A Note on Minimum-Segment Drawings of Planar Graphs

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Abstract

A straight-line drawing of a planar graph G is a planar drawing of G such that each vertex is mapped to a point on the Euclidean plane, each edge is drawn as a straight line segment, and no two edges intersect except possibly at a common endpoint. A segment in a straight-line drawing is a maximal set of edges that form a straight line segment. A k-segment drawing of G is a straight-line drawing of G such that the number of segments is at most k. A plane graph is a fixed planar embedding of a planar graph. In this paper we prove that it is NP-hard to determine whether a plane graph G with maximum degree four has a k-segment drawing, where $k \geq 3$. The problem remains NP-hard when the drawing is constrained to be convex. We also prove that given a partial drawing Γ of a plane graph G, it is NP-hard to determine whether there exists a k-segment drawing of G that contains all the segments specified in Γ, even when G is outerplanar. The problem remains NP-hard for planar graphs with maximum degree three in $\mathbb{R}^3$ when given subsets of the vertices are restricted to be coplanar. Finally, we investigate a worst-case lower bound on the number of segments required by straight-line drawings of arbitrary spanning trees of a given planar graph.

1   Introduction

A graph is *planar* if it admits an embedding in the plane without edge crossings. A *plane graph* G is a fixed planar embedding of the underlying planar graph, which can be specified by the cyclic ordering of the incident edges at each vertex of G and the outer face of G. A *straight-line drawing* Γ of G is an embedding of G in the Euclidean plane such that each vertex is mapped to a distinct point, each edge is drawn as a straight line segment, and no two edges intersect except possibly at a common endpoint. A *segment* of Γ is a maximal set of edges in Γ that form a straight line segment. Γ is called a *k-segment drawing* of G if the number of segments in Γ is at most k. Γ is called a *minimum-segment drawing* of G if the number of segments in Γ is the minimum among all possible straight-line drawings of G. A *k-segment drawing* is *convex* if all the faces in the drawing are convex polygons. Figure 1(a) depicts a plane graph G, Figure 1(b) depicts its straight-line drawing with thirteen segments, and Figure 1(c) shows a minimum-segment drawing of G with seven segments, which is also convex.
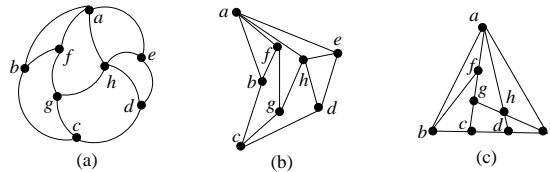

Figure 1: (a) A plane graph G. (b) A straight-line drawing of G. (c) A minimum-segment drawing of G.

Straight-line drawings of plane graphs are aesthetically pleasing. We can sometimes further enhance the beauty of a straight-line drawing by minimizing the number of segments in the drawing because a minimum-segment drawing sometimes captures properties related to a graph's symmetry and its induced path covers. A drawing that reflects symmetry information helps us to visually identify the constituent components and characteristics of the underlying graph [16, 17]. Minimum-segment drawings sometimes produce symmetric drawings as shown in Figures 2(a)–(c).

The induced path vertex cover number (respectively, induced path edge cover number) of a graph G is the smallest number ρ such that the vertices (respectively, edges) of G can be partitioned into ρ sets, where each set induces a path in G. Many researchers have studied various types of path covering problems due to theoretical interest [20, 21] and applications in communication networks [14], program testing [24] and vehicle routing [28]. Since every segment in a straight-line drawing is an induced path, the number of segments in a minimum-segment drawing gives an upper bound on the induced path edge cover number of the corresponding graph.

Dujmović *et al.* [5] presented a comprehensive study on drawings of planar graphs with few segments. They showed that $\eta/2$ segments are necessary
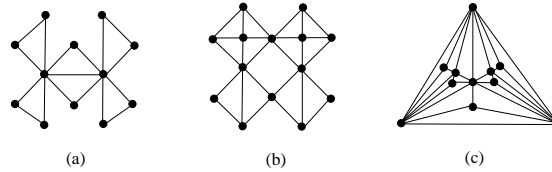
Figure 2: Three minimum-segment drawings reflecting symmetry information.

and sufficient for a straight-line drawing of a tree, where η is the number of odd degree vertices in the tree. They gave optimal bounds on the number of segments in straight-line drawings of outerplanar graphs, plane 2-trees and plane 3-trees, as well as polynomial-time algorithms for constructing straight-line drawings of planar 3-connected graphs with at most $5n/2$ segments, where n is the number of vertices. Later, Samee *et al.* [26] gave a linear-time algorithm for computing minimum-segment drawings of series-parallel graphs with maximum degree three. Recently, Biswas *et al.* [2] gave a linear-time algorithm to obtain minimum-segment convex drawings of 3-connected cubic plane graphs.

A natural question is: What is the time complexity of computing a minimum-segment drawing of a planar graph [2]? Dujmović *et al.* [5] posed the following related questions:

(a) Is there a polynomial-time algorithm to compute a minimum-segment drawing of an outerplanar graph?

(b) What is the minimum c such that every n-vertex planar graph has a plane drawing with at most $cn + O(1)$ segments?

We prove that the problem of determining a minimum-segment drawing is NP-hard for graphs with maximum degree four, even when the drawing is constrained to be convex. This suggests the above questions of Dujmović *et al.* could be potentially difficult to answer.

De Fraysseix and de Mendez [4] examined the following problem. Given a set P of paths in a connected planar graph G such that no two paths have any internal vertex in common, decide whether G admits a straight-line drawing such that each path in P is contained in a segment. They gave necessary and sufficient conditions to solve the problem in polynomial time. We study the case when the paths in P are allowed to cross, but are restricted to form a segment in the output drawing. We prove that even if G admits such a straight-line drawing, it is NP-hard to find such a drawing with the minimum number of segments. We consider the paths in P as pre-determined segments that have been specified by giving a partial drawing of G as input. We prove that the problem is NP-hard for outerplanar graphs. We then show that the problem remains NP-hard for planar graphs with maximum degree three in $\mathbb{R}^3$ when given subsets of the vertices are restricted to be coplanar.

In many applications a graph is drawn to emphasize one of its spanning trees, and the other edges are displayed on demand [15, 23, 29]. Given an

arbitrary spanning tree, one may want to draw it with the minimum number of segments, where the edges that are not in the spanning tree are to be drawn with polylines or curves. Given a planar graph G, we investigate a worst-case lower bound on the number of segments required by straight-line drawings of arbitrary spanning trees of G. For this purpose, we introduce a new graph parameter for planar graphs, which we call "segment complexity" and define as follows. The *segment complexity* of a planar graph G is the minimum positive integer C such that every spanning tree of G admits a drawing with at most C segments. Observe that any lower bound on C is a lower bound on the number of segments required by straight-line drawings of those spanning trees of G that determine the spanning-tree segment complexity of G.

Results: We briefly describe the main results shown in this paper.

(1) In Section 3, we prove that given an arbitrary integer $k \geq 3$, it is NP-hard to decide if a given plane graph with maximum degree four has a k-segment drawing. In Section 4, we show that the problem remains NP-hard when the drawing is constrained to be convex.

(2) In Section 5, we prove that given a partial drawing Γ of a plane graph G, it is NP-hard to determine a k-segment drawing of G that contains all the segments specified in Γ, even when G is outerplanar. In Section 6, we show that the problem remains NP-hard for planar graphs with maximum degree three in $\mathbb{R}^3$, if given subsets of the vertices are restricted to be coplanar.

(3) In Section 7, we derive lower bounds on segment complexities of different classes of planar graphs. See Table 1.

Graph Class	Lower Bound on C
Maximal outerplanar	$n/6$
Plane 2-tree	$n/6$
Plane 3-tree	$(2n-5)/6$
Plane 3-connected	$n/8$
Plane 4-connected	$n/5$

Table 1: Lower bounds on the segment complexity C. Here n denotes the number of vertices.

The rest of this paper is organized as follows. Section 2 gives some definitions and presents some preliminary results. Sections 3–7 establish the main results of the paper as described in (1)–(3) above. Finally, Section 8 concludes the paper with some open questions.

2 Preliminaries

Here we introduce some preliminary definitions.

Let $G = (V, E)$ be a connected simple graph with vertex set V and edge set E. Let v be a vertex in G. We denote the degree of v by $\deg(v)$. Let k be the minimum positive integer such that there exists a set of k vertices in G whose removal results in a disconnected graph or a single-vertex graph. Then G is called *k-connected*. An *independent set* S is a subset of V, such that no two vertices of S are adjacent in G.

A plane graph partitions the plane into connected regions, called *faces*. The unbounded face is called the *outer face* and all other faces are called the *inner faces*. The vertices on the boundary of the outer face are called the *outer vertices* and all other vertices are called the *inner vertices*. A *maximal planar graph* is a planar graph such that the addition of any edge results in a nonplanar graph.

An *outerplanar graph* is a planar graph that admits a plane embedding, where all its vertices are on the outer face. We call such an embedding an *outerplanar embedding*. An outerplanar graph G is called a *maximal outerplanar graph* if no edge can be added to G so that outerplanarity is preserved.

As in, for example [8], we define a *pseudoline* to be a simple (i.e., non-self-intersecting) curve in $\mathbb{R}^2$ that is infinite in both directions and whose removal leaves two unbounded connected components. An *arrangement* $A(L)$ of a set L of n pseudolines is the subdivision of the plane induced by L, where the vertices are the intersection points of the pseudolines and each pair of pseudolines has a unique point of intersection, where the two pseudolines cross. As in, for example [9], we say an arrangement $A(L)$ is *simple* if no three pseudolines share a common point. From now on, we assume all pseudoline arrangements are simple and that all graphs therefore have maximum degree 4. Given an arrangement $A(L)$ of pseudolines, we define the *pseudo-arrangement graph of $A(L)$* as follows. The vertices correspond to the intersection points of the pseudolines and two vertices are adjacent in the graph if and only if they are adjacent along some pseudoline. A *pseudo-arrangement graph* is a graph that arises from some arrangement $A(L)$ of pseudolines. An *arrangement graph* is a graph that arises from some arrangement $A(L)$ of pseudolines, where all the pseudolines in $A(L)$ are straight lines. See Figure 3.
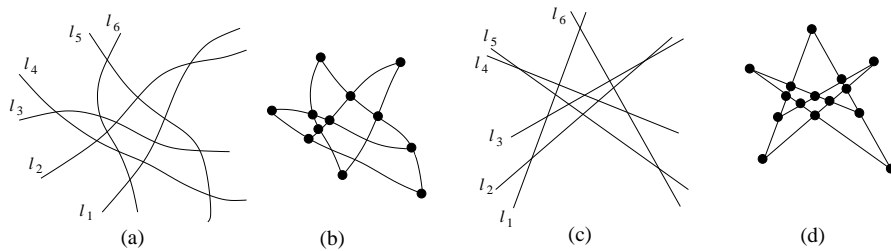

Figure 3: (a) An arrangement of six pseudolines. (b) Their pseudo-arrangement graph. (c) An arrangement of six straight lines. (d) Their arrangement graph.

We now give a characterization of arrangement graphs in terms of their

drawability, which follows easily from Bose et al. [3].

Lemma 1 *A graph G is an arrangement graph of a set of $l \geq 3$ straight lines if and only if G admits a straight-line drawing Γ such that:*

1. *Each segment contains $l - 2$ edges.*

2. *The degree of each vertex in Γ is at most four, and all vertices of degree two and degree three in Γ are on the outer face.*

3. *Each vertex of degree two is the endpoint of exactly two segments, and each vertex of degree three is the endpoint of exactly one segment. No segment has an endpoint that is a vertex of degree four.*

4. *The number of segments is $l = n_2 + n_3/2$, where n_2 and n_3 are the number of vertices of degree two and degree three, respectively.*

Proof: Assume first that G is an arrangement graph. Then by definition, G admits a straight-line drawing Γ' with l segments. We now verify that Γ' satisfies Properties 1–4, and hence it is the required straight-line drawing Γ. Bose et al. [3] observed that Γ' can be decomposed into l edge disjoint paths, corresponding to the straight lines of the underlying arrangement, each of length $l-1$. Hence, Property 1 follows. Since we consider only simple arrangements, the degree of each vertex in Γ' is at most four. Bose et al. [3] proved that all vertices of degree two or three must be on the outer face of Γ', which implies Property 2. Finally, Properties 3–4 are direct consequences of Proposition 1 of Bose et al. [3].

Now let Γ be a straight-line drawing of G that satisfies Properties 1–4. By Property 2, all the vertices of degree two are on the outer face, and by Property 3, all those vertices are the endpoints of exactly two segments. Therefore, no segment has an inner vertex of degree two. By Property 1, every segment has exactly $l-2$ edges. Since there is no inner vertex of degree two on a segment, each inner vertex is the intersection point of more than one segments as well. Since l segments can have at most $l(l-1)/2$ intersection points, Γ contains all possible intersections of its l segments and no additional intersections are created if we extend each segment of Γ to a 2-way infinite straight line. Thus, the set of intersection points of this arrangement of l straight lines determines the graph G, implying that G is an arrangement graph of a set of l lines.   □

We define an *arrangement drawing* to be a straight-line drawing that satisfies Properties 1–4. Since every face of an arrangement drawing can be defined as the intersection of a set of half-planes, we have the following.

Observation 1 *Every inner face in an arrangement drawing is convex.*

Let D be a straight-line drawing of a connected planar graph. A path in D is an *outer path* if it lies on the outer face. Let $v_1, v_2, \ldots, v_k$ be an outer path of D, where the vertices are in clockwise order. We call $\angle v_{i-1} v_i v_{i+1}$, $1 < i < k$, a *concave corner at v_i* if the angle $\angle v_{i-1} v_i v_{i+1}$ on the outerface is more than $180°$. We call the path $v_1, v_2, \ldots, v_k$ *convex* if no $v_i, 1 < i < k$, contains a concave corner. We call the path $v_1, v_2, \ldots, v_k$ *straight* if for each $v_i, 1 < i < k$, the angle $\angle v_{i-1} v_i v_{i+1}$ on the outerface is equal to $180°$.

3   Minimum-Segment Drawings

In this section we prove that it is NP-hard to decide whether a plane graph has a straight-line drawing with a given number of segments. We first need the following two lemmas.

Lemma 2 *Let G be a graph with $l(l-1)/2$ vertices and $l(l-2)$ edges, where $l \geq 3$, such that the degree of each inner vertex of G is four and all the vertices of degree two and degree three are on the outer face. Let the number of degree two and degree three vertices be n_2 and n_3, respectively. Then G is an arrangement graph if and only if G admits a straight-line drawing Γ with l segments, where $l = n_2 + n_3/2$.*

Proof: By Lemma 1, if G is an arrangement graph, then G admits a drawing with $l = n_2 + n_3/2$ segments.

We thus assume that Γ is a straight-line drawing of G with l segments and then prove that Γ is an arrangement drawing of G. By Lemma 1, this will imply that G is an arrangement graph. Since Properties 2 and 4 of Lemma 1 are satisfied by Γ already, it suffices to prove that Γ satisfies Properties 1 and 3.

We first prove that Γ satisfies Property 3 of Lemma 1. The number of vertices in G is $l(l-1)/2$ and the number of segments in Γ is l. Thus, Γ contains all pairwise intersections of the l segments and each vertex v in Γ must be an intersection point of two different segments. Consequently, if $\deg(v) = 4$, then v cannot be an endpoint of any of those two different segments. Similarly, if $\deg(v) = 3$, then v is the endpoint of one of those two different segments. If $\deg(v)$ is two, then v must be a common endpoint of those two different segments.

We then prove that Γ satisfies Property 1 of Lemma 1. Suppose for a contradiction that there exists a segment s that contains at least $l-1$ edges. Since s cannot have an inner vertex of degree two, then s must be intersected by at least l other lines. Thus the number of segments in Γ is at least $l+1$, a contradiction. Thus each segment contains at most $l-2$ edges. Since the number of edges in Γ is $l(l-2)$ and there are l segments, each segment contains exactly $l-2$ edges, which proves the property. $\square$

Lemma 3 *An arrangement drawing of an arrangement graph G is a minimum-segment drawing of G.*

Proof: Let n_2 and n_3 be the number of vertices of degree two and degree three, respectively. From the proof of Lemma 2 we can observe that G admits an arrangement drawing D in which the number of segments is $l = n_2 + n_3/2$. Suppose for a contradiction that D is not a minimum-segment drawing of G. Then some drawing D' of G has l' segments, where $l' < l$. Let $l = l' + k$.

In any drawing of G, the vertices of degree three are endpoints of at least $n_3/2$ segments. Therefore, there exist vertices of degree two in D' that are not an endpoint of any segment. Let p be the number of such degree two vertices.

The number of intersection points in D' is at most $l'(l'-1)/2$. Since G is an arrangement of l segments, the upper bound on p is l. In any straight-line drawing of G, at least three vertices on the outer face must have concave corners. We now consider the following two cases depending on the characteristics of these corners.

(1) All the concave corners have vertices of degree two. Then $p \leq l-3$ and the number of vertices in D' is at most $l'(l'-1)/2 + l - 3$.

(2) At least one concave corner has a vertex of degree three. Since it is a concave corner, the angle between any two edges at that point cannot be $180°$. Therefore, at least three segments have a common intersection point, and the number of intersection points in D' is at most $l'(l'-1)/2 - 2$ and $p \leq l$. Hence, the number of vertices in D' is at most $l'(l'-1)/2 - 2 + l = l'(l'-1)/2 + l - 2$.

Therefore, the number of vertices in D' is at most $l'(l'-1)/2 + l - 2 = l'(l'-1)/2 + l' + k - 2$.

The number of vertices in D is $l(l-1)/2 = (l'+k)(l'+k-1)/2 = l'(l'-1)/2 + (2l'k+k^2-k))/2 = l'(l'-1)/2 + l'k + k(k-1)/2$. Since $l'+k-2 < l'k + k(k-1)/2$ for any $0 < k, l' < l$, the number of vertices in D' is therefore less than the number of vertices in D, a contradiction. $\square$

Observe that the drawing D' in the proof of Lemma 3 cannot be a drawing of G even when G is a pseudo-arrangement graph. Therefore, any straight-line drawing of a pseudo-arrangement graph G with $l(l-1)/2$ vertices contains at least l segments. By Lemma 2, if a pseudo-arrangement graph G admits a straight-line drawing with exactly l segments, then G is an arrangement graph. Consequently, we have the following corollary.

Corollary 1 *Let G be a pseudo-arrangement graph of $l(l-1)/2$ vertices that is not an arrangement graph. Then any straight-line drawing of G contains more than l segments.*

We are now ready to prove that it is NP-hard to decide whether a plane graph admits a straight-line drawing with a given number of segments. A formal definition of the problem is as follows.

Problem : MIN-SEG-DRAW

Instance : A plane graph G and an integer $k \geq 3$.

Question : Is there a straight-line drawing Γ of G with at most k segments?

We reduce an NP-hard problem, ARRANGEMENT-GRAPH-RECOGNITION [3], to MIN-SEG-DRAW.

Problem : ARRANGEMENT-GRAPH-RECOGNITION

Instance : A plane pseudo-arrangement graph G such that the degree of each inner vertex of G is four and all the vertices of degree two and degree three are on the outer face.

Question : Is G an arrangement graph?

We now show the following theorem.

Theorem 1 Min-Seg-Draw *is NP-hard.*

Proof: To prove the problem is NP-hard we reduce Arrangement-Graph-Recognition to Min-Seg-Draw. Let G be an instance of Arrangement-Graph-Recognition. We take $G' = G$ as an instance of Min-Seg-Draw and define k such that $k(k-1)/2 = n$, where n is the number of vertices in G.

By Lemma 2, Lemma 3 and Corollary 1, G' is an arrangement graph if and only if G' admits a straight-line drawing with at most k segments. Therefore, the answer to the instance of Min-Seg-Draw is the answer to the instance of Arrangement-Graph-Recognition. $\square$

It is not clear whether Min-Seg-Draw is in NP. If there is a k-segment drawing of G, then we do not know whether it (or any other k-segment drawing of G) can be described to a Turing machine in a number of bits that is polynomial in the size of the input instance. Proving membership in NP does not appear to be straightforward since Goodman et al. [13] showed that there are pseudo-arrangement graphs of l lines whose arrangement drawings require 2^{cl} bits for coordinate representation, where $c > 0$ is a fixed constant.

Before ending this section, it is worth mentioning that k-segment drawings have a connection with layered polyline drawings of planar graphs [6]. In a *layered polyline drawing* of a planar graph, each vertex is drawn on a horizontal line called a layer and each edge is drawn as a polygonal chain whose bends lie on the layers of the drawing. Every arrangement of k lines admits a layered polyline drawing (also known as wiring diagram) on $O(k)$ layers [12]. Since the extension of the segments in a k-segment drawing creates an arrangement of k lines, every graph that admits a k-segment drawing, admits a layered polyline drawing on $O(k)$ layers.

4   Minimum-Segment Convex Drawings

In this section we examine the problem of computing minimum-segment convex drawings.

Recently, Biswas *et al.* [2] gave a linear-time algorithm to obtain minimum-segment convex drawings of 3-connected cubic (also known as 3-regular) plane graphs. Here we prove that the problem of computing a minimum-segment convex drawing is NP-hard for plane graphs with maximum degree four. The decision version of the problem is as follows.

Problem : Min-Seg-Convex

Instance : A plane graph G and an integer $k \geq 3$.

Question : Is there a convex drawing Γ of G with at most k segments?

We prove the NP-hardness of MIN-SEG-CONVEX by reducing the ARRANGEMENT-GRAPH-RECOGNITION problem in a similar way as in Theorem 1. However, we now perform some modifications in the reduction so that the minimum-segment convex drawing contains an arrangement drawing. We first need the following lemma.

Lemma 4 *Let G be an arrangement graph with $l(l-1)/2$ vertices, where $l \geq 3$. Let D be an arrangement drawing of G. Let $\sigma = (v_1, v_2, \ldots, v_m)$ be the vertices of degree two on the outer face of D in clockwise order. Then D satisfies the following properties.*

(a) The outer path in D between any two consecutive vertices v_i and v_j of σ is either straight or convex.

(b) Every vertex of the convex hull[1] of D is a vertex of degree two in G.

Proof: Assume that the outer path $v_i(= w_1), w_2, \ldots, w_{q-1}, v_j(= w_q)$ is neither straight nor convex. Then the outer path must contain a vertex $w_x, 1 < x < q$, that contains a concave corner. See Figure 4(a). Since v_i and v_j are two distinct vertices of degree two that are consecutive in the sequence $v_1, v_2, \ldots, v_m$, therefore $deg(w_x) = 3$. Since w_x contains a concave corner and $deg(w_x) = 3$, the point corresponding to w_x must be an intersection point of three lines, which contradicts that G is a simple arrangement graph.

We now prove that every point on the boundary of the convex hull of D is a vertex of degree two. Suppose for a contradiction that w is a vertex on the boundary of the convex hull, where $deg(w) \geq 3$. Then vertex w must contain a concave corner. Consequently, the point corresponding to w must be an intersection point of three segments, which contradicts that G is a simple arrangement graph. □

A *wedge W with respect to an angle $\theta, 0 < \theta < 180°$*, is the closed region of the plane bounded by two non-collinear rays that determine θ. Its *complementary wedge* is the wedge with respect to the opposite angle of θ. Let H be a convex polygon such that no three vertices are collinear. Let $h_1, h_2, \ldots, h_m$ be the vertices of H in clockwise order. We denote by W_i a wedge at vertex $h_i, 1 \leq i \leq m$, which is determined by edges incident to h_i. A ray r_i emanating from h_i is *outgoing* if r_i is contained in the complementary wedge of W_i. Note that we allow r_i to lie on the boundary of its corresponding wedge. We call h_i the *origin* of r_i. Let p be the intersection point of the two straight-lines containing two such rays r_i and r_j, $i \neq j$. The *angle $\theta(r_i, r_j)$ between r_i and r_j* is the clockwise angle through which r_i must be rotated around p to make it coincide with r_j. See Figure 4(b).

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[1] We do not regard interior points of convex hull edges as vertices of the hull, even though they may be vertices of the drawing.
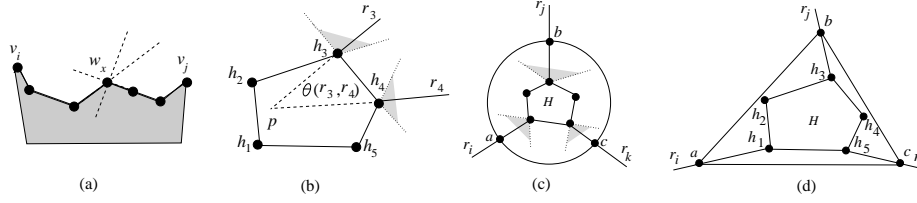
Figure 4: (a) Illustration for the proof of Lemma 4. (b) Illustration for wedge, complementary wedge and angle between two outgoing rays r_3 and r_4. Complementary wedges of h_3 and h_4 are shown in light-gray. (c) Example of a covering triple (r_i, r_j, r_k) and a covering triangle abc.

Observation 2 *Let H be a convex polygon such that no three vertices are collinear. Let r_i, r_j and r_k be three distinct outgoing rays in clockwise order around H such that each of the angles $\theta(r_i, r_j), \theta(r_j, r_k)$ and $\theta(r_k, r_i)$ is less than $180°$. Then there exist three points a, b and c on r_i, r_j and r_k, respectively, such that the triangle abc contains H. See Figure 4(c).*

We call the triple (r_i, r_j, r_k) and the triangle abc a *covering triple* and a *covering triangle*, respectively. See Figure 4(d). We now show the following theorem.

Theorem 2 MIN-SEG-CONVEX *is NP-hard.*

Proof: To prove the problem is NP-hard we reduce ARRANGEMENT-GRAPH-RECOGNITION to MIN-SEG-CONVEX. Let G be an instance of ARRANGEMENT-GRAPH-RECOGNITION, where the number of vertices in G is $l(l-1)/2$. Let n_2 and n_3 be the numbers of vertices of degree two and three in G, respectively, where $l = n_2 + n_3/2$. Take an embedding Γ_G of G such that all the vertices of degree two are on the outer face. Let $v_1, v_2, \ldots, v_{n_2}$ be the vertices of degree two on the outer face of G in clockwise order. Create a cycle $c_1, c_2, \ldots, c_{n_2}$ of n_2 vertices on the plane enclosing Γ_G, and add the edges (c_i, v_i), $1 \le i \le n_2$. The resulting plane graph G' is an instance of MIN-SEG-CONVEX. See Figure 5(a). It is straightforward to construct G' in polynomial time in the size of the input G.

By construction, the number of vertices in G' is $l(l-1)/2 + n_2$ and G' is a graph of maximum degree four. The number of vertices of degree three in G' is $2n_2 + n_3$. We now ask whether G' admits a convex drawing with at most $k = 3 + n_2 + n_3/2 = 3 + l$ segments. Observe that one can compute G' and k in polynomial time in the size of the input G. In the following we prove that G is an arrangement graph if and only if G' admits such a drawing.

We first assume that G is an arrangement graph and then define a convex drawing of G' with at most $3 + l$ segments. By Lemma 1, we observe that G admits an arrangement drawing Γ with l segments, where all the vertices of degree two and three are on the outer face. By property (b) of Lemma 4, the convex hull H of Γ is determined by the vertices of degree two. In the following we use Γ to define the required convex drawing of G' using Steps (a)–(d). In

Step (a) we show the existence of a covering triple (r_i, r_j, r_k); in Step (b) we show that the origins h_i, h_j, h_k of this covering triple must be distinct; in Step (c) we use Observation 2 to show the existence of a covering triangle abc, which we then use to define a drawing Γ' for G' with $(3+l)$ segments; finally in Step (d) we verify that Γ' is a convex drawing.

(a) Let $h_1, h_2, \ldots, h_m$ be the sequence of vertices on the boundary of H in clockwise order. Let $\sigma = (r_1, r_1, r_2, r_2, \ldots, r_m, r_m)$ be a tuple of the $2m$ outgoing rays obtained by extending the segments adjacent to $h_i, 1 \leq i \leq m$, in clockwise order. For simplicity, we use the same notation for the rays that have the same origin. See Figure 5(b). We now claim that there exists a covering triple (r_i, r_j, r_k) such that $1 \leq i, j, k \leq m$. Suppose for a contradiction that no such covering triple exists. In the following we show that for any two consecutive rays r_t and $r_{t'}$ in σ, the inequality $\theta(r_t, r_{t'}) < 180°$ holds. We then use this observation to find a covering triple, which gives us a contradiction.

Assume that there exist two rays r_t and $r_{t'}$, where r_t and $r_{t'}$ are consecutive in σ, such that $\theta(r_t, r_{t'}) \geq 180°$. Let h_t and $h_{t'}$ be the origins of r_t and $r_{t'}$, respectively. Since the angle formed by any two rays with the same origin is less than $180°$, t and t' must be distinct. Moreover, since r_t and $r_{t'}$ are consecutive in σ, $(h_t, h_{t'})$ must determine an edge of the convex hull H. Since each of r_t and $r_{t'}$ is an extension of some segment of Γ, they must intersect at some point p inside H. Consequently, $\angle h_t p h_{t'} = \theta(r_t, r_{t'})$ must be less than $180°$.

We now find a covering triple as follows. Choose one of the two r_1s arbitrarily. Let r_x be the outgoing ray among the two r_1s that maximizes $\theta(r_1, r_x)$, where $\theta(r_1, r_x) < 180°$. Similarly, let r_y be the outgoing ray that maximizes $\theta(r_y, r_1)$, where $\theta(r_y, r_1) < 180°$. See Figure 5(b). Now (r_1, r_x, r_y) is a covering triple if $\theta(r_x, r_y) < 180°$. We show that this is indeed the case. Suppose for a contradiction that $\theta(r_x, r_y) \geq 180°$. Since r_x and r_y intersects inside H, their corresponding origins h_x and h_y cannot form an edge of the convex hull H. Therefore, if we start traversing the vertices of H from h_x in clockwise order, then we must visit some convex hull vertex $h_z \notin \{h_x, h_y\}$ before we visit h_y. Since at most one of the two rays originating from h_z can be collinear with r_1, either $\theta(r_1, r_x)$ or $\theta(r_y, r_1)$ is not maximal, which is a contradiction. See Figure 5(c). Consequently, $\theta(r_x, r_y)$ is less than $180°$, and by definition (r_1, r_x, r_y) must be a covering triple.

(b) We have proved that there exists a covering triple (r_i, r_j, r_k) among the outgoing rays obtained by extending the segments adjacent to $h_i, 1 \leq i \leq m$. We now prove that the corresponding origins h_i, h_j, h_k are distinct. Suppose for a contradiction that the origins of r_i, r_j, r_k are not distinct. Since the outgoing rays are obtained by the extensions of the segments adjacent to a vertex on the boundary of H (which is a vertex of degree two by Property (b) of Lemma 4), at most two outgoing rays can have a common origin.
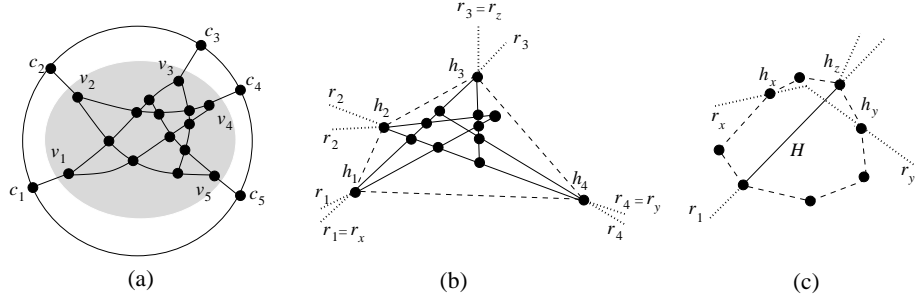
Figure 5: (a) Illustration for G', where Γ_G is shown in the shaded region. (b) An arrangement drawing and the outgoing rays obtained by extending the segments adjacent to the vertices on the convex hull. (c) Illustration for the proof of Theorem 2.

Without loss of generality assume that h_i coincides with h_k. Let s_i, s_j and s_k be the complementary rays of r_i, r_j and r_k, respectively.

If s_j coincides with either s_i or s_k, then either $\theta(r_i, r_j) = 180°$ or $\theta(r_j, r_k) = 180°$, which contradicts that (r_i, r_j, r_k) is a covering triple. If s_j is different from s_i and s_k, then s_j intersects s_i and s_k inside H. It is now straightforward to observe that if s_j hits s_i before s_k, then $\theta(r_i, r_j) > 180°$. Similarly, if s_j hits s_k before s_i, then $\theta(r_j, r_k) > 180°$. This contradicts that (r_i, r_j, r_k) is a covering triple. See Figure 6(a).

(c) By Observation 2, there exists a covering triangle abc that contains Γ. We now repeatedly choose a vertex v of degree two in Γ such that $v \notin \{h_i, h_j, h_k\}$ and extend one of its adjacent segments until the endpoint of that segment crosses the boundary of abc. It is now straightforward to verify that the resulting drawing Γ' is a $(3 + l)$-segment drawing of G'. See Figures 6(b) and (c).

(d) We now only need to show that Γ' is a convex drawing. By Observation 1, the inner faces in Γ' that correspond to the inner faces in Γ are convex. By the construction of Γ' along with Property (a) of Lemma 4, we can verify that all the newly created inner faces of Γ' are convex. Finally, the outer face of Γ' is the triangle abc, which is also convex.

We now assume that G' admits a convex drawing Γ' with at most $k = 3 + n_2 + n_3/2$ segments and then show that G is an arrangement graph. We delete all the vertices of the outer cycle C from Γ' along with their incident edges. It is straightforward to check that the resulting drawing Γ'' is a straight-line drawing of G. We now count the number of segments in Γ''. The drawing of C in Γ' must contain at least three segments. Therefore, the number of segments in Γ'' is less than or equal to $l = n_2 + n_3/2$. By Lemma 2, Lemma 3 and Corollary 1, G is an arrangement graph. □
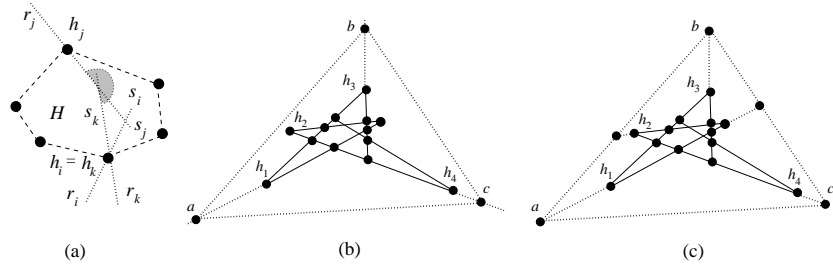
Figure 6: (a) Illustration for the proof of Theorem 2 with the angle $\theta(r_j, r_k)$ shown in light-gray. (b) A covering triangle abc of Γ. (c) Construction of Γ'.

5   Minimum-Segment Drawings with Prescribed Segments

Drawing a graph respecting the properties of a partial drawing is a well-studied problem. For example, Angelini et al. [1] gave a polynomial-time algorithm to test the planarity of partially embedded graphs. Di Giacomo et al. [11] examined the problem of determining a point-set embedding of a tree that includes a given partial drawing of the tree. Patrignani [25] proved the NP-hardness of computing a straight-line drawing of a planar graph that includes a given partial drawing of the graph.

In this section, we prove that given a partial drawing Γ' of a graph G, it is NP-hard to determine a k-segment drawing of G that contains all the segments specified in Γ'. The problem remains NP-hard even when G is outerplanar and a straight-line drawing of G containing all the segments specified in Γ' is computable in polynomial time. A formal definition of the decision problem is as follows.

Problem : PARTIAL-MIN-SEG

Instance : An outerplanar graph G, a straight-line drawing Γ' of a subgraph G' of G such that a straight-line drawing of G containing all the segments specified in Γ' is computable in polynomial time, and an integer $k \geq 1$.

Question : Is there a k-segment drawing Γ of G that contains all the segments specified in Γ'?

We prove the NP-hardness of PARTIAL-MIN-SEG by reducing a strongly NP-complete problem 3-PARTITION [10], which is defined as follows.

Problem : 3-PARTITION

Instance : A set of $3m$ positive integers $S=\{a_1, a_2, \ldots, a_{3m}\}$ and an integer $B > 0$, where $a_1 + a_2 + \ldots + a_{3m} = mB$ and $B/4 < a_i < B/2$, $1 \leq i \leq 3m$.

Question : Can S be partitioned into m subsets $S_1, S_2, \ldots, S_m$ such that $|S_1|=|S_2|=\ldots=|S_m|=3$ and the sum of the integers in each subset is equal to B?

Observe that the NP-completeness of 3-PARTITION holds even when each integer of S is greater than one, which we will use in our reduction.

A *fan* f is a maximal outerplanar graph with n vertices such that some vertex v has degree $n-1$. We call v the *apex* of f and all the other vertices the *path vertices* of f. We call the edges that are incident to v the *ribs* of f. We now show the following theorem.

Theorem 3 PARTIAL-MIN-SEG *is NP-hard.*

Proof: To prove the problem is NP-hard we reduce 3-PARTITION to PARTIAL-MIN-SEG. We create an instance of PARTIAL-MIN-SEG from an instance $B, S = \{a_1, \ldots, a_{3m}\}$ of 3-PARTITION, where each integer of S is greater than one.

We construct in polynomial time an outerplanar graph G as in Figure 7(a), where $3m + 2$ fans have a common apex v. Each fan f_i, $1 \leq i \leq 3m$, contains exactly a_i path vertices. There are two more fans f' and f'' which contain $m+1$ path vertices and $mB+m+1$ path vertices, respectively. The size of G is polynomial since 3-PARTITION is strongly NP-complete. We denote by G' the subgraph of G induced by the vertices of f' and f''. We construct a straight-line outerplanar drawing Γ' of G' that satisfies the following (a)–(b).

(a) Let $w_1, \ldots, w_{m+1}$ be the path vertices of f' ordered clockwise around v and let $u_1, u_2, \ldots, u_{mB+m+1}$ be the path vertices of f'' ordered clockwise around v. For each j, $1 \leq j \leq m+1$, rib (w_j, v) of f' and rib (v, u_i) of f'' form a segment, $i=B(j-1)+j$. These segments are shown in bold lines in Figure 7(a).

(b) The edges between path vertices of f' and f'' are drawn on two different segments. All the other edges of f'' are drawn as separate segments, which are shown as thin lines in Figure 7(a).

The gray region in Figure 7(a) shows Γ'. By construction, the number of segments in Γ' is $k' = mB + m + 3$. We can observe that G admits some straight-line drawing containing all the segments of Γ'. We now ask whether G admits a k-segment drawing that includes all the segments specified in Γ', where $k = mB+m+3+3m$. In the following we prove that such a drawing exists if and only if the given instance of 3-PARTITION has a positive answer.

We first assume that the 3-PARTITION we considered has a positive answer. In other words, S can be partitioned into m subsets $S_1, S_2, \ldots, S_m$ such that each S_i, $1 \leq i \leq m$, contains exactly three integers and the sum of the integers in S_i is equal to B. Since we assume that each integer in S is greater than one, each fan f_i, $1 \leq i \leq 3m$, contains at least two path vertices and requires at least one new segment to draw the edges between the path vertices. Consequently, any straight-line drawing of G that contains the segments of Γ' requires at least $k' + 3m = k$ segments. Let E' be the set of ribs of f'' that are not drawn on the

same segment as any rib of f'. To obtain a k-segment drawing of G, we draw each rib of each f_i on the same segment as one of the ribs in E'. Let e_1 and e_2 be any two consecutive ribs of f' in Γ' and let e'_1 and e'_2 be the ribs of f'' that are on the same segments as e_1 and e_2, respectively. Then the number of ribs between e'_1 and e'_2 is B. Let the integers in any S_i, $1 \leq i \leq m$, be a, b and c, where $a + b + c = B$. We place the fans that have a, b and c path vertices inside the face bounded by the ribs e_1 and e_2 in Γ' in such a way that each rib of a, b and c shares a segment with one of the ribs of f'' between e'_1 and e'_2. In this way, we place the three fans with path vertices corresponding to the integers in S_i in the face bounded by the pair of ribs e_i and e_{i+1}, where $1 \leq i \leq m$. It is now straightforward to check that the final drawing Γ of G that contains all the segments of Γ' is a k-segment drawing. See Figure 7(b).

We now assume that the given instance of 3-PARTITION has a negative answer and hence the set S cannot be partitioned into m subsets as described above. We prove that in that case G does not have a drawing with k or fewer segments that includes all the segments of Γ'. Recall that any straight-line drawing of G that includes all the segments of Γ' requires at least k segments. Suppose for a contradiction that G has a drawing Γ that includes all the segments of Γ' with exactly k segments. Then each rib of each f_i, $1 \leq i \leq 3m$, must be drawn on the same segment as one of the ribs of E'. Since Γ is a planar drawing of G, each f_i must be placed inside a face bounded by two consecutive ribs of f'. Therefore, the fans $f_1, \ldots, f_{3m}$ are partitioned into m subsets and the total number of ribs for each set of fans must be B. Since $a_i < B/2$, no two fans can together have B ribs. Similarly, since $B/4 < a_i$, four or more fans cumulatively have more than B ribs. Therefore, each subset must contain exactly three fans. Hence each subset of fans corresponds to a subset S_i of S that contains three integers whose sum is B. This gives a solution to the given instance of 3-PARTITION, a contradiction. Therefore, G cannot have a k-segment drawing that includes all the segments of Γ'. $\qquad\square$
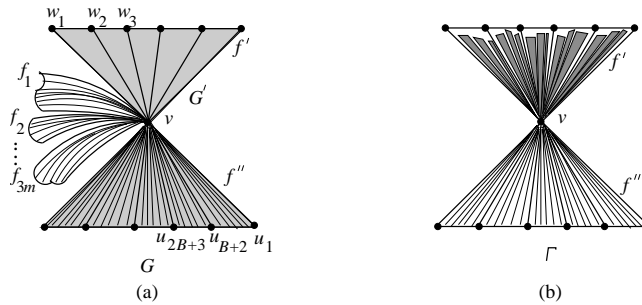

Figure 7: Illustration for the proof of Theorem 3.

6   Minimum-Segment Drawings in $\mathbb{R}^3$

In this section, we examine the complexity of computing a minimum-segment drawing of a graph in $\mathbb{R}^3$. It is straightforward to verify that The proof of Theorem 3 holds in $\mathbb{R}^3$. The proof of Theorem 1 also holds in $\mathbb{R}^3$ since every pseudo-arrangement graph with l lines that is not an arrangement graph must require more than l segments in any of its straight line drawing in $\mathbb{R}^3$. We suspect that the problem remains NP-hard in $\mathbb{R}^3$ even when the graph is subcubic, i.e., a graph with maximum degree three.

However, we have not been able to prove the NP-hardness of the problem in its original form. We impose some constraints on the problem and show that it is NP-hard to compute a minimum-segment drawing of a maximum degree three graph in $\mathbb{R}^3$, if a set of segments is pre-specified and some sets of vertices are restricted to be coplanar. A formal definition of the decision problem is as follows.

Problem : 3D-Partial-Min-Seg

Instance : An integer $k \geq 1$, a graph G with maximum degree three, a straight-line drawing Γ of G in $\mathbb{R}^3$ and a subgraph G' of G, and some subsets $S_1, S_2, \ldots, S_z$ of the vertices of G.

Question : Is there a k-segment drawing Γ^* of G in $\mathbb{R}^3$ that contains all the segments of Γ' and keeps the vertices of each $S_i, 1 \leq i \leq z$, coplanar? Here, Γ' denotes the restriction of Γ to G'.

We prove the NP-hardness of 3D-Partial-Min-Seg by reduction from an NP-complete problem Monotone-Not-All-Equal-3-SAT [27], which is defined as follows.

Problem : Monotone-Not-All-Equal-3-SAT

Instance : A set U of variables and a collection C of clauses over U such that no clause contains a negated literal and each clause $c \in C$ has exactly three literals.

Question : Is there a truth assignment for U such that each clause in C has at least one true literal and at least one false literal?

Let F be an instance of Monotone-Not-All-Equal-3-SAT with m literals $x_1, x_2, \ldots, x_m$ and m' clauses $c_1, c_2, \ldots, c_{m'}$. We define a corresponding graph G_F, a drawing Γ of G_F, a subgraph G' of G_F, some subsets $S_i, 1 \leq i \leq z$, of vertices of G_F and an integer k such that Monotone-Not-All-Equal-3-SAT has a positive solution if and only if G_F admits a k-segment drawing Γ^* in $\mathbb{R}^3$ that contains all the segments specified in the restriction Γ' of Γ to G', and keeps all the vertices of each S_i coplanar. We first define G_F following the steps described below.
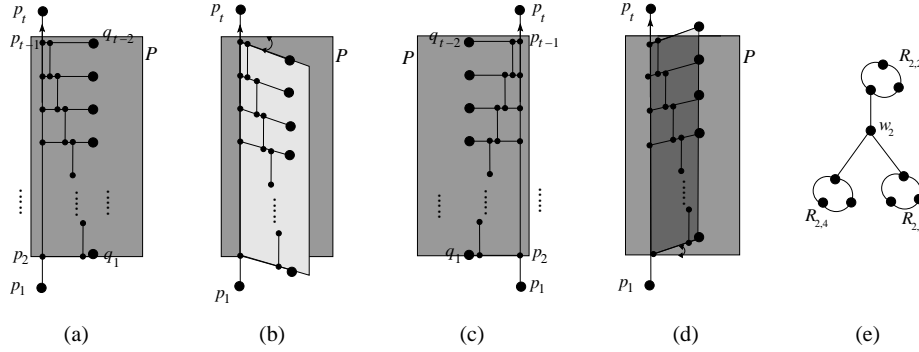
Figure 8:   A staircase Γ_t in (a)–(b) positive orientations, and (c)–(d) negative orientations. (e) An example blossom for the clause $c_2 = x_2 \vee x_4 \vee x_5$.

Step 1. Observe the *staircase* structure $\Gamma_t, t \geq 2$, in Figure 8(a). We call the leftmost vertical segment with vertices $p_1, p_2, \ldots, p_t$ the *spine of* Γ_t. The vertices p_1 and p_t are the *end vertices of* Γ_t. The vertex $q_i, 1 \leq i \leq t-2$, is the *i-th contact of* Γ_t. Let θ be the counterclockwise angle between a pre-specified plane P containing the spine and the plane containing the contact vertices when looking along the directed line from p_1 to p_t. A staircase is in *positive orientation with respect to plane P* if $0° \leq \theta < 180°$ as shown in Figures 8(a) and (b). Similarly a staircase is in *negative orientation with respect to plane P* if $180° \leq \theta < 360°$ as shown in Figures 8(c) and (d).

In our reduction, each positive and negative orientation will correspond to a true and a false literal, respectively.

Step 2. Let D be a straight-line drawing of a path $v_0, v_1, \ldots, v_{m+1}$ of $m + 2$ vertices on the XY-plane, where the coordinate of v_0 is $(0, 0, 0)$ and the coordinates of $v_i, 1 \leq i \leq m+1$, are $(m' + i, m' + i, 0)$. Similarly, let D' be a straight-line drawing of a path $v'_0, v'_1, \ldots, v'_{m+1}$ of $m + 2$ vertices on the XY-plane, where v'_0 coincides with v_0 and the coordinates of $v'_i, 1 \leq i \leq m+1$, are $(m' + i, -m' - i, 0)$. Let D'' denote the single-edge path v_{m+1}, v'_{m+1}. See Figure 9(a).

Step 3. For each $j, 1 \leq j \leq m$, add a copy of staircase $\Gamma_{m'+2}$ such that v_j and v'_j coincide with p_1 and $p_{m'+2}$ of $\Gamma_{m'+2}$, respectively. It is straightforward to add the staircases such that the resulting drawing is a straight-line drawing in $\mathbb{R}^2$ and each staircase in the drawing contains the same set of segments as in Figure 8(a). Let the resulting straight-line drawing be Γ''. See Figure 9(a). By *the j-th staircase* of Γ'' we mean the staircase with end vertices v_j and v'_j.

Step 4. For each clause $c_i, 1 \leq i \leq m'$, of F we construct a *blossom* as shown in Figure 8(e). We take a vertex w_i, and for each literal x_j that is in
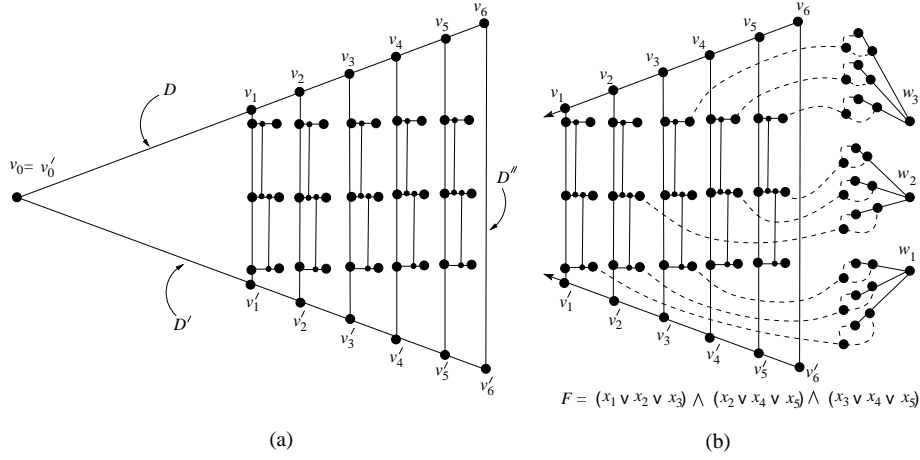
Figure 9: (a) A hypothetical illustration for Γ'' with $m = 5$ and $m' = 3$. The positions of the vertices do not reflect the actual coordinates. (b) Illustration for G_F, where the edges not in Γ' are shown in dashed lines.

> c_i, $1 \leq j \leq m$, we add a three cycle $R_{i,j}$ and connect one of its vertices to w_i. We then connect one of the remaining two vertices of $R_{i,j}$ to the i-th contact vertex q_i of j-th staircase as shown in Figure 9(b). We call the degree two vertex of $R_{i,j}$ the *free vertex*. We denote the resulting graph by G_F.

We now define $S_i, 1 \leq i \leq m + 1$, Γ and Γ'. Each subset $S_i, 1 \leq i \leq m$, consists of the vertices of the j-th staircase. The subset S_{m+1} contains the vertices on the segments D, D', D'', the vertices on the spines and the vertices $w_1, w_2, \ldots, w_{m'}$. Drawing Γ' contains all the segments in Γ'' and the segments determined by the shortest path between each w_i and its corresponding free vertices. Figure 8(f) depicts an embedding of G_F, where the edges do not belong to Γ' are shown as dashed lines. It is easy to obtain a drawing Γ of G_F by embedding the dashed edges above the plane containing all the segments of Γ'. See Figure 10. Observe that one can construct G_F and Γ', hence Γ, in $f(m, m')$ time, where the function f is a polynomial in m and m'.

We now prove the following theorem.

Theorem 4 3D-Partial-Min-Seg *is NP-Complete.*

Proof: Given a drawing Γ of G, in polynomial time in the input length of the 3D-Partial-Min-Seg instance we can certify whether Γ is a k-segment drawing containing all the pre-specified segments and whether all the vertices of each pre-specified subset are coplanar. Thus the problem is in NP. To prove the problem is NP-hard we reduce Monotone-Not-All-Equal-3-SAT to 3D-Partial-Min-Seg. Let F be an instance of Monotone-Not-All-Equal-3-SAT. Let $x_1, x_2, \ldots, x_m$ and $c_1, c_2, \ldots, c_{m'}$ be the distinct literals and clauses in
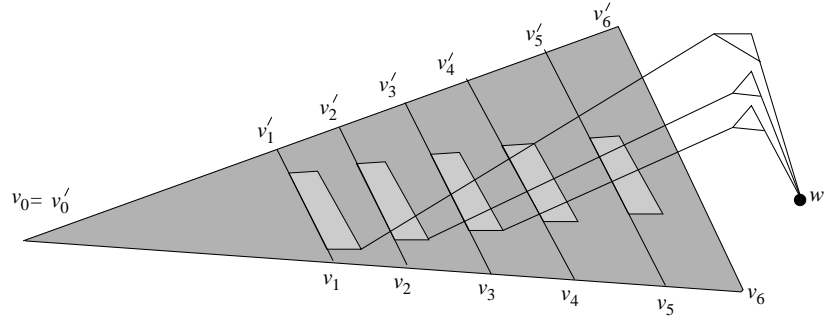
Figure 10: A sketch for the drawing Γ of G_F, where only one blossom is illustrated.

F, respectively. We construct the corresponding graph G_F, subsets $S_i, 1 \leq i \leq m+1$, a drawing $\Gamma(G_F)$ of G and a drawing $\Gamma' \subseteq \Gamma(G_F)$ of a subgraph of G_F. Let k' be the number of segments in Γ'. We prove that F has an affirmative solution if and only if G_F admits a k-segment drawing, $k = k' + 2m'$, that contains all the segments of Γ' and keeps the vertices of each S_i coplanar.

Consider first the case when G_F admits a k-segment drawing Γ that contains all the segments of Γ' and keeps the vertices of each S_i coplanar. We now prove that F has an affirmative solution. Let w_i be the vertex in Γ that corresponds to the clause c_i. By construction, each edge adjacent to w_i is incident to a vertex of some cycle $R_{i,j}$, where $R_{i,j}$ contains exactly three vertices. In any straight-line drawing, $R_{i,j}$ is drawn as a triangle. The three sides of that triangle correspond to three distinct segments in Γ. Observe that at least one of these segments consists of only one edge and hence, that segment does not belong to Γ'. See Figure 11(a). Since there are $3m'$ such cycles, there are $3m'$ such segments in Γ that are not in Γ'. On the other hand, since $deg(w_i) = 3$, at most two segments of Γ' that are incident to w_i can share a single segment in Γ. See Figure 11(b). Hence the number of segments in Γ may be decreased by at most m'. Therefore, Γ cannot contain fewer than $k = k' + 3m' - m' = k' + 2m'$ segments.

We may thus assume that Γ contains exactly k-segments. Consequently, the following properties hold.

(a) For each $i, 1 \leq i \leq m'$, two segments among the three segments that are incident to w_i in Γ' form a single segment in Γ.

(b) Every contact vertex becomes an interior vertex of some segment in Γ.

(c) Every cycle $R_{i,j}$ contributes exactly one segment in Γ that does not belong to Γ'.

Let l and l' be the two segments that are incident to w_i in Γ' and form a single segment in Γ. Let P be the plane that contains the vertices of S_{m+1}. We consider two cases.

Case 1. If l and l' both lie on P, then Properties (b) and (c) ensure that the two staircases connected to w_i through l and l' lie on P. To avoid edge crossings in Γ, one of these two staircases must be in positive orientation and the other staircase must be in negative orientation and lying on P. An example is shown in Figure 11(c).
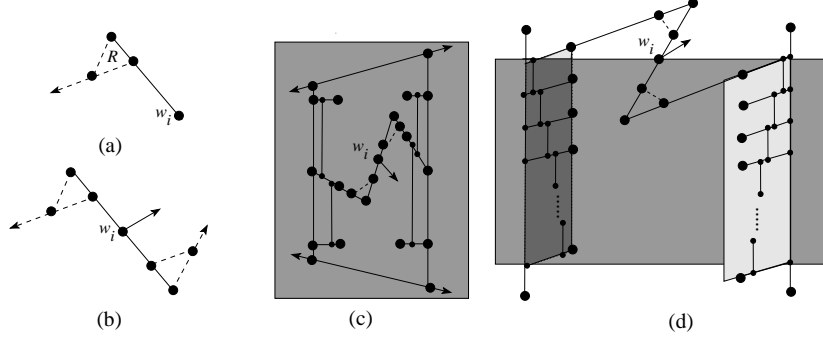

Figure 11: (a) A drawing of R, where a segment with only one edge is shown in dashed line. (b) Illustration for Γ'. The positions of the vertices do not reflect the actual coordinates. (c)–(d) Illustration for truth value assignment.

Case 2. If l does not lie on P, then l and l' must lie on opposite sides of P. Without loss of generality assume that the staircase connected to w_i through l is in positive orientation with respect to P. Then the staircase connected to w_i through l' is in negative orientation with respect to P. An example is shown in Figure 11(d).

In both cases, each w_i must be connected with at least one positively oriented staircase and one negatively oriented staircase. Thus each clause can be satisfied by setting the literals associated with the positively oriented staircases to "true" and the literals associated with the negatively oriented staircases to "false". The restriction that the vertices of each S_i lie on the same plane ensures that the truth values are assigned consistently. Hence, we obtain the required truth assignment for F.

We now assume that F has an affirmative solution. In the following we construct a k-segment drawing Γ of G that contains all the segments of Γ' and keeps the vertices of each S_i on the same plane. Figure 12 shows a sketch for an example of Γ.

Step 1. For each true literal (false literal) in F, rotate the corresponding staircase in Γ around its spine through 45° anticlockwise (clockwise).

Step 2. Place each vertex w_i to the right of D'' on P.

Step 3. Since each clause in F contains a positive and a negative literal, we can take the corresponding pair of segments that are incident to w_i and
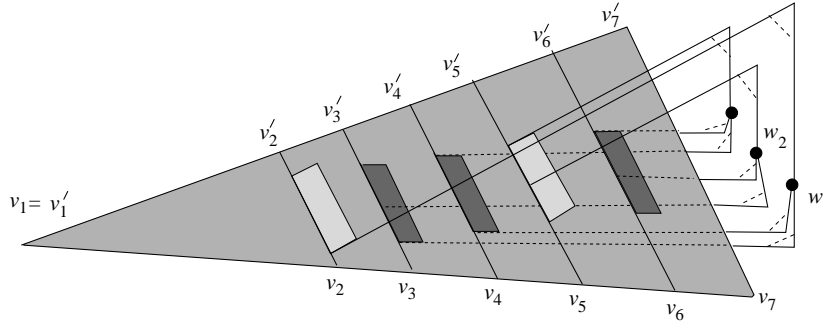
Figure 12: A sketch for Γ, where $F = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_4 \vee x_5) \wedge (x_3 \vee x_4 \vee x_5)$ and $x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1, x_5 = 0$. The staircases in positive and negative orientations are shown in light-gray and dark-gray, respectively.

align them vertically so that they form a single segment. Now complete the connection with the staircases in such a way that the corresponding cycle $R_{i,j}$ contributes at most one segment that does not belong to Γ'.

Step 4. The remaining segment s that is incident to w_i in Γ' does not require any special treatment. Consequently, it is straightforward to connect w_i with the corresponding staircase through s, creating at most one new segment.

Observe that Γ has k' segments of Γ', and $2m'$ more segments by Steps 3–4. Consequently, Γ is a $k = k' + 2m'$ segment drawing. □

7 Segment Complexity of Planar Graphs

In this section we give lower bounds on the segment complexities of different classes of planar graphs. Recall that the segment complexity of a planar graph G is the minimum positive integer C, such that any spanning tree of G admits a drawing with at most C segments. Dujmović *et al.* [5] proved that if the number of odd degree vertices in a tree is η, then any straight-line drawing of the tree requires at least $\eta/2$ segments. If a spanning tree T of G has x leaves, then $x \leq \eta$ and any straight-line drawing of the tree requires at least $x/2$ segments. Thus we have the following observation.

Observation 3 *Let G be a planar graph with a spanning tree T, where the number of leaves is x. Then $x/2$ is a lower bound on the segment complexity of G.*

By Observation 3, we obtain a lower bound on the segment complexity of a planar graph by finding a spanning tree with many leaves. A *maximum-leaf spanning tree* of a graph G is a spanning tree of G, where the number of leaves is the maximum possible. It is NP-hard to find a maximum-leaf spanning tree

in a graph G, even when G is a planar bipartite graph with maximum degree four [19]. In the following we obtain lower bounds on segment complexities for maximal outerplanar graphs, plane 2-trees, plane 3-trees, plane 3-connected graphs and plane 4-connected graphs.

Recall that a graph G with n vertices is a *k-tree* if G satisfies the following properties:

(a) If $n = k$, then G is the complete graph K_n.

(b) If $n > k$, then G can be constructed from a k-tree G' with $n-1$ vertices by adding a vertex adjacent to exactly k vertices of G', where the induced graph of these k-vertices is a complete graph.

Every k-tree $G = (V, E)$ admits an ordered partition $\pi = (V_1, V_2, ..., V_m)$ of V that satisfies the following:

(a) V_1 contains k vertices inducing a complete graph and every other partition contains only one vertex.

(b) Let G_k, $1 \le k \le m$, be the subgraph of G induced by $V_1 \cup V_2 \cup ... \cup V_k$. Then G_k, $k > 1$, is a k-tree obtained by adding V_k to G_{k-1}.

Every 2-tree is 3-colorable. The following lemma finds a spanning tree of a plane 2-tree using graph coloring.

Lemma 5 *Let G be a 2-tree with $n \ge 3$ vertices. Let S be a set of vertices that are assigned the same color c in a 3-coloring of G. Then $G - S$ is a tree.*

Proof: Let $\pi = (V_1, V_2, ..., V_m)$ be an ordered partition of V and let S_i denote a set of vertices in G_i that are assigned the same color c in a 3-coloring of G_i. We now use induction on m. The case when $m = 1$ is straightforward since G_1 is K_2. We thus assume that for each G_i, $1 \le i \le m - 1$, $G_i - S_i$ is a tree. Now consider $G_m = G$. Let z be the vertex in V_m and let x and y be its neighbors. By the definition of plane 2-tree, x and y are adjacent. We assume that G is colored with colors c_1, c_2, c_3 such that color(x)=c_1, color(y)=c_2 and color(z)=c_3. If c=c_3, then $G-S$=$G_{m-1}-S_{m-1}$ is a tree by induction. If c= c_1 or $c = c_2$, then $G-S$ is formed by connecting vertex z to $G_{m-1}-S_{m-1}$ with exactly one edge. Since $G_{m-1}-S_{m-1}$ is a tree, $G-S$ is a tree. $\square$

We use Lemma 5 to prove the following lemma.

Lemma 6 *Let G be a plane 2-tree with $n \ge 3$ vertices. Then the segment complexity of G is at least $n/6$.*

Proof: We show that every plane 2-tree G with $n \ge 3$ vertices has a spanning tree T, where the number of leaves in T is at least $n/3$. By Observation 3, this will prove that the segment complexity of G is at least $n/6$.

Every plane 2-tree admits a 3-coloring. Let S_i, $1 \le i \le 3$, be a set of vertices that are assigned color i in a 3-coloring of G. The set with the maximum cardinality among S_1, S_2 and S_3 contains at least $n/3$ vertices. Without loss of generality assume that the set with the maximum cardinality is S_1, that is, $|S_1| \ge n/3$. By Lemma 5, $G - S_1$ is a tree, which we denote by T'.

Let v be a vertex in S_1. Since S_1 is an independent set and G is connected, there exists an edge (x, v), where x is a vertex of T'. Therefore, we can connect v to x to obtain another tree that contains v as one of its leaves. By making the vertices of S_1 leaves in T', we can obtain a spanning tree T with at least $n/3$ leaves. $\qquad\square$

Since every outerplanar graph is a 2-tree, the following corollary follows from Lemma 6.

Corollary 2 *Let G be a maximal outerplanar graph with $n \geq 3$ vertices. Then the segment complexity of G is at least $n/6$.*

Every plane 3-tree G has a spanning tree with at least $(2n-5)/3$ leaves [30]. Furthermore, Kleitman and West [18] proved that every plane 4-connected graph has a spanning tree with at least $2n/5$ leaves, and every plane 3-connected graph has a spanning tree with at least $n/4$ leaves. We combine these results with Observation 3 to obtain the following theorem.

Theorem 5 *The segment complexities of plane 2-trees, plane 3-trees, plane 4-connected graphs and plane 3-connected graphs are at least $n/6$, $(2n-5)/6$, $n/5$ and $n/8$, respectively.*

8   Conclusion

Among other results, we have proved that it is NP-hard to decide whether a plane graph G has a straight-line drawing with k segments. This motivates us to ask the following questions.

Open Problem 1: Is there a constant-factor approximation algorithm for minimum-segment drawings of planar graphs?

We proved that it is NP-hard to compute a minimum-segment drawing of a maximum degree three graph in $\mathbb{R}^3$, if a set of segments is pre-specified and some vertices are restricted to lie on the same plane. However, the problem remains open when no constraints are imposed.

Open Problem 2: What is the complexity of deciding whether a graph with maximum degree three has a k-segment drawing in $\mathbb{R}^3$?

A minimum-segment drawing becomes more visually coherent if we minimize the number of distinct lines that contain the segments of the drawing. We call such a drawing a *minimum-line drawing*. Figures 13(a) and (b) depict two different minimum-segment drawings of a tree, where the number of lines are 7 and 6, respectively. Since the number of distinct slopes used in both figures is two, the problem of computing a minimum-line drawing is different from the problem of minimizing the number of distinct slopes.

Open Problem 3: Compute non-trivial upper bounds on the number of lines required for minimum-line drawings of different classes of planar graphs.
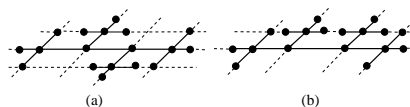
Figure 13: (a) A minimum-segment drawing. (b) A minimum-segment drawing, which is also a minimum-line drawing. Lines are shown in dotted lines.

Acknowledgment

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