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ON COMBINATORIAL DEPTH MEASURES*

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Given a set $P = \{p_1, \ldots, p_n\}$ of points and a point q in the plane, we define a function $\psi(q)$ that provides a combinatorial characterization of the multiset of values $\{|P \cap H_i|\}$, where for each $i \in \{1, \ldots, n\}$, H_i is the open half-plane determined by q and p_i. We introduce two new natural measures of depth, perihedral depth and eutomic depth, and we show how to express these and the well-known simplicial and Tukey depths concisely in terms of $\psi(q)$. The perihedral and eutomic depths of q with respect to P correspond respectively to the number of subsets of P whose convex hull contains q, and the number of combinatorially distinct bisections of P determined by a line through q. We present algorithms to compute the depth of an arbitrary query point in $O(n \log n)$ time and medians (deepest points) with respect to these depth measures in $O(n^4)$ and $O(n^{8/3})$ time respectively. For comparison, these results match or slightly improve on the corresponding best-known running times for simplicial depth, whose definition involves similar combinatorial complexity.

Keywords: depth measure; Tukey depth; simplicial depth

1. Introduction

This paper presents new work on measuring the degree to which a point q is interior or central relative to a given set P of n points in $\mathbb{R}^d$, i.e., the *depth* of q with respect to P. Depth measures are widely studied in statistics and data analysis, and many depth measures have been defined to quantify the centrality or eccentricity of a given point q relative to a given data set P. When $d = 1$ and P is a set of points in $\mathbb{R}$, natural measures for the depth of q relative to P include the minimum of the

*A preliminary version of some of these results appeared in the Proceedings of the Canadian Conference on Computational Geometry (CCCG 2014) [17].

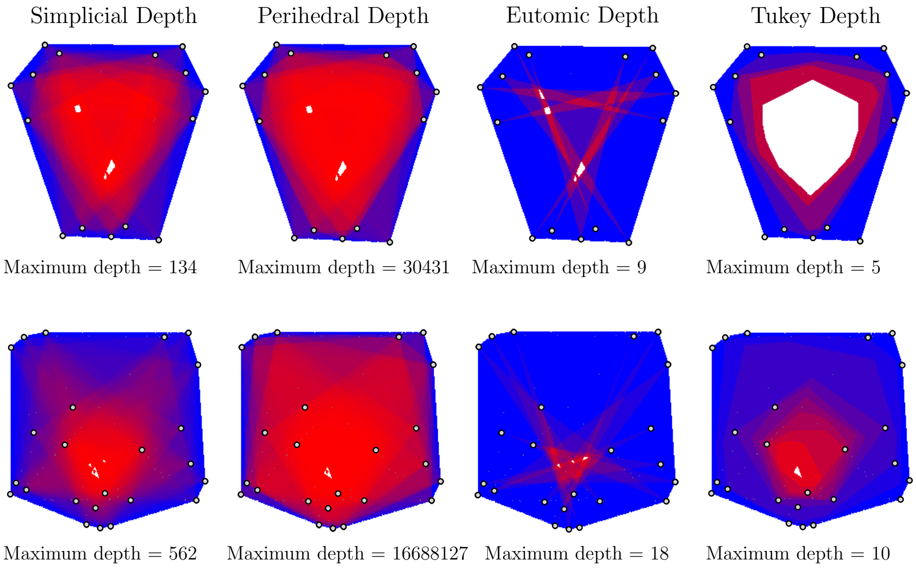

Fig. 1: Simplicial, perihedral, eutomic and Tukey depths are shown for two point sets. The colours progress from blue to red for increasing depth, and regions of maximum depth are shown in white. Note that in the bottom figures, the region of maximum depth is unique for each depth measure.

number of points of P to the left and right of q, and the number of subsets of P whose interval contains q. That subset count is proportional to the probability that q is contained within the interval, or one-dimensional convex hull, determined by a randomly selected subset of P. All these are maximized by a median of P.

The one-dimensional median has many possible generalizations to two and higher dimensions. We focus on the case $d = 2$ (P is a set of points in the plane $\mathbb{R}^2$) and examine combinatorial measures of depth, specifically, functions of the numbers of points of P contained in convex regions defined by half-planes through pairs of points of P, and whether those regions contain q. Such measures of depth include Tukey depth and simplicial depth. Our examination captures some previous definitions of depth, leads us to define two new depth measures, and provides algorithms for efficiently computing these measures of depth and the corresponding *medians*, defined as points of maximal depth.

In Section 2 we review results related to common notions of depth, including Tukey depth, simplicial depth, and half-space counts. Building on the half-space counts of Rousseeuw and Ruts [33], we introduce the function $\psi_j(q) : \mathbb{R}^2 \to \{0, \ldots, n\}$ in Section 3: given any $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^2$, $q \in \mathbb{R}^2$, and $j \in \{0, \ldots, n-1\}$, $\psi_j(q)$ is the number of open half-planes H_i determined by q and some $p_i \in P$ such that $|P \cap H_i| = j$. The n values $\psi(q) = \langle \psi_0(q), \ldots, \psi_{n-1}(q) \rangle$

summarize pertinent combinatorial properties of the set P, allowing simple computation of various depth measures as functions of $\psi(q)$. Moreover, $\psi(q)$ is itself easy to calculate, and it has attractive geometric properties. Specifically, we show how to express simplicial depth and Tukey depth in terms of $\psi(q)$.

We introduce two new natural measures of depth, perihedral depth in Section 4 and eutomic depth in Section 5, each defined in terms of $\psi(q)$. We present algorithms to compute eutomic and perihedral medians in $O(n^4)$ and $O(n^{8/3})$ time respectively, and the depth of an arbitrary query point q in $O(n \log n)$ time for both. For comparison, these results match or slightly improve on the corresponding best-known running times for simplicial depth, whose definition involves similar combinatorial complexity. We show examples of these depth measures in Figure 1. The perihedral and eutomic depths of q with respect to P correspond respectively to the number of subsets of P whose convex hull contains q, and the number of distinct bisections of P determined by lines through q. These have intuitive probabilistic interpretations: after scaling by a normalizing factor (dependent on n, but independent of q) the perihedral depth of q with respect to P is equal to the probability that q lies inside the convex hull of a subset of P selected at random, whereas the eutomic depth of q with respect to P is equal to the probability that q lies on a halving line of P. Both of these are reasonable measures of centrality, further motivating their definitions as depth measures. In Section 6 we briefly compare properties of these depth measures, discuss generalizations, and suggest directions for future research.

2. Definitions and Related Work

Consider a set of n points P in $\mathbb{R}^d$ and a point q in $\mathbb{R}^d$ in general position (specifically, no $k+1$ points of P and q lie on any $(k-1)$-flat for any $k \leq d$). Let $CH(P)$ denote the convex hull of set P. A number of functions of the form $f(q, P) : (\mathbb{R}^d \times (\mathbb{R}^d)^n) \to \mathbb{R}$ that define a measure of the depth of q relative to P have been studied extensively. We describe these functions in terms of general d, although our main algorithmic results are for the two-dimensional case.

In particular, Tukey (or half-space) depth [37] and simplicial depth [26] are well understood and related to the depth measures introduced in this paper. See Subsections 2.2 and 2.3. Other measures of this form are described briefly below. For additional details on depth measures, see the reviews of Aloupis [1] for a computational geometry perspective, Small [35] for a statistical perspective, and Zuo and Serfling [39] for formal expressions of generalized depth measures, including several discussed in this paper.

In addition to Tukey depth and simplicial depth, several other common depth measures are primarily combinatorial. Convex hull peeling [5, 34] expresses the depth of q as the number of times the convex hull of P must be removed before q appears on the hull. The ray shooting depth [30] is the minimum number of $(d-1)$-simplices on P hit by any ray from q. Oja depth [31] computes the sum of the volumes of the simplices defined by $P' \cup \{q\}$ for every subset $P' \subseteq P$ of cardinality

d, where $P \subseteq \mathbb{R}^d$. For majority depth [12, 27], any d points in P determine a hyperplane and its two associated half-spaces; a half-space containing at least half of the points in P is called a *majority side*. The majority depth of q is the number of majority sides of P that contain q.

Many measures seek to determine a point that minimizes the sum of the distances to the points of the input set. The sum of Euclidean distances (or more generally, the L_p depth) [39] is a primary example. The concept of expressing the distance from q to P as the sum of Euclidean distances is over a century old [38]. The geometric median with respect to this measure (also known as the Fermat-Weber point) is the point which minimizes the Euclidean distance to all points in P.[a] Because of the relationship with the Weber point, this may also be referred to as the *Weber depth*. Squaring the distances would yield a sum of squared distances depth, for which the point of minimum depth is uniquely realized by the average of the points in P [16]. As a measure of distance, Mahalanobis generalizes Euclidean distance, in that a (variance-)covariance matrix S is applied: $\text{dist}_M(q, P) = \sqrt{(q-\mu)^T S^{-1}(x-\mu)}$, where μ is the geometric mean of P (if S is the identity matrix, this reduces to Euclidean distance) [14]. Occasionally this measure is squared, see e.g. [21].

In a sense, zonoid depth [24] is a generalization of convex hull peeling. First, a value $\alpha \in [0, 1]$ is used to bound the region of the plane into a *zonoid α-trimmed region*, where larger values of α constrain the plane more. The zonoid depth measures the depth of the point in these regions. Ellipsoids have been used for similar purposes, see e.g. [32, 36]. Other statistical measures of data depth include the approximate likelihood (APL) depth [20, 21] and the transformation and retransformation estimate of depth [9, 10].

In addition to the algorithmic problem of evaluating a depth measure $f(q, P)$ for an arbitrary point q, the problem of identifying a point $q_{\max}$ of maximum depth with respect to a given set P is typically difficult. Such a point $q_{\max}$ is a *median* of P with respect to the given depth measure f, where $q_{\max} = \arg\max_{q \in \mathbb{R}^d} f(q, P)$. If, however, $q_{\max}$ is constrained to be selected from a discrete set of $Q \in \mathbb{R}^d$, then $q_{\max}$ is a *medoid* of P.

2.1. *Half-Space Counts*

Half-space counts, denoted $h_i(q)$, were introduced by Rousseeuw and Ruts [33] for bivariate depth ($d = 2$), and are fundamental to previous work in computing Tukey and simplicial depths. In Section 3 we define a summary statistic of $\langle h_1(q), \ldots, h_n(q)\rangle$ which we call the ψ histogram.

Consider a point q whose depth is queried, and for each $i \in \{1, \ldots, n\}$, let α_i denote the angle between the vectors $p_i - q$ and $(1, 0)$. Without loss of generality, assume the points $p_1, \ldots, p_n$ are sorted in angular order such that $0 \leq \alpha_1 < \ldots <$

[a]In general, it is not known how to compute the Fermat-Weber point exactly on sets of cardinality 5 or greater [4, Table 1], but it can be approximated [6, 22].

$\alpha_n < 2\pi$. This assumes that the set $P \cup \{q\}$ is in general position and therefore the angles are all unique.

For each $i \in \{1, \ldots, n\}$, the half-space count $h_i(q)$ is the largest integer such that $\alpha_i < \alpha_{i+1} \leq \alpha_{i+h_i(q)} < \alpha_i + \pi$, where $\alpha_{n+j} = \alpha_j + 2\pi$ for all j. This definition of $h_i(q)$ is equivalent to counting the number of points of P in the right open half-plane defined by the line through the points (p_i, q) and the vector $p_i - q$. Given any arbitrarily ordered set P of n points in $\mathbb{R}^2$ and any $q \in \mathbb{R}^2$, Rousseeuw and Ruts [33] also give a method for computing $h_i(q)$ in $O(n)$ time for any fixed i, and an optimal $\Theta(n \log n)$-time algorithm for computing $\langle h_1(q), \ldots, h_n(q)\rangle$. The optimality of their algorithm in the real RAM model was proven by Aloupis et al. [2].

2.2. *Tukey (Half-Space) Depth*

The Tukey depth of a point q, denoted $TD(q, P)$, is the minimum number of points of P in any closed half-space containing q on its boundary. Aloupis [1, §2.2] provides an excellent summary of Tukey depth. A Tukey median of P is a point of maximum Tukey depth. Rousseeuw and Ruts [33] express the Tukey depth of $q \in P$ in terms of $h_i(q)$ as $\min_i\{\min(h_i(q), n - h_i(q) - 1)\}$. Since their algorithm for computing all n values $\langle h_1(q), \ldots, h_n(q)\rangle$ requires only $O(n \log n)$ time for any given point q, the computation of the Tukey depth of q may be determined in the same time, which is optimal [2]. The best known algorithms for computing a Tukey median require $O(n \log^3 n)$ time (deterministic) [25] and $O(n \log n)$ time (randomized) [11]. Recently, Chen et al. [13] presented a randomized generalization of the algorithm of Rousseeuw and Ruts [33] for computing an approximation to the Tukey depth. Finally, Bremner et al. [7] show that the Tukey depth of a given point q can be computed in $O(n + k \log k)$ time, where $k = TD(q, P)$.

2.3. *Simplicial Depth*

Every subset of P of cardinality $d + 1$ determines a simplex, where d denotes the number of dimensions ($P \subseteq \mathbb{R}^d$); the simplicial depth of q, denoted $SD(q, P)$, is the number of open simplices determined by points in P that contain q (some definitions use closed simplices). Aloupis [1, §2.5] again provides a nice overview. A point of maximum simplicial depth with respect to P is a simplicial median. Aloupis et al. [2] use half-space counts to define the simplicial depth of a point in $\mathbb{R}^2$:

$$SD(q, P) = \binom{n}{3} - \sum_{i=1}^{n} \binom{h_i(q)}{2}.$$

This formulation subtracts the number of simplices that do not strictly contain q, and again requires $O(n \log n)$ time to compute the depth of a point q. If a median of P is not restricted to the set of input points, then there are $\Theta(n^4)$ combinatorially distinct regions to consider for selecting a median; the best known algorithm for computing a simplical median in this setting requires $O(n^4)$ time [3]. The only known lower bound related to the simplicial depth is $\Omega(n \log n)$ time to calculate it for a single point [2] (matching the upper bound). Elbassioni et al. [19] show that the simplicial depth of a given point q can be computed in $O(n+k)$ time, where $k = SD(q,P) \in O(n^3)$.

3. ψ Histograms

In this section we introduce a summary statistic that can be used to compute several combinatorial depth measures. Given a point q and a point set P in $\mathbb{R}^2$, we define a function $\psi(q)$ such that the Tukey depth, the simplicial depth, and the perihedral and eutomic depths of Sections 4 and 5 can all be calculated easily once $\psi(q)$ is known. Moreover, $\psi(q)$ is itself easy to calculate, and it has attractive geometric properties.

Given a set P of points, a point q, and an integer j, $\psi_j(q)$ counts the number of points p_i in P for which the directed line from q to p_i has exactly j points of P strictly on the right side. Specifically, $\psi_j(q) = \sum_{i=1}^{n} I(h_i(q), j)$, where $I(h_i(q), j)$ denotes an indicator function equal to 1 when $h_i(q) = j$ and 0 otherwise. Thus, for any q, $\psi(q) = \langle \psi_0(q), \ldots, \psi_{n-1}(q) \rangle$ can be interpreted as a histogram of the $h_i(q)$ values. Depending on the point arrangement, $\psi_j(q)$ could be non-zero for any integer value of $j \in [0, n-1]$; it is zero everywhere else. Given any point q, it is straightforward to calculate $\psi_0(q), \ldots, \psi_{n-1}(q)$ by computing $h_1(q), \ldots, h_n(q)$ in $O(n \log n)$ time; a histogram could be constructed along the way or in a separate pass requiring only linear additional time, so the overall time remains $O(n \log n)$.

If P' is a subset of P chosen uniformly at random from all 2^n subsets, the probability that a given q is interior to $CH(P')$ is given by

$$\mathbf{Pr}\big(q \text{ is interior to } CH(P')\big) = 1 - 2^{-n} - \sum_{j=0}^{n-1} \psi_j(q) 2^{-(n-j)} \,. \tag{1}$$

This is just a sum of $\psi_j(q)$ values normalized to probabilities. If we generalize it to the case where each element of P is selected as an element of P' independently with probability $\xi \in (0,1)$, then the probability that p_i is selected and that q is exterior to the selected subset is $\xi(1-\xi)^{n-1-h_i(q)}$. It follows that (1) generalizes to

$$\mathbf{Pr}\big(q \text{ is interior to } CH(P'); \xi\big) = 1 - (1-\xi)^n - \xi \sum_{j=0}^{n-1} \psi_j(q)(1-\xi)^{n-j-1} \,. \tag{2}$$

This expression leads to an interesting probabilistic intepretation of perihedral depth, as well as being useful in proving the following result.

Since $\psi_j(q)$ counts the n values of $h_i(q)$, we have that $\sum_{j=0}^{n-1} \psi_j(q) = n$. Less obvious is the symmetry of $\psi(q)$:

Theorem 1. *If P and q are in general position, then $\psi_j(q) = \psi_{n-1-j}(q)$.*

Proof. Consider the point q for which the histogram $\psi(q) = \langle \psi_0(q), \ldots, \psi_{n-1}(q) \rangle$ is to be constructed. Recall that $\psi(q)$ is the histogram of the half-space counts

$h_i(q)$, where each $h_i(q)$ is the number of points of P in the right open half-plane determined by the line through the points p_i and q, and the vector $p_i - q$. Without loss of generality, the points $p_1, p_2, \ldots, p_n$ are assumed to be sorted in angular order; see Subsection 2.1.

Now, let $g_i(q)$ denote the number of points of P in the left open half-space defined by the line through the points p_i and q, and the vector $p_i - q$. By construction, $g_i(q) = n - h_i(q) - 1$. We denote the histogram of the left half-space counts as $\Psi(q) = \langle \Psi_0(q), \ldots, \Psi_{n-1}(q) \rangle$, whose elements satisfy $\Psi_j(q) = \psi_{n-1-j}(q)$ for $j = 0, 1, \ldots, n-1$. We next prove that $\Psi_j(q) = \psi_j(q)$, implying the stated symmetry of the ψ histogram.

To see this, consider the probability that the query point q is interior to $CH(P')$ when the subset P' is constructed by selecting each point of P with probability $\xi \in (0, 1)$ independently of all others. Working from the (right) half-space counts $h_i(q)$, this probability was derived in (2) in terms of the ψ histogram. Working from the left half-space counts, the symmetry of the problem implies that

$$\mathbf{Pr}\big(q \text{ is interior to } CH(P'); \xi\big) = 1 - (1-\xi)^n - \xi \sum_{j=0}^{n-1} \Psi_j(q)(1-\xi)^{n-j-1},$$

is also valid. These two equations together imply, after letting $t = 1 - \xi$, that

$$\sum_{j=0}^{n-1} [\Psi_j(q) - \psi_j(q)] t^{n-j-1} = 0,$$

for all $t \in (0, 1)$. The left hand side of this equality is a polynomial in t that is uniformly equal to zero on the unit interval, implying that its coefficients are all zero, in turn implying that $\Psi_j(q) = \psi_j(q)$ for $j = 0, 1, \ldots, n-1$. This follows, for instance, by considering all derivatives of the polynomial. □

Another useful property of $\psi(q)$ is that the values j for which $\psi_j(q)$ is non-zero are a contiguous interval of integers.

Theorem 2. *If P and q are in general position, and $a \leq b \leq c$ are integers such that $\psi_a(q) > 0$ and $\psi_c(q) > 0$, then $\psi_b(q) > 0$.*

Proof. Sweeping a ray from q counterclockwise through a circle, pausing when it reaches each point in P to record the number of elements of P strictly to the right of the line containing the ray, gives a cyclic sequence of n integers which are the half-space counts h_i. Let $h(\theta)$ be the number of points in P strictly to the right of the line containing the ray as a function of the ray's angle. This function increases by one immediately after the ray encounters a point, and decreases by one immediately after another ray from q in the opposite direction (the other half of the line) encounters a point. General position implies that no two of these events coincide. In the complete sweep, the line encounters each point once on each side, so that $h(\theta)$ returns to its starting value. Only the values immediately after each increase are recorded in the sequence h_i.

If a, b, and c are not all distinct then the theorem is trivially true, so the interesting case is when $a < b < c$. Having $\psi_a(q)$ and $\psi_c(q)$ both non-zero means that the number of points to the right of the ray must have assumed both the values a and c during the sweep. Then because $h(\theta)$ can only increase or decrease by one at a time, it must have also assumed all integer values between a and c, notably including b. Moreover, if we choose the starting point for the cyclic sweep such that $h(\theta) = a$ is true before $h(\theta) = b$, then there must be a first time it assumes the value b after a. If we consider the first time it reaches b after reaching a, and the last time it reaches a before that, then between those points $h(\theta)$ must be strictly between a and b. The move immediately prior to $h(\theta) = b$ must be an increase. Therefore on that occasion b will be included in the h_i sequence and counted in the histogram, and so $\psi_b(q) > 0$. □

Well-known depth measures, as well as our new ones, can be expressed concisely in terms of $\psi_j(q)$. In particular, the Tukey depth of a point q with respect to P is the least number of points in P strictly on one side of a line through q. That is the least element in the h_i sequence and therefore the minimum index with a non-zero count in the histogram. This gives the following proposition.

Proposition 1. *Given a set of points $P \subseteq \mathbb{R}^2$ and a point $q \in \mathbb{R}^2$, the Tukey depth of q with respect to P can be expressed as*

$$TD(q, P) = \min\{j \mid \psi_j(q) > 0\}\,. \tag{3}$$

The simplicial depth of q in two dimensions with respect to P is the number of triangles with vertices in P that contain q, which can be found by subtracting the triangles that do not contain q from the total of $\binom{n}{3}$ triangles. To count exactly once each triangle that does not contain q, we count it in the unique h_i for which one vertex is the point p_i defining the sweep line and the other two vertices are on the right (and thus counted in h_i). Each h_i corresponds to $\binom{h_i}{2}$ of these triangles, giving a total of $\sum_{i=1}^{n} \binom{h_i}{2}$ triangles that do not include q. Since $\psi_j(q)$ is the number of values of i for which $h_i = j$, we can group terms in the sum and use the $\psi_j(q)$ values to count instead, giving the following proposition.

Proposition 2. *Given a set of n points $P \subseteq \mathbb{R}^2$ and a point $q \in \mathbb{R}^2$, the simplicial depth of q with respect to P can be expressed as*

$$SD(q, P) = \binom{n}{3} - \sum_{j=0}^{n-1} \binom{j}{2} \psi_j(q)\,. \tag{4}$$

In addition to Tukey depth and simplicial depth, other combinatorial depth measures might seem like natural candidates for being expressed in terms of $\psi(q)$, including majority depth [12, 27], convex hull peeling depth [5, 34], zonoid depth [29, 24], and ray shooting depth [30]. Unlike majority depth whose value varies outside $CH(P)$, $\psi(q)$ remains invariant for all q outside $CH(P)$. Similarly, for some point q, the convex hull peeling depth and zonoid depth at q can be altered by moving points onto the convex hull of P along rays emanating from q without changing $\psi(q)$. Consequently, none of majority depth, convex hull peeling depth, or zonoid depth can be expressed in terms of $\psi(q)$. Although it seems unlikely, it remains to be determined whether ray shooting depth is expressible in terms of $\psi(q)$. Depth measures whose definitions involve volumes (e.g., Oja depth [31]), distances (e.g., Mahalanobis depth [14, 39]), or curves (e.g., lens depth [28]) are not determined exclusively by the arrangement of $\binom{n}{2}$ lines through pairs of points in P and, consequently, cannot be expressed in terms of $\psi(q)$.

4. Perihedral Depth

Definition 1 (perihedral depth). Given a set of points $P \subseteq \mathbb{R}^2$ and a point $q \in \mathbb{R}^2$, the *perihedral* ("around the shape") *depth* of q, denoted $PD(q, P)$, is the number of subsets of P whose convex hull contains q.

We assume that P and q are in general position. An alternative definition is given by counting the number of distinct convex subsets of P whose convex hull contain q; our definition, which is not restricted to convex subsets of P, counts all distinct subsets of P whose convex hull contains q. When normalized, this has a natural interpretation as the probability that q is contained in the convex hull of a subset of P selected uniformly at random.

A simple brute force method for determining $PD(q, P)$ is to enumerate all subsets of P and count those that contain q in their convex hull. While such an algorithm would require tremendous time (there are 2^n subsets of P), it does give an idea as to the nature of the computation and the importance of using the geometric properties of P.

We compute $PD(q, P)$ without explicit enumeration by finding the number of subsets of P whose convex hull does not contain q. Consider an arbitrary subset P' of P. Note that q is not interior to $CH(P')$ if and only if it is on the boundary of $CH(P' \cup \{q\})$. Let $Q = \{P' \subseteq P \mid q \text{ is on the boundary of } CH(P' \cup \{q\})\}$. By construction, $PD(q, P) = 2^n - |Q|$. Label the points of P according to their angular ordering from the horizontal line passing through the point q. The empty set $\varnothing$ must be in Q, and any other P' uniquely determines one element of P which is the clockwise immediate predecessor of q on the boundary of $CH(P' \cup \{q\})$. For each p_i let $Q_i = \{P' \subseteq P \mid p_i \text{ precedes } q \text{ on the boundary of } CH(P' \cup \{q\})\}$. Then Q is the disjoint union of $\{\varnothing\}, Q_1, Q_2, \ldots, Q_n$. This partition of Q implies

$$PD(q, P) = 2^n - 1 - \sum_{i=1}^{n} |Q_i|.$$

By construction, for $1 \leq i \leq n$, the sets in Q_i contain only points that are to the right of the line from q to p_i. However, the number of points in P to the right of this line is just the half-space count $h_i(q)$, and so Q_i contains exactly $2^{h_i(q)}$ sets, all containing p_i but otherwise formed from those $h_i(q)$ other points. This gives the following theorem:

Theorem 3. *Given a set of n points $P \subseteq \mathbb{R}^2$ and a point $q \in \mathbb{R}^2$, the perihedral depth of q with respect to P can be expressed as*

$$PD(q, P) = 2^n - 1 - \sum_{i=1}^{n} 2^{h_i(q)} = 2^n - 1 - \sum_{j=0}^{n-1} \psi_j(q) 2^j. \tag{5}$$

Calculating $h_i(q)$ for all i gives $PD(q, P)$, which leads to (5) in $O(n)$ additional time. As discussed in Section 3, the complete list of values $\langle h_1(q), \ldots, h_n(q)\rangle$ can be computed in $O(n \log n)$ time and $O(n)$ space.

It is possible to find the maximum attainable depth, given that P contains n points in general position. This maximal depth, which is not attainable for all sets P containing n points, is given by following corollary.

Corollary 1. *The maximum achievable perihedral depth of a point $q \in \mathbb{R}^2$ with respect to a set of n points $P \subseteq \mathbb{R}^2$, n being odd, is*

$$\max_{q,\, P} PD(q, P) = 2^n - 1 - n\, 2^{(n-1)/2},$$

achieved when q and P are such that $\psi_{(n-1)/2}(q) = n$ and $\psi_j(q) = 0$ otherwise.

When n is even, the maximum achievable perihedral depth is

$$\max_{q,\, P} PD(q, P) = 2^n - 1 - 3n\, 2^{n/2-2},$$

achieved when $\psi_{n/2-1}(q) = \psi_{n/2}(q) = n/2$ and $\psi_j(q) = 0$ otherwise.

Proof. Let $\mathcal{H}_n$ denote the set of possible ψ-histograms for a point q with respect to a set of n points $P \subseteq \mathbb{R}^2$ and consider the case where n is odd.

Define the function $\Phi : \mathcal{H}_n \to \mathbb{R}$ such that

$$\Phi(\psi) = 2^n - 1 - \sum_{j=0}^{n-1} \psi_j 2^j.$$

It should be clear that

$$\max_{q,\, P} PD(q, P) = \max_{\psi \in \mathcal{H}_n} \Phi(\psi).$$

At this point, let $\psi \in \mathcal{H}_n$ be such that $\psi_j = k > 0$ for some index $0 \le j < (n-1)/2$. Assume also that j is the smallest such index. Note that Theorem 1 implies that $\psi_{n-j-1} = k$ and that it is the last nonzero component of ψ. Theorem 2, on the other hand, implies that $\psi_i > 0$ for $j \le i \le n - j - 1$.

Now, let ψ' be the histogram obtained by perturbing ψ such that

$$\psi'_i = \begin{cases} k - 1 & \text{for } i = j, n - j - 1, \\ \psi_i + 1 & \text{for } i = j + \ell, n - j - \ell - 1, \\ \psi_i & \text{otherwise,} \end{cases}$$

for some integer $1 \le \ell < (n-1)/2 - j$, and ψ^* be given by

$$\psi_i^* = \begin{cases} k-1 & \text{for } i = j, n-j-1, \\ \psi_i + 2 & \text{for } i = (n-1)/2, \\ \psi_i & \text{otherwise.} \end{cases}$$

We now show that

$$\Phi(\psi^*) > \Phi(\psi') > \Phi(\psi) \tag{6}$$

for any ℓ such that $1 \le \ell < (n-1)/2 - j$. To see this, we first note that

$$\begin{aligned} \Phi(\psi') - \Phi(\psi) &= 2^n - 1 - \sum_{i=0}^{n-1} \psi_i' 2^i - \left(2^n - 1 - \sum_{i=0}^{n-1} \psi_i 2^i \right) \\ &= -2^{j+\ell} - 2^{n-j-\ell-1} + 2^j + 2^{n-j-1} \\ &\ge 2^j + 2^{n-j-1} - 2\big(2^{n-j-\ell-1}\big) = 2^j + 2^{n-j-1} - 2^{n-j-\ell} \ge 2^j > 0, \end{aligned}$$

since $j + \ell < n - j - \ell - 1$ and $\ell \ge 1$. Also, using similar arguments, we have that

$$\begin{aligned} \Phi(\psi^*) - \Phi(\psi') &= -2(2^{(n-1)/2)}) + 2^{n-j-\ell-1} + 2^{j+\ell} \\ &= 2^{j+\ell} + 2^{n-j-\ell-1} - 2^{(n+1)/2} \ge 2^{j+\ell} > 0, \end{aligned}$$

since $j + \ell \le (n-1)/2 - 1$. The conclusion now follows from (6), which implies that Φ is maximized when $\psi_{(n-1)/2} = n$ and $\psi_j = 0$ otherwise.

The case where n is even is treated in a similar way, the main difference being in how ψ^* is defined in that case. □

We now examine a different approach for calculating $PD(q, P)$. Consider the subsets of P of cardinality k for some $k \le n$ and let $PD_k(q, P)$ denote the number of these subsets whose convex hull contains q. Observe that

$$PD(q, P) = \sum_{k=1}^{n} PD_k(q, P). \tag{7}$$

Note that $PD_3(q, P) = SD(q, P)$, the simplicial depth of q. If $PD_k(q, P) = \binom{n}{k}$, then q has Tukey depth less than k because the convex hull of every subset of size k contains q. Arguments similar to those used to derive (4) allow us to write

$$PD_k(q, P) = \binom{n}{k} - \sum_{i=1}^{n} \binom{h_i(q)}{k-1} = \binom{n}{k} - \sum_{j=k-1}^{n-1} \binom{j}{k-1} \psi_j(q)\,. \tag{8}$$

Substituting (8) into (7) gives (5), our previous form for $PD(q, P)$.

Consider that choosing k defines the size of the randomly chosen subset to k, whereas simplicial depth defines that subsets of exactly size 3 must be chosen. Then $PD_k(q, P)$, scaled to the number of subsets, is the probability that q is interior to the convex hull of any random subset P' with a cardinality of k. For $k = 3$ this is simplicial depth. However, for $k = n$ this depth becomes an indicator for

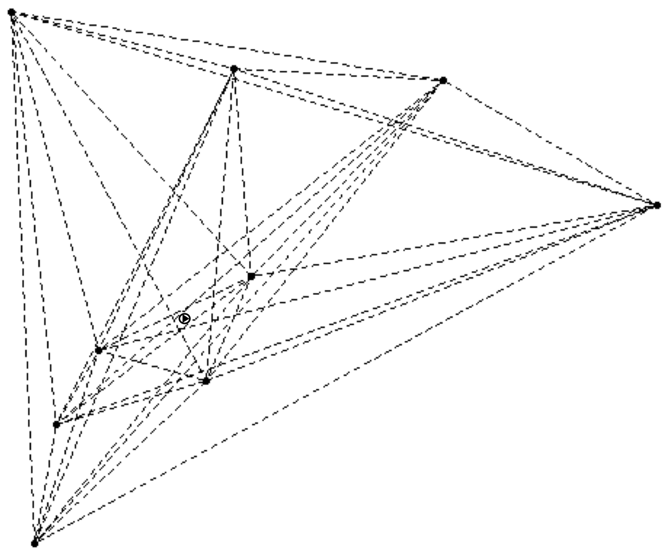

Fig. 2: An example of nine points that achieves maximum perihedral depth at the query point (triangle in circle).

being interior to the convex hull of P. More interesting is that $PD_k(q, P)$ returns a probability of 1 if and only if the point q has a Tukey depth greater than $\frac{n-k}{n}$ with respect to P, because the convex hull of every subset of size k contains q. This depth is computed using (8), which is both a generalization of simplicial depth and a specialization of perihedral depth.

For any fixed point q such that $P \cup \{q\}$ is in general position, the computation of the depths presented in this section can be done by first computing $\psi_j(q)$ for all j in $O(n \log n)$ time and $O(n)$ space by using the computational method for halfcounts presented by Rousseeuw and Ruts [33].

Computing a perihedral median, i.e., a point $\arg\max_{q \in \mathbb{R}^2} PD(q, P)$, is significantly more difficult than computing the depth of an arbitrary query point. The fastest known algorithm for computing a simplicial median ($\arg\max_{q \in \mathbb{R}^2} PD_3(q, P)$) requires $\Theta(n^4)$ time in the worst case [3]. Essentially a similar algorithm can also compute a perihedral median in $O(n^4)$ time. Since the definition of halfspace counts by Rousseeuw and Ruts [33] is combinatorial in nature and therefore depends only on the arrangement of all the lines connecting pairs of points in P, the ψ histogram is also constant within each cell. When we step across an edge l in the arrangement, the h_i sequence changes by (at most) the two elements corresponding to the points that determine l. That constant-sized change corresponds to changing at most four elements in the ψ histogram, and then in constant time we can compute the resulting change in the perihedral depth. The arrangement is of size $O(n^4)$ because it is determined by the pairwise intersections of $\binom{n}{2}$ lines, and we can traverse it in $O(n^4)$ steps while maintaining the h_i sequence, the ψ histogram, and the perihedral depth in constant time at each step.

Theorem 4. *Given a set of n points $P \subseteq \mathbb{R}^2$ and a point $q \in \mathbb{R}^2$, the perihedral depth of q may be computed in $O(n \log n)$ time using $O(n)$ space, and a perihedral median of P can be found in $O(n^4)$ time using $O(n^2)$ space.*

As the information required is identical in content and differs by a constant in complexity, the median for the $PD_k(q, P)$ depths can also be computed in $O(n^4)$ time and $O(n^2)$ space. Note that for values of $k > 2n/3$, the algorithm of Chan [11] computes a Tukey median in $O(n \log n)$ expected time. The difference in computation of medians for different parameters is remarkable and suggests that some improvement might be possible. On the other hand, the fastest known algorithm for computing a simplicial median requires $O(n^4)$ time [3].

The expression (1) gives a normalized formulation for perihedral depth, i.e., one for which the depth of every point lies in $(0, 1)$. The generalized version (2) suggests a parameterized family of normalized perihedral depths, and a possible link to applications in which the question of a point being inside a random convex hull may be relevant.

5. Eutomic Depth

Given a set P of n points in $\mathbb{R}^2$, a line L is a *halving line* if it partitions P into two sets whose cardinalities differ by at most one. This definition implies that for any point q such that P and q are in general position, the line determined by q and any point $p_i \in P$ (which we denote L_i) is a halving line of $P \setminus \{p_i\}$ if and only if n is odd and $h_r(q) = (n-1)/2$ or n is even and $h_i(q) \in \{n/2 - 1, n/2\}$.

Definition 2 (eutomic depth). Given a set of n points $P \subseteq \mathbb{R}^2$ and a point $q \in \mathbb{R}^2$, the *eutomic* ("good cutting") *depth* of q, denoted $ED(q, P)$, is the number of halving lines of P, among $L_1, \ldots, L_n$. That is,

$$ED(q, P) = \begin{cases} \psi_{(n-1)/2}(q) & \text{if } n \text{ is odd} \\ \psi_{n/2-1}(q) + \psi_{n/2}(q) & \text{if } n \text{ is even.} \end{cases} \tag{9}$$

Consequently, if the Tukey depth of a point q is $\lfloor n/2 \rfloor$ (implying that q is a Tukey median), then q is also a eutomic median because the only non-zero ψ values are used in computing $ED(q, P)$. Generally, the Tukey median can have Tukey depth less than $\lfloor n/2 \rfloor$, and the eutomic and Tukey medians may differ.

Computing $ED(q, P)$ is achieved by calculating $h_i(q)$ for all i, giving (9). Therefore the eutomic depth of any point q can be computed in $O(n \log n)$ time and $O(n)$ space. Finding a eutomic median, however, is significantly more difficult. First we briefly examine how eutomic depth is related to simplicial and Tukey depths. If the maximal Tukey depth for P is $\lfloor n/2 \rfloor$ then a eutomic median is any Tukey median and vice versa, because all splitting planes are halving lines. Although an analogous approach to that used for finding simplicial and perihedral medians provides a $O(n^4)$-time algorithm for finding a eutomic median, we describe a more efficient algorithm that runs in $O(n^{8/3})$ time.

First, find the set of all *near-halving* lines passing through two points of P. Specifically, find all pairs (p_a, p_b), $a \neq b$, such that

$$h_a(p_b) \in \begin{cases} \{(n-3)/2, (n-1)/2\} & \text{if } n \text{ is odd,} \\ \{n/2-2, n/2\} & \text{if } n \text{ is even.} \end{cases}$$

These pairs can all be identified in $O(n^2 \log n)$ time and $O(K_n)$ space, where K_n is the number of such pairs, each corresponding to a near-halving line. From Dey [15] we know that $K_n \in O(n^{4/3})$. This is equivalent to finding the lines between points of P that define change in the most central portions of the ψ histogram.

Consider the arrangement of lines defined by points in P. As mentioned in Section 4 the ψ histogram is constant throughout each cell, i.e., it can only change when crossing a line. Furthermore, since the eutomic depth of a point q is defined by the central values of the ψ histogram, only the lines across which the central values change are relevant. Those are exactly the near-halving lines. Where p_a and p_b are the points defining a near-halving line, the half-counts h_a and h_b both change by one between query points q and q' on either side of the segment between p_a and p_b, resulting in a change of two in eutomic depth on crossing the segment (if p_a and p_b both move into or out of the central bin or bins) or no change (if p_a and p_b exchange places between the two central bins, possible only in the case where n is even). Therefore the arrangement of near-halving lines partitions the plane into cells, each of which is a locus of points that have the same $\psi_{\lfloor (n-1)/2 \rfloor}(q)$ and $\psi_{\lfloor n/2 \rfloor}(q)$ values and hence the same eutomic depth.

The arrangement of $O(n^{4/3})$ near-halving lines partitions the plane into $O(n^{8/3})$ cells, and can be constructed in $O(n^{8/3})$ time using $O(n^{8/3})$ space [18]. To check the depth of any single point q in the arrangement requires $O(n \log n)$ time and $O(n)$ space. To identify a eutomic median of P it suffices to traverse the arrangement to find a cell of maximum depth. When moving from one cell to a neighbouring cell, the eutomic depth changes by either -2, 0, or 2, depending on the point at which the near-halving line is crossed. For each near-halving line we store the two points of P that define it and the number of points of P on its right side. Thus, a eutomic median can be computed in time proportional to the size of the arrangement, i.e., $O(n^{8/3})$ time and $O(n^{8/3})$ space.

Theorem 5. *Given a set of n points $P \subseteq \mathbb{R}^2$ and a point $q \in \mathbb{R}^2$, the eutomic depth of q may be computed in $O(n \log n)$ time, and a eutomic median of P can be found in $O(n^{8/3})$ time and $O(n^{8/3})$ space.*

6. Discussion and Directions for Future Research

Corollary 1 describes the special form of the ψ histogram that maximizes the perihedral depth. If a set P of n points in general position allows some query point (necessarily a perihedral median) to attain that perihedral depth, then that query point must have the ψ histogram described with all the weight concentrated in the central one (for odd n) or two (for even n) entries. It follows trivially from the symmetry and sum properties of ψ histograms that this same histogram uniquely maximizes the Tukey and eutomic depths. We also know from the properties of Tukey medians [7] that the set of all Tukey medians is convex. Therefore we have the following corollary, which links all three depth measures.

Corollary 2. *Let P be a set of n points in general position such that one of Tukey, perihedral, or eutomic depth at some point q is maximized over all q and all sets P of n points in general position. Then q is also a median of P in the other two depths; such medians maximize all three depths over sets P of n points in general position; and the set of medians is convex.*

Examples like those of Figure 1 show that the medians are not always the same among all three depths, but it remains open that there may be useful relationships among them in other cases beyond the global maximum of Corollary 2.

Table 1 compares properties by which depth measures are commonly evaluated. These include that a median occurs at the centre of symmetry when P is centrally symmetric (P1); the depth is invariant under affine transformations (P2); the depth approaches zero as q moves away from P (P3); the depth is non-increasing along any ray rooted at a median (P4); the contour lines bounding adjacent regions of different depth are convex (P5); the depth of q is equal under the k- and d-dimensional definitions of the depth measure when P lies in a k-flat of $\mathbb{R}^d$ for all $k < d$ (P6); and a depth measure's breakdown point, i.e., the fraction α of P that must be displaced before the median moves away from the unperturbed points of P (P7). Note that P1 is not required to hold under other symmetries, e.g., rotational symmetry.

Although P1–P7 are commonly evaluated for depth measures, these provide only a limited classification of depth measures (e.g., the trivial function $f(q, P) = 1$ if q is in the convex hull of P and 0 otherwise satisfies properties P1–P6). Furthermore, convexity and monotonicity arguably limit the ability of a depth measure to express a refined characterization of the relative position of a query point within a point set. Thus, while these properties help compare aspects of different depth measures, this classification alone is insufficient to quantify a given depth measure's ability to provide a good high-dimensional median.

By definition of central symmetry about a point q', every line through q' and some point $p \in P$ is a halving line. Therefore, P1 holds for eutomic depth. P2 follows immediately from the fact that convex hulls and halving lines are invariant under affine transformations. P3 is also straightforward (although in the case of eutomic depth, the minimum depth attainable is 1 instead of 0). Like simplicial

	Properties	Simplicial	Tukey	Perihedral	Eutomic
P1	median at centre of symmetry	unknown†	✓[35]	unknown	✓
P2	affine invariance	✓[39]	✓[1]	✓	✓
P3	vanishing at ∞	✓[39]	✓[1]	✓	✓
P4	monotonicity relative to median	×[39]	✓[7]	×	×
P5	convexity of depth contours	×[39]	✓[7]	×	×
P6	consistency across dimensions	×	✓	✓	✓
P7	breakdown point	unknown	1/3 [1]	unknown	unknown

Table 1: Comparing Properties of Depth Measures
† Some previous results evaluate P1 for simplicial depth with respect to rotational [8] or other symmetries [39], instead of central symmetry.

medians, the set of eutomic medians is not necessarily a convex set (and, therefore, not monotonic). A well-known example [8] is to position a cluster of $n/3$ points near each of the vertices of an equilateral triangle. The maximum simplicial, perihedral, and eutomic depths are typically near the clusters and comprise at least three disconnected sets. Therefore, neither P4 nor P5 hold for perihedral or eutomic depths. P6 follows from the generalizations of the definitions of perihedral and eutomic depths to higher dimensions (see below).

The definitions of perihedral depth and eutomic depth should be straightforward to generalize to sets of points that are not in general position and to arbitrary data sets that include collocated points. The generalization of perihedral depth to d dimensions follows immediately by counting the subsets of P whose d-dimensional convex hull contains the query point q. Similarly, the generalization of eutomic depth to d dimensions follows by counting the halving hyperplanes of P through q. The algorithms described in this paper apply to $P \subseteq \mathbb{R}^2$; it remains to examine how to efficiently compute the depth of an arbitrary query point q, and how to find a median of a given set $P \subseteq \mathbb{R}^d$ with respect to these new depth measures.

Suppose the point set P is known ahead of a sequence of point depth queries for perihedral or eutomic depth. Instead of computing the depth of each query point independently, requiring $O(n \log n)$ time per query, a query data structure could be constructed on P to provide more efficient depth queries subsequently. The usual time-space trade-off applies, where a larger data structure permits faster query time. Thus, the respective arrangements of all lines determined by pairs of points in P and all near-halving lines of P can be stored using $O(n^4)$ and $O(n^{8/3})$ space, and paired with a point-location query data structure (e.g., Kirkpatrick's planar subdivision algorithm [23]) to support arbitrary depth queries in $O(\log n)$ time per query. The space can be reduced slightly by stopping the recursion in Kirkpatrick's algorithm when each triangular region contains $O(\log n)$ triangles of the arrangement. A simple table lookup suffices once the query point is located within the arrangement. In particular, $O(\log n)$ computation time is available without any increase in query time. The space requirements for these data structures are likely prohibitively large for many applications. Determining structures which use improved space (e.g., quadratic or linear in n) is an interesting direction for future research.

As mentioned in Section 3, depth measures whose definitions involve volumes, distances, or curves cannot be expressed in terms of $\psi(q)$. Furthermore, none of majority depth, convex hull peeling depth, or zoonoid depth can be expressed in terms of $\psi(q)$. It remains to be determined whether ray shooting depth (a combinatorial depth measure) is expressible in terms of $\psi(q)$.

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