let u be the vertex adjacent to v on the upper chain and let ℓ be the vertex adjacent to v that is furthest right on the lower chain. If u is in $S(s,t)$ then forward the message to u, otherwise forward it to ℓ. This decision can be made locally given the following information stored in the header: the source s, the destination t and $N(v)$, the set of vertices adjacent to v. Let σ be the path followed by the message.

For ease of exposition, we assume that st is horizontal with s to the left of t. The initial call to the algorithm is LocalConvexRoute(s,t,m). Let σ be the path followed by the message from s to t (refer to Figure 7).

Algorithm 2 LocalConvexRoute(s, t, m).

Input: c is the current node, $N(c)$ is the set of vertices adjacent to c. The header contains s the source node, t the destination node and m the message. Recall that we reorient the graph such that st is horizontal.

Output: Forward message and header from c to one of the vertices in $N(c)$ until t is reached.

Let $N'(c) = \{u, \ell\}$ where u is the furthest neighbour adjacent to c on the upper chain and ℓ is the furthest neighbour adjacent to c on the lower chain where distance is measured by the x-coordinate of the vertex.
if $t \in N(c)$ **then**
    Forward s, t, m to vertex t. EXIT
end if
if $c = t$ **then**
    Destination Reached. EXIT
end if
if $c = s$ **then**
    Forward s, t, m to a vertex of $N'(c)$ that is on the Voronoi path. EXIT
end if
Let $x \in N'(c)$ be the vertex such that cx does not intersect st.
Let $y \in N'(c)$ be the vertex such that cy intersects st.
if cx is in $S(s, t)$ **then**
    Forward s, t, m to vertex x
else
    Forward s, t, m to vertex y
end if

Lemma 16 *The path σ taken by the message m crosses the line segment st at most three times before reaching t.*

Proof Notice that prior to crossing the boundary of the square, the path σ crosses st. Without loss of generality, assume that σ crosses st for the first time from a vertex on the upper chain to a vertex on the lower chain. Let x_1y_1 be this edge with $x_1 \in U$ and $y_1 \in L$. Since the path crosses st, x_1 must be adjacent to a vertex $x'_1 \in U$ that is outside $S(s, t)$. By Lemma 15, y_1 must be in $\ominus st$ since it is also adjacent to x'_1. By Observation 14, the portion of the upper chain from x_1 to t in clockwise order and the portion of the lower chain from y_1 to t in counter-clockwise order intersects $S(s, t)$ a total of 6 times.

Suppose, for a contradiction, that σ crossed st four times with the first edge as above from x_1 to y_1. Let the other three edges be x_2y_2, x_3y_3, and x_4y_4 with $x_i \in U$ and $y_i \in L$. This means that the upper chain intersects $S(s, t)$ twice from x_1 to x_2 since x'_1 is outside $S(s, t)$ and x_2 is inside $S(s, t)$ by Lemma 15. Similarly, the lower chain between y_2 and y_3 intersects $S(s,t)$ twice. The upper chain from x_3 to x_4 intersects $S(s,t)$ twice. Finally, the edge on the lower chain adjacent to y_4 intersects $S(s,t)$ since this is what prompted the algorithm to cross to x_4. However, this is at least 7 intersections which is a contradiction. □

Lemma 17 *The length of the path σ is at most $(11+3\sqrt{2})|st|/2$.*

Proof Let U' be the sequence $s = u'_0, u'_1, ...u'_k = t$ of vertices followed by the message on U and L' be $s = \ell'_0, \ell'_1, ...\ell'_b = t$ be the sequence followed by the message on L. By construction, neither U' nor L' go outside $S(s,t)$. Since the union of these two sequences is a convex polygon inside $S(s,t)$, its perimeter is at most the perimeter of the square which is $4|st|$. This accounts for all of σ except for the crossing edges.

By Lemma 16, σ crosses st at most 3 times. Each of those edges has one endpoint in $S(s,t)$ and one endpoint in $\ominus st$. Therefore, its length is at most $(\sqrt{2}/2 + 1/2)|st|$ since the longest such edge has one endpoint on the corner of the square and the other diametrically opposed on the boundary of the circle. Summing the components gives an upper bound on σ of $(11+3\sqrt{2})|st|/2$. □

Theorem 6 follows from Lemma 17:

Theorem 6 *There is a $(11+3\sqrt{2})/2$-competitive online routing algorithm for Delaunay triangulations of convex point sets.*

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