Kinetic Maintenance of Mobile k-Centres on Trees ⋆

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Abstract

Given a set P of points (clients) on a weighted tree T, a k-centre of P corresponds to a set of k points (facilities) on T such that the maximum graph distance between any client and its nearest facility is minimized. We consider the *mobile* k-centre problem on trees. Let C denote a set of n mobile clients, each of which follows a continuous trajectory on a weighted tree T. We establish tight bounds on the maximum relative velocity of the 1-centre and 2-centre of C. When each client in C moves with linear motion along a path on T, the motions of the corresponding 1-centre and 2-centre are piecewise linear; we derive a tight combinatorial bound of $\Theta(n)$ on the complexity of the motion of the 1-centre and corresponding bounds of $O(n^2\alpha(n))$ and $\Omega(n^2)$ for a 2-centre, where $\alpha(n)$ denotes the inverse Ackermann function. We describe efficient algorithms for calculating the trajectories of the 1-centre and 2-centre of C: the 1-centre can be found in optimal time $O(n \log n)$ and a 2-centre can be found in time $O(n^2 \log n)$. These algorithms lend themselves to implementation within the framework of kinetic data structures. Finally, we examine properties of the mobile 1-centre on graphs and describe an optimal unit-velocity 2-approximation.

Key words: algorithms, trees, k-centre, kinetic data structures

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1 Introduction

Motivation. Finding a set of k points that are central to a collection of data points drawn from a metric space is a fundamental problem of geometry and data analysis. Within the context of facility location, this problem is commonly known as the k-centre problem; given a set P of points (clients) in a metric space S, a k-centre of P is a set of k points (facilities) such that the maximum distance from any client to its nearest facility is minimized. Two common choices for S are a Minkowski distance (typically ℓ_1, ℓ_2, or ℓ_∞) in Euclidean space and graph distance on a weighted graph.

Recently, the k-centre problem has been explored under mobility. In one dimension, the mobile 1-centre problem reduces to maintaining the extrema of a set of mobile clients as these move along the real line [1,2,5,22]. Natural generalizations of this problem to higher dimensions in $\mathbb{R}^d$ lead to the mobile Euclidean 1-centre [2,8,14], the mobile rectilinear 1-centre [2,9], and the kinetic convex hull [5,6,22]. Although some mobile k-centre problems can be modelled by motion in Euclidean space, several applications are better represented by motion on a graph. That is, the underlying graph remains fixed while clients and facilities move along its edges and vertices. Examples include vehicles moving along a road network or mobile robots following defined routes in an industrial setting [10]. In this paper, we consider the mobile k-centre problem on the metric space of graph distance on a weighted graph and, in particular, on a weighted tree.

Although the static k-centre problem on graphs is well understood, the corresponding mobile problem remained unexplored. Any path in a weighted graph is isometric to a line segment; we generalize the motion of a single client on the line to motion on a path in a graph. That is, given a weighted graph G, each mobile client follows a continuous trajectory along the edges and vertices of G. Continuity and bounded velocity are natural constraints on any physical moving object. As we show in Section 6, for any graph G that contains a cycle, there exist sets of mobile clients on G whose 1-centre is discontinuous. As such, we primarily focus our attention on metric spaces for which the k-centre is continuous. In particular, graph distance on a tree maintains many properties of Euclidean distance in $\mathbb{R}^d$, such as a unique shortest path between two points and a unique, continuous 1-centre, while introducing interesting algorithmic challenges to the problem of maintaining a mobile k-centre.

Main Results. The 1-centre on a tree is unique [25]. We show its motion is continuous and has relative velocity at most one when the motion of clients is continuous. Since a 2-centre of a tree is not unique, we identify a particular 2-centre which we call the *equidistant 2-centre* and show that its motion is continuous and has relative velocity at most two when the motion of clients is continuous. The 3-centre is discontinuous even on a line segment; furthermore, no bounded-velocity approximation is possible for the mobile 3-centre [13]. We consider values of k for which the mobile k-centre is continuous: $k \leq 2$.

When each client in C moves with linear motion along a path on T, the motions of the corresponding 1-centre and equidistant 2-centre are piecewise linear. We derive a tight combinatorial bound of $\Theta(n)$ on the complexity of the motion of the 1-centre of C, an upper bound of $O(n^2\alpha(n))$ on the complexity of the motion of the equidistant 2-centre of C, and a worst-case lower bound of $\Omega(n^2)$ on the complexity of the motion of any 2-centre of C, where $\alpha(n)$ denotes the inverse Ackermann function. We describe efficient algorithms for calculating the trajectories of the 1-centre of C in optimal time $O(n \log n)$ and the equidistant 2-centre of C in time $O(n^2 \log n)$. Moreover, our algorithms have natural implementations as kinetic data structures (KDS). Although previous applications of KDSs have been to mobile problems in Euclidean space (e.g., [1,2,5,6,8,9,13–15,20–22,31]), as we demonstrate, the KDS framework lends itself naturally to mobile problems on graphs.

Finally, we show that the 1-centre is discontinuous on graphs that contain cycles. We describe a unit-velocity 2-approximation and show that no $(2-\epsilon)$-approximation is possible for any $\epsilon > 0$ and any fixed upper bound on velocity.

2 Definitions

Since a *point* refers to a fixed position in a metric space, we refer to a *client* in the context of motion. Let $C = \{c_1, \ldots, c_n\}$ denote a set of mobile clients, where $I = [0, t_f]$ denotes a time interval, U_T denotes the continuum of points [2] defined by a weighted tree $T = (V, E)$, and each c_i is a continuous function $c_i : I \to U_T$. For every $t \in I$, let $C(t) = \{c(t) \mid c \in C\}$ denote the set of

[2] A point in U_T is uniquely defined by an edge (u, v) on which it lies and the distance it lies from u (equivalently, from v) along that edge.

points in U_T that corresponds to the positions of clients in C at time t. The position of a mobile *facility* f is a function of the positions of a set of clients, $f : \mathscr{P}(U_T) \to U_T$, where $\mathscr{P}(A)$ denotes the power set of set A.

A common assumption in kinetic problems involving motion in Euclidean space is that the position of a mobile client can be represented as a bounded-degree polynomial function over time. For comparison against other kinetic data structures, performance bounds are typically derived in terms of motion that is linear, or piecewise linear, where motion plan updates allow the trajectory of a client to be modified (e.g., [1,2,5]). We make a similar assumption and consider clients with linear motion on trees to establish combinatorial bounds. A mobile client or facility a has *linear motion* if for all $t \in I$, $d(a(0), a(t)) = t \cdot v_a$, where v_a is a non-negative fixed real number and $d(b, c)$ denotes the graph distance between points b and c in U_T. We refer to v_a as the *velocity* of a. That is, a follows a continuous trajectory along the path on T between $a(0)$ and $a(t_f)$ with velocity v_a.

Each client's trajectory can be specified by its endpoints in U_T. The distance $d(a(t), b(t))$ can be calculated in constant time for any two mobile clients a and b and any time t. This is achieved by selecting an arbitrary vertex of T as a root, precomputing distances from the root to all vertices in T, and precomputing a lowest common ancestor (LCA) data structure for T (e.g., [7]). The precomputation takes $O(|T|)$ time; the resulting data structure requires $O(|T|)$ space and provides constant-time queries. The distance between two vertices in T corresponds to the sum of their distances to the root minus twice the distance from their LCA to the root. The segments of the spanning tree of $a'(0)$, $a'(t_f)$, $b'(0)$, and $b'(t_f)$ in T can be identified using LCA queries, where $p'(t)$ denotes a vertex of T closest to client $p(t)$. Within this spanning tree, it is straightforward to identify the segment and the distance from each endpoint of the segment in which each of $a(t)$ and $b(t)$ lie, and from this calculate the distance between $a(t)$ and $b(t)$.

We assume an upper bound of one on the velocity of clients since we are interested in *relative velocity*. Unlike mobile clients, a mobile facility is not required to travel along a path in T nor is its velocity required to remain constant. A mobile facility f has *maximum velocity* bounded by v_f if

$$\forall t_1, t_2 \in I,\ d(f(C(t_1)), f(C(t_2))) \leq v_f |t_1 - t_2|, \tag{1}$$

for all sets of mobile clients C defined on any tree T and any time interval I. Continuity is a necessary condition for any fixed upper bound on velocity. Similarly, we say the *rate of change* of function r is bounded by r_f if

$$\forall t_1, t_2 \in I,\ |r(C(t_1)) - r(C(t_2))| \leq r_f |t_1 - t_2|, \tag{2}$$

for all sets of mobile clients C defined on any tree T and any time interval I.

We say that two clients a and b *cross* at time t_0 if

$$a(t_0) = b(t_0) \text{ and } \exists \epsilon > 0 \text{ s.t. } \forall t \in (t_0 - \epsilon, t_0),\ a(t) \neq b(t).$$

In most cases, clients a and b coincide only at the instant t_0. However, if a and b have the same velocity, then their trajectories may merge such that the positions of a and b coincide until their trajectories diverge again. We define the crossing event as the instant t_0 when their two positions first coincide. Since clients a and b have constant velocity and their trajectories intersect in a path, a and b may cross at most once.

We say client $c \in C$ is *extreme* at time t if $c(t)$ does not lie in the interior of any path through T between two clients in $C(t)$. The *convex hull* of $C(t)$ corresponds to the union of all paths between any two clients in $C(t)$. Whereas some definitions of the convex hull on a graph refer to a subset of the vertices [11], we refer to the continuous subset of U_T.

We recall the definition of a (static) k-centre of a client set on a tree.

Definition 1 *Given a weighted tree T and a set of points C in U_T, a k-*centre *of C is a set of k points in U_T, denoted $\Xi_1(C), \ldots, \Xi_k(C)$, that minimizes*

$$\max_{c \in C} \min_{1 \leq i \leq k} d(c, \Xi_i(C)). \tag{3}$$

When $k = 1$, we omit the subscript and write $\Xi(C)$. Similarly, we write simply Ξ_i when C is implicit. The definition of a mobile k-centre of a set of mobile clients C follows directly from this static definition. That is, the instantaneous positions of a mobile k-centre of C at time t is given by Definition 1 in terms of $C(t)$.

We refer to the value of (3) as the *k-radius* of C or simply as its *radius* when $k = 1$. The diameter of C is twice the radius of C [26] (for graphs, the diameter is at most twice the radius). A *diametric path* of C is a path between two clients c_1 and c_2 in C such that the distance between them is the diameter of C. We refer to $\{c_1, c_2\}$ as a *diametric pair* and to c_1 and c_2 as *diametric clients*. The 1-centre of C is the unique midpoint of all diametric paths of C [25].

The 1-centre problem on graphs is also known as the absolute centre [25–27], single centre [26], and minimax location problem [12,25]. A common variation of the k-centre problem on graphs is known as the vertex k-centre or discrete k-centre problem, for which the choice of locations for the facility is restricted to vertices (clients) of the graph G. Maintaining continuity in the motion of a mobile facility is impossible in the vertex centre model, as a facility could be required to jump discontinuously from vertex to vertex (client to client).

3 Related Work

Handler [25] gives linear-time algorithms for identifying the 1-centre and 2-centre of a tree. Frederickson gives a linear-time algorithm for finding a k-centre of a tree when k is fixed [19]. Kariv and Hakimi [32] provide an $O(mn + n^2 \log n)$-time algorithm for the 1-centre problem on graphs, where $n = |V|$ and $m = |E|$. Tamir [33] gives an $O(m^k n^k \log^2 n)$-time algorithm for the k-centre on graphs, where k is fixed. The problem is NP-hard if k is an input parameter [32]. A review of 1-centre and k-centre problems on trees and on graphs can be found in [17,24,28,32,34,35].

Kinetic data structures (KDS), introduced by Basch et al. [5], allow the maintenance of an attribute (called the configuration function) of a set of mobile objects moving continuously in some metric space. To do so, a KDS maintains a dynamic set of certificates that guarantees the correctness of the configuration function at any time during the motion. Each certificate c is associated with a small set of mobile objects for which some property is verified. The failure time of certificate c (called an event) is calculated as a function of the motion of these objects. The failure time is added to a priority queue. Restoring the configuration function following a certificate failure requires updating the set of certificates (and the corresponding events in the queue).

Guibas [23] describes four properties used to evaluate the quality of a KDS. A KDS is *compact* if the maximum number of certificates active at any given time is linear or near-linear in the degrees of freedom of the set of moving objects. A KDS is *local* if the maximum number of certificates associated with any one mobile object is polylogarithmic in the problem size. A KDS is *responsive* if at most a small number of certificates require updating as a result of a certificate failure. A KDS is *efficient* if the total number of certificate failures is proportional to the number of external events (changes to the configuration function). See [4–6,22,23] for a more complete description of the KDS framework.

In relation to our work on the mobile k-centre, KDSs have been constructed to maintain various attributes of a set of mobile clients; these include extremal elements in $\mathbb{R}$ [1,2,5,22], the extent and approximate extent (e.g., diameter and width) in $\mathbb{R}^2$ [1,2], approximations of the mobile 1-centre in $\mathbb{R}^2$ [2,8,13,14], approximations of mobile 2-centres in $\mathbb{R}^2$ [13,15], the mobile rectilinear 1-centre in $\mathbb{R}^2$ [2,9], the kinetic convex hull [5,6,22], an approximation of mobile k-centres in $\mathbb{R}^d$ [21], and approximations of discrete rectilinear k-centres [20,31].

In any metric space, identifying a pair of furthest clients in a set of mobile clients corresponds to finding the upper envelope (the maximum function) of a set of distance functions. This problem is related to Davenport-Schinzel sequences [3,18,29,30,36]. In particular, the upper (lower) envelope of a set of n line segments is a piecewise-linear function that consists of $\Theta(n\alpha(n))$ linear segments [29] in the worst case. Hershberger [30] provides an algorithm for computing the upper envelope in optimal $O(n \log n)$ time.

4 The Mobile 1-Centre on Trees

4.1 Properties of the Mobile 1-Centre

The mobile 1-centre is continuous in $\mathbb{R}^d$ [13]. Although the mobile 1-centre has at most unit relative velocity in $\mathbb{R}$, its relative velocity is unbounded in $\mathbb{R}^2$ [9]. As we show in Section 6, the mobile 1-centre is discontinuous on graphs. Restricted to trees, however, we show that the mobile 1-centre remains continuous and has at most unit relative velocity.

Theorem 1 *The mobile 1-centre has relative velocity at most one on trees. This bound is tight.*

Proof. Choose any $t_1, t_2 \in I$ and let $\delta = |t_1 - t_2|$. If $\Xi(t_1) = \Xi(t_2)$, then (1) holds trivially. Therefore, assume $\Xi(t_1) \neq \Xi(t_2)$. Let P denote the interior of the path in U_T between $\Xi(t_1)$ and $\Xi(t_2)$. Let r_1 and r_2 denote the respective radii of $C(t_1)$ and $C(t_2)$. Let L_1 denote the subtree of $U_T \setminus P$ incident to $\Xi(t_1)$. Similarly, let L_2 denote the subtree of $U_T \setminus P$ incident to $\Xi(t_2)$. See Fig. A.1.

Let a be a client in C such that $a(t_1) \in L_1$ and $d(a(t_1), \Xi(t_1)) = r_1$. Similarly, let b be a client in C such that $b(t_2) \in L_2$ and $d(b(t_2), \Xi(t_2)) = r_2$. Such clients must exist since $\Xi(t)$ is the midpoint of a diametric path of $C(t)$ for all t. Therefore,

$$\begin{aligned}
d(a(t_1), b(t_2)) &\leq d(a(t_1), \Xi(t_1)) + d(\Xi(t_1), b(t_1)) + d(b(t_1), b(t_2)) \\
&\leq 2r_1 + \delta, && \text{(4a)} \\
\text{and } d(a(t_1), b(t_2)) &\leq d(a(t_1), a(t_2)) + d(a(t_2), \Xi(t_2)) + d(\Xi(t_2), b(t_2)) \\
&\leq 2r_2 + \delta. && \text{(4b)}
\end{aligned}$$

Consequently,

$$\begin{aligned}
d(a(t_1), b(t_2)) &= d(a(t_1), \Xi(t_1)) + d(\Xi(t_1), \Xi(t_2)) + d(\Xi(t_2), b(t_2)), \\
\Rightarrow d(\Xi(t_1), \Xi(t_2)) &= d(a(t_1), b(t_2)) - d(a(t_1), \Xi((t_1)) - d(\Xi(t_2), b(t_2)) \\
&= d(a(t_1), b(t_2)) - r_1 - r_2 \\
&\leq \delta,
\end{aligned}$$

by (4a) and (4b). The bound is realized when the two diametric clients move in a parallel direction with equal velocity. □

Corollary 2 *The mobile 1-centre is continuous on trees.*

Since clients move with at most unit velocity, the relative rate of change of the diameter is at most two. Consequently:

Observation 3 *The relative rate of change of the radius is at most one on trees.*

We refer to the following lemma by Handler:

Lemma 4 (Handler 1973 [25]) *Given a set of clients C on a tree T, clients $a, b \in C$ are a diametric pair of C if and only if $d(a, b) \geq \max\{d(a, c), d(b, c)\}$ for all $c \in C$.*

4.2 Complexity of the Motion of the 1-Centre

When n clients move along the real line, each with some constant velocity, the identity of the client that realizes either extremum changes $\Theta(n)$ times in the worst case [5]. In particular, any given client realizes each extremum at most once in the sequence of changes. When n clients move in $\mathbb{R}^2$ along linear trajectories with constant velocity, the diametric pair of clients changes $\Omega(n^2)$ times in the worst case [1]. As we show in Theorem 11, for a set C of n clients with linear motion on a tree T, the identity of the diametric pair of C changes $\Theta(n)$ times in the worst case. We assume linear motion of a set of clients C on a tree T throughout Section 4.2. We begin with a definition.

Definition 2 *The* outward velocity *of client c at time t, denoted $\vec{v}_c(t)$, is given by*

$$\vec{v}_c(t) = \lim_{\epsilon \to 0^+} \frac{d(\Xi(t), c(t+\epsilon)) - d(\Xi(t), c(t))}{\epsilon}. \tag{5}$$

Observe that $\vec{v}_c(t) = \pm v_c$, where v_c denotes the velocity of c. Specifically, the outward velocity of client c assigns an orientation to its velocity relative to $\Xi(t)$. That is, $\vec{v}_c(t) = v_c$ if $c(t)$ moves toward the boundary of the convex hull (away from $\Xi(t)$) and $\vec{v}_c(t) = -v_c$ otherwise.

Lemma 5 *If $c(t) \neq \Xi(t)$ for all $t \in [t_1, t_2]$, then $\vec{v}_c(t)$ is non-decreasing over $t \in [t_1, t_2]$.*

Proof. For all $t \in [t_1, t_2]$, $\vec{v}_c(t) = \pm v_c$.

Case 1. Suppose $\vec{v}_c(t_1) = v_c$. Let P denote the path in U_T between $c(t_1)$ and $c(t_2)$. For any $t \in [t_1, t_2]$, the subpath of P that remains to be travelled by

c lies opposite $c(t)$ from $\Xi(t)$ since $c(t)$ and $\Xi(t)$ do not cross. Therefore, c continues moving away from $\Xi(t)$ and $\vec{v}_c(t)$ remains constant.

Case 2. Suppose $\vec{v}_c(t_1) = -v_c$. The outward velocity of c remains constant until some $t \in [t_1, t_2]$ when c branches and turns away from $\Xi(t)$. The remainder of the motion corresponds to Case 1. $\square$

Corollary 6 *The outward velocity of client c is non-decreasing while c remains diametric and the diameter of C is non-zero.*

By Lemma 5, it follows that the average outward velocity of a client over any subinterval of $[t_1, t_2]$ is bounded from below by $\vec{v}_c(t_1)$ and from above by $\vec{v}_c(t_2)$, assuming $c(t) \neq \Xi(t)$ for all $t \in [t_1, t_2]$. That is:

Observation 7 *If $c(t) \neq \Xi(t)$ for all $t \in [t_1, t_2]$, then*

$$\forall [t_1', t_2'] \subseteq [t_1, t_2],\ \vec{v}_c(t_1') \leq \frac{d(\Xi(t_1'), c(t_2')) - d(\Xi(t_1'), c(t_1'))}{|t_2' - t_1'|} \leq \vec{v}_c(t_2'). \tag{6}$$

Let $D(t) \subseteq C$ denote the set of diametric clients of $C(t)$; that is, $c \in D(t)$ if and only if $c(t)$ is diametric in $C(t)$. Let $D'(t) = \lim_{\epsilon \to 0^+} D(t + \epsilon)$. This limit exists since D changes discretely. Let $\vec{V}(t)$ denote the set of outward velocities of $D'(t)$; that is, $\vec{V}(t) = \{\vec{v}_c(t) \mid c \in D'(t)\}$.

As we now show, if multiple pairs of clients remain diametric throughout some time interval, then the corresponding pairs of outward velocities coincide. In other words, $\vec{V}(t)$ has cardinality at most two: one value for each client in a diametric pair.

Lemma 8 *If the diameter of $C(t)$ is non-zero and $\{a_1, b_1\}$ and $\{a_2, b_2\}$ are diametric pairs of $C(t)$ for all $t \in [t_1, t_2]$, then $\{\vec{v}_{a_1}(t), \vec{v}_{b_1}(t)\} = \{\vec{v}_{a_2}(t), \vec{v}_{b_2}(t)\}$ for all $t \in [t_1, t_2)$.*

Proof. Choose any $t \in [t_1, t_2)$. Choose any $\epsilon \in (0, \min(r(t)/2, t_2 - t))$, where $r(t)$ denotes the radius of $C(t)$. Since a_1, b_1, a_2, and b_2 are diametric for the duration of the time interval $[t_1, t_2]$, therefore, for all $t' \in [t_1, t_2]$,

$$d(a_1(t'), \Xi(t')) = d(b_1(t'), \Xi(t')) = d(a_2(t'), \Xi(t')) = d(b_2(t'), \Xi(t')). \tag{7}$$

Case 1. Suppose $\Xi(t) = \Xi(t + \epsilon)$. Therefore, By (7),

$$d(a_1(t+\epsilon), \Xi(t)) = d(a_1(t+\epsilon), \Xi(t+\epsilon)) = d(a_2(t+\epsilon), \Xi(t+\epsilon)) = d(a_2(t+\epsilon), \Xi(t)).$$

Similarly, we get $d(b_1(t + \epsilon), \Xi(t)) = d(b_2(t + \epsilon), \Xi(t))$.

Case 2. Suppose $\Xi(t) \neq \Xi(t+\epsilon)$. Let P denote the path in U_T between $\Xi(t)$ and $\Xi(t+\epsilon)$. Since $\epsilon < r(t)/2$ and by Theorem 1, every client $c \in \{a_1, b_1, a_2, b_2\}$ must lie outside P during the time interval $[t, t+\epsilon]$. Furthermore, since c is diametric at times t and $t+\epsilon$, either $\Xi(t)$ lies on the path between $c(t)$ and $\Xi(t+\epsilon)$, or $\Xi(t+\epsilon)$ lies on the path between $c(t)$ and $\Xi(t)$. The same holds for $c(t+\epsilon)$, $\Xi(t)$, and $\Xi(t+\epsilon)$. Without loss of generality, assume a_1 and a_2 lie on the same side of P, say the side nearest to $\Xi(t)$, while b_1 and b_2 lie on the opposite side, nearest to $\Xi(t+\epsilon)$. By (7),

$$\begin{aligned} d(a_1(t+\epsilon), \Xi(t)) &= d(a_1(t+\epsilon), \Xi(t+\epsilon)) + d(\Xi(t), \Xi(t+\epsilon)) \\ &= d(a_2(t+\epsilon), \Xi(t+\epsilon)) + d(\Xi(t), \Xi(t+\epsilon)) \\ &= d(a_2(t+\epsilon), \Xi(t)). \end{aligned}$$

Similarly,

$$\begin{aligned} d(b_1(t+\epsilon), \Xi(t)) &= d(b_1(t+\epsilon), \Xi(t+\epsilon)) - d(\Xi(t), \Xi(t+\epsilon)) \\ &= d(b_2(t+\epsilon), \Xi(t+\epsilon)) - d(\Xi(t), \Xi(t+\epsilon)) \\ &= d(b_2(t+\epsilon), \Xi(t)). \end{aligned}$$

Therefore, in all cases, $d(a_1(t+\epsilon), \Xi(t)) = d(a_2(t+\epsilon), \Xi(t))$ and $d(b_1(t+\epsilon), \Xi(t)) = d(b_2(t+\epsilon), \Xi(t))$. Consequently, by (7) and Definition 2, for all $\epsilon \in (0, \min(r(t)/2, t_2 - t))$,

$$\begin{gathered} \frac{d(\Xi(t), a_1(t+\epsilon)) - d(\Xi(t), a_1(t))}{\epsilon} = \frac{d(\Xi(t), a_2(t+\epsilon)) - d(\Xi(t), a_2(t))}{\epsilon} \\ \Rightarrow \vec{v}_{a_1}(t) = \vec{v}_{a_2}(t). \end{gathered}$$

Similarly, $\vec{v}_{b_1}(t) = \vec{v}_{b_2}(t)$. $\square$

By Lemma 8, $|\vec{V}(t)| \leq 2$. Let $\{\vec{v}_{\min}(t), \vec{v}_{\max}(t)\} = \vec{V}(t)$ such that $\vec{v}_{\min}(t) \leq \vec{v}_{\max}(t)$. If $|\vec{V}(t)| = 1$, then $\vec{v}_{\min}(t) = \vec{v}_{\max}(t)$.

As we now show, the pair of outward velocities of diametric clients is non-decreasing over time and, furthermore, any change in diametric clients corresponds to an increase in one or both outward velocities.

Lemma 9 *If the set of diametric clients of C changes at time t_0 and the diameter of $C(t_0)$ is non-zero, then $\exists \epsilon > 0$ such that $\forall t_1 \in (t_0 - \epsilon, t_0)$, $\forall t_2 \in (t_0, t_0 + \epsilon)$,*

$$\vec{v}_{\min}(t_1) < \vec{v}_{\min}(t_2) \wedge \vec{v}_{\max}(t_1) \leq \vec{v}_{\max}(t_2), \text{ or} \tag{8}$$

$$\vec{v}_{\min}(t_1) \leq \vec{v}_{\min}(t_2) \wedge \vec{v}_{\max}(t_1) < \vec{v}_{\max}(t_2). \tag{9}$$

Proof. The distance function between any two clients is piecewise-linear, consisting of at most three linear segments. Clients realizing the maximum of these $\binom{n}{2}$ functions at time t correspond to the set of diametric clients at time t. Consequently, the set of changes to the set of diametric clients is discrete and has bounded cardinality.

Select $\epsilon > 0$ such that the following properties hold:

(1) the set of diametric clients of $C(t_1)$ remains unchanged for all $t_1 \in (t_0 - \epsilon, t_0)$,
(2) the set of diametric clients of $C(t_2)$ remains unchanged for all $t_2 \in (t_0, t_0 + \epsilon)$,
(3) If $a(t)$ is diametric in $C(t)$ for some $t \in (t_0 - \epsilon, t_0 + \epsilon)$, then $\Xi(t') \neq a(t'')$ for all $t', t'' \in (t_0 - \epsilon, t_0 + \epsilon)$.

Properties 1 and 2 are easily satisfied since the set of diametric clients changes discretely. Since the diameter of $C(t_0)$ is non-zero, and clients and Ξ move with bounded velocity, it follows that for some $\epsilon > 0$, any client that is diametric during the time interval $(t_0 - \epsilon, t_0 + \epsilon)$ will not intersect the subset of U_T covered by Ξ during that time; therefore, Property 3 can also be satisfied.

Choose any $t_1 \in (t_0 - \epsilon, t_0)$ and any $t_2 \in (t_0, t_0 + \epsilon)$.

Case 1. Suppose some client a_1 is diametric in $C(t_1)$ but not in $C(t_2)$. Let b_1 denote a client that forms a diametric pair of $C(t_1)$ with a_1 and let $\{a_2, b_2\}$ denote a diametric pair of $C(t_2)$. Therefore,

$$\forall \{c_1, c_2\} \in C,\ d(a_1(t_1), b_1(t_1)) \geq d(c_1(t_1), c_2(t_1)). \tag{10a}$$
$$\forall \{c_3, c_4\} \in C,\ d(a_2(t_2), b_2(t_2)) \geq d(c_3(t_2), c_4(t_2)). \tag{10b}$$
$$\forall c_5 \in C,\ d(a_1(t_2), c_5(t_2)) < d(a_2(t_2), b_2(t_2)). \tag{10c}$$
$$\text{(10a) and (10b)} \Rightarrow d(a_1(t_0), b_1(t_0)) = d(a_2(t_0), b_2(t_0)). \tag{10d}$$

$\Xi(t_0)$ lies between $a_1(t_0)$ and $b_1(t_0)$ and, similarly, $\Xi(t_0)$ lies between $a_2(t_0)$ and $b_2(t_0)$. Consequently, either $\Xi(t_0)$ lies between $b_1(t_0)$ and $a_2(t_0)$ or $\Xi(t_0)$ lies between $b_1(t_0)$ and $b_2(t_0)$ (or both). Without loss of generality, assume $\Xi(t_0)$ lies between $b_1(t_0)$ and $a_2(t_0)$. It follows that $\Xi(t_0)$ lies between $a_1(t_0)$ and $b_2(t_0)$. Furthermore, by Property 3, $\Xi(t_0)$ lies between $a(t)$ and $b(t)$, for any $a \in \{a_1, a_2\}$, any $b \in \{b_1, b_2\}$, and any $t \in (t_0 - \epsilon, t_0 + \epsilon)$. By (10c), this gives,

$$\begin{aligned}
d(a_2(t_2), \Xi(t_0)) + d(\Xi(t_0), b_2(t_2)) &= d(a_2(t_2), b_2(t_2)) \\
&> d(a_1(t_2), b_2(t_2)) \\
&= d(a_1(t_2), \Xi(t_0)) + d(\Xi(t_0), b_2(t_2)) \\
\Rightarrow d(a_2(t_2), \Xi(t_0)) &> d(a_1(t_2), \Xi(t_0)) \\
\Rightarrow d(a_2(t_2), b_1(t_2)) &> d(a_1(t_2), b_1(t_2)).
\end{aligned} \tag{11}$$

By a similar argument, we get

$$d(b_2(t_2), a_1(t_2)) \geq d(b_1(t_2), a_1(t_2)). \tag{12}$$

Given a mobile client c, let

$$f(c) = \begin{cases} 1 & \text{if } d(c(t_1), \Xi(t_0)) > d(c(t_2), \Xi(t_0)), \\ -1 & \text{otherwise.} \end{cases} \tag{13}$$

By Observation 7 and by (10a), (10c), and (11),

$$\begin{aligned}
\vec{v}_{a_1}(t_1) &\leq \frac{d(a_1(t_2), \Xi(t_1)) - d(a_1(t_1), \Xi(t_1))}{t_2 - t_1} \\
&= \frac{d(a_1(t_2), b_1(t_1)) - d(a_1(t_1), b_1(t_1))}{t_2 - t_1} \\
&= \frac{d(a_1(t_2), b_1(t_2)) + f(b_1)d(b_1(t_1), b_1(t_2)) - d(a_1(t_1), b_1(t_1))}{t_2 - t_1} \\
&< \frac{d(a_2(t_2), b_1(t_2)) + f(b_1)d(b_1(t_1), b_1(t_2)) - d(a_2(t_1), b_1(t_1))}{t_2 - t_1} \\
&= \frac{d(a_2(t_2), b_1(t_1)) - d(a_2(t_1), b_1(t_1))}{t_2 - t_1} \\
&= \frac{d(a_2(t_2), \Xi(t_1)) - d(a_2(t_1), \Xi(t_1))}{t_2 - t_1} \\
&\leq \vec{v}_{a_2}(t_2).
\end{aligned} \tag{14}$$

By Observation 7 and by (10b) and (12),

$$\begin{aligned}
\vec{v}_{b_1}(t_1) &\leq \frac{d(b_1(t_2), \Xi(t_1)) - d(b_1(t_1), \Xi(t_1))}{t_2 - t_1} \\
&= \frac{d(b_1(t_2), a_1(t_1)) - d(b_1(t_1), a_1(t_1))}{t_2 - t_1} \\
&= \frac{d(b_1(t_2), a_1(t_2)) + f(a_1)d(a_1(t_1), a_1(t_2)) - d(b_1(t_1), a_1(t_1))}{t_2 - t_1} \\
&\leq \frac{d(b_2(t_2), a_1(t_2)) + f(a_1)d(a_1(t_1), a_1(t_2)) - d(b_2(t_1), a_1(t_1))}{t_2 - t_1} \\
&= \frac{d(b_2(t_2), a_1(t_1)) - d(b_2(t_1), a_1(t_1))}{t_2 - t_1} \\
&= \frac{d(b_2(t_2), \Xi(t_1)) - d(b_2(t_1), \Xi(t_1))}{t_2 - t_1} \\
&\leq \vec{v}_{b_2}(t_2).
\end{aligned} \tag{15}$$

By (14) and (15),

$$\forall t_1 \in (t_0 - \epsilon, t_0), \forall t_2 \in (t_0, t_0 + \epsilon),\ \vec{v}_{a_1}(t_1) < \vec{v}_{a_2}(t_2) \text{ and } \vec{v}_{b_1}(t_1) \leq \vec{v}_{b_2}(t_2).$$

Observe that $\{\vec{v}_{\min}(t_i)\vec{v}_{\max}(t_i)\} = \{\vec{v}_{a_i}(t_i)\vec{v}_{b_i}(t_i)\}$ for $i \in \{1, 2\}$. Consequently, (8) or (9) holds.

Case 2. Suppose all clients that are diametric in $C(t_1)$ remain diametric in $C(t_2)$. Since the set of diametric clients changes at time t_0, some client a_2 must be diametric in $C(t_2)$ but not in $C(t_1)$. Let b_2 denote a client that forms a diametric pair of $C(t_2)$ with a_2 and let $\{a_1, b_1\}$ denote a diametric pair of $C(t_1)$. The result follows by an argument analogous to Case 1.

In both cases we get that (8) or (9) holds. □

Lemma 10 *While the diameter remains non-zero, a client $c \in C$ becomes an endpoint of a diametric path of C at most four times.*

Proof. The outward velocity of a diametric client c is one of two values: $\pm v_c$. By Lemma 9, a change in a diametric pair corresponds to an increase in outward velocity. Therefore, for any client $c \in C$, $\vec{v}_{\min}$ assumes the value $-v_c$ at most once and the value v_c at most once. Similarly, $\vec{v}_{\max}$ assumes each of these values at most once. The result follows. □

Theorem 11 *When each client in C moves with linear motion along a path on T, the motion of the 1-centre of C is piecewise linear and is composed of $\Theta(n)$ linear segments in the worst case, where $n = |C|$.*

Proof. *Case 1.* Suppose the diameter of C is non-zero throughout the motion. The upper bound $O(n)$ follows from Corollary 6, Lemmas 9 and 10, and the fact that the 1-centre of C is the midpoint of a diametric pair.

Case 2. Suppose the diameter of C is zero at some time during the motion. A zero diameter implies that all clients in C coincide at a point; that is, all clients cross simultaneously. This degeneracy occurs at most once since any two clients cross at most once. Since clients in C have linear motion, the 1-centre of C has linear motion while all clients coincide. Before and after the degeneracy, the motion of clients in C corresponds to Case 1. Therefore, the sum of the number of linear segments of the motion of the 1-centre remains $O(n)$.

The worst-case lower bound of $\Omega(n)$ follows from the corresponding result in one dimension [5]. □

Given a set C of n mobile clients, each moving with linear motion in $\mathbb{R}$, the 1-centre of C is the midpoint of the extrema of C. The position of each extremum is given by the upper (respectively, lower) envelope of the set of n linear functions that correspond to the positions of clients in C relative to a fixed point in $\mathbb{R}$. Hershberger [30] gives an $O(n \log n)$ time algorithm which finds the upper envelope by dividing the set of linear functions in two, recursively finding the upper envelope of each set, and recombining the two envelopes to give the upper envelope of the union of the two sets.

Using a related idea, we describe an algorithm for identifying a sequence of diametric pairs of a set of mobile clients, each moving with linear motion on a tree. We then describe how to implement the algorithm as a KDS. We begin with the following lemma upon which our algorithm relies.

Lemma 12 *Let C_1 and C_2 be sets of points on U_T for some tree T. Let $\{a_i, b_i\}$ denote a diametric pair of C_i, for $i = 1, 2$. Set $\{e, f\}$ is a diametric pair of $C_1 \cup C_2$, where*

$$\{e, f\} = \operatorname*{argmax}_{\{e', f'\} \subseteq \{a_1, b_1, a_2, b_2\}} d(e', f'). \tag{16}$$

Proof. By Lemma 4 it suffices to show

$$\forall c \in C_1 \cup C_2,\ \max\{d(c, e), d(c, f)\} \leq d(e, f).$$

We will show $d(c, e) \leq d(e, f)$; an analogous argument can be used to show $d(c, f) \leq d(e, f)$. Choose any $c \in C_1 \cup C_2$. Without loss of generality, assume $c \in C_1$.

Case 1. Suppose $e \in \{a_1, b_1\}$. By Lemma 4 and the definition of e and f we get that $d(c, e) \leq d(a_1, b_1) \leq d(e, f)$.

Case 2. Suppose $e \in \{a_2, b_2\}$. Without loss of generality, assume $e = a_2$. Let T' denote the spanning tree of a_1, b_1, and a_2 in U_T. Let q denote the point of T' that is closest to c (note, $q = c$ if $c \in T'$). By Lemma 4,

$$\begin{aligned} & d(b_1, c) \leq d(b_1, a_1), \\ \Rightarrow \quad & d(b_1, q) + d(q, c) \leq d(b_1, q) + d(q, a_1), \\ \Rightarrow \quad & d(q, c) \leq d(q, a_1). \end{aligned} \tag{17}$$

By an analogous argument, we get that

$$d(q, c) \leq d(q, b_1). \tag{18}$$

By definition of T', q must lie on the path between a_1 and a_2 or on the path between b_1 and a_2 (or both).

Case 2a. Suppose q lies on the path between a_1 and a_2. By (17) and the definition of e and f, therefore,

$$\begin{aligned} d(c,e) &= d(c,a_2) \\ &= d(c,q) + d(q,a_2) \\ &\leq d(a_1,q) + d(q,a_2) \\ &= d(a_1,a_2) \\ &\leq d(e,f). \end{aligned}$$

Case 2b. Suppose q lies on the path between b_1 and a_2. The result follows by (18) and an argument analogous to Case 2a. □

Algorithm Description. The set of mobile clients C is partitioned arbitrarily into sets C_1 and C_2 of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. For each $i = 1, 2$, the algorithm is called recursively to find a sequence of diametric pairs of C_i, denoted $\{a_{i,1}, b_{i,1}\}, \ldots, \{a_{i,m_i}, b_{i,m_i}\}$, and a corresponding partition of the time interval I, denoted $I_{i,1}, \ldots, I_{i,m_i}$, such that for each j, $\{a_{i,j}(t), b_{i,j}(t)\}$ is a diametric pair of $C_i(t)$ for all $t \in I_{i,j}$. The recursion terminates when $n \leq 2$, in which case each client in C is in a diametric pair. We now describe how to compute a corresponding sequence for C.

Consider a third partition of the time interval I, denoted $I_1, \ldots, I_m$, such that for each i, $I_i = I_{1,j} \cap I_{2,k}$, for some j, k. For all $t \in I_i$, diametric pairs of $C_1(t)$ and $C_2(t)$ consist of four clients in C, say a_1, b_1, a_2, and b_2. Let e and f be defined as in (16). By Lemma 12, e and f are a diametric pair of $C(t)$. The sequence of pairs of clients in $\{a_1, b_1, a_2, b_2\}$ that realize e and f corresponds to the sequence of pairs whose relative distance is maximized. That is, there are six combinations of pairs in $\{a_1, b_1, a_2, b_2\}$, each of which corresponds to an inter-client distance function. The upper envelope of these six functions determines the sequence of identities of e and f during I_i. Thus, solutions to the recursive subproblems are combined to find the sequence of diametric pairs of C.

Time Complexity. By Theorem 11, the complexity of the motion of the 1-centres of C_1 and C_2 is $O(n)$. That is, the time interval I can be partitioned into $O(n)$ subintervals such that the motion of each 1-centre is linear within every subinterval (i.e., $m \in O(n)$). Within each subinterval, we find the maximum of six piecewise-linear functions, each composed of at most four linear segments. Therefore, the maximum function is also piecewise linear, consists of at most

24 linear segments, and can be found in constant time. Thus, the solutions to the two subproblems are combined in $O(n)$ time. The recursion tree has depth $\lfloor \log_2 n \rfloor$, resulting in a total runtime of $O(n \log n)$. The worst-case lower bound of $\Omega(n \log n)$ follows from the corresponding one-dimensional problem [30].

KDS Implementation. We describe a KDS that maintains a diametric pair over time along with a set of certificates that validates the identity of the pair at any time during the motion.

Theorem 13 *Given a tree T and a set of mobile clients C, each moving with linear motion on a path of T, there exists a KDS to maintain the mobile 1-centre of C that is local, responsive, efficient, and compact.*

Proof. The set of certificates corresponds to the recursive hierarchy described in our algorithm. At any time t, for each set C in the hierarchy, the certificate for $C(t)$ consists of five inequalities that confirm the maximum of six functions. That is, the certificate verifies the identity of a diametric pair of $C(t)$ in terms of the diametric pairs of the subsets $C_1(t)$ and $C_2(t)$ by Lemma 12. The corresponding properties are certified recursively for $C_1(t)$ and $C_2(t)$. Each set maintains a single certificate defined in terms of four clients and the total number of certificates is $O(n)$; therefore, the KDS is compact. Each client is contained in at most $O(\log n)$ sets and, consequently, is associated with at most $O(\log n)$ certificates. As a result, a motion plan update for a client results in changes to the failure times of $O(\log n)$ certificates; therefore, the KDS is local.

A certificate failure occurs whenever the diametric pair of a set C changes. Locally, the certificate for C is restored in constant time; however, a change in the diametric pair of C may percolate upwards in the tree, resulting in $O(\log n)$ additional certificate updates; therefore, the KDS is responsive. By Theorem 11, each set C contributes at most $O(|C|)$ certificate failures, resulting in a total of $O(n \log n)$ certificate failures over the entire motion. Although this value is asymptotically greater than $\Theta(n)$ (the worst-case number of external events for a set of n clients), any offline algorithm for finding the trajectory of the 1-centre requires $\Omega(n \log n)$ time in the worst case, even in one dimension [30]. Therefore, the KDS is efficient. □

Note, linear motion is not required by this KDS. In particular, the KDS applies to any algebraic motion for which the client-to-client distance function permits calculating the failure time of a certificate. In general, the combinatorial bounds and running times mentioned earlier do not apply to non-linear motion.

5 The Mobile 2-Centre on Trees

5.1 Properties of the Mobile 2-Centre

Although a 2-centre of a set of clients C on a tree is not unique (this is the case even in one dimension [13]), any 2-centre of C, $\Xi_1(C)$ and $\Xi_2(C)$, defines a natural bipartition of C, denoted $\{C_1, C_2\}$, such that

$$\forall c \in C_1,\ d(c, \Xi_1(C)) \leq d(c, \Xi_2(C)) \quad \text{and} \quad \forall c \in C_2,\ d(c, \Xi_1(C)) \geq d(c, \Xi_2(C)). \tag{19}$$

We refer to $\{C_1, C_2\}$ as a *diametric partition* of C and to C_1 and C_2 as *diametric subsets* of C. A diametric partition induced by a given 2-centre is not unique. Since the 2-radius of C is at most the radius, it follows that there exists a diametric pair $\{a, b\}$ such that $a \in C_1$ and $b \in C_2$. As shown by Handler [26], the following property is equivalent to (19):

$$\forall c \in C_1,\ d(c, a) \leq d(c, b) \quad \text{and} \quad \forall c \in C_2,\ d(c, a) \geq d(c, b). \tag{20}$$

We refer to the *local 1-centre*, *local radius*, and *local diametric pair/path*, respectively, in reference to the 1-centre, radius, and diametric pair/path of C_1 or C_2. The local 1-centres of C_1 and C_2 are a 2-centre of C [26].

We refer to the following lemma by Handler:

Lemma 14 (Handler 1978 [26]) *Any local diametric pair includes one diametric client in C.*

5.2 Equidistant 2-Centre

Even in one dimension the motion of a 2-centre defined by two local 1-centres is not continuous. This is easily demonstrated by an example: position a client at each endpoint of a line segment and let a third client move from one endpoint to the other. Not all 2-centres are discontinuous; we describe a strategy for defining the positions of a 2-centre on a tree whose motion is continuous and whose relative velocity is at most two. We refer to this particular 2-centre as the *equidistant 2-centre*:

Definition 3 *Let $\{a, b\}$ be a diametric pair of C. An* equidistant 2-centre *of C, denoted $\{\dot{\Xi}_1(C), \dot{\Xi}_2(C)\}$, is a pair of points that lie on the path between a and b at a distance ρ from a and b, respectively, where ρ denotes the 2-radius of C.*

See Fig. A.2 for an example. As we did for Ξ_i, we write simply $\dot{\Xi}_i$ when C is implicit. As we now demonstrate, the equidistant 2-centre is independent of the choice of the diametric pair $\{a, b\}$.

Lemma 15 *The equidistant 2-centre is unique.*

Proof. If C has a unique diametric pair, then the equidistant 2-centre is also unique by Definition 3. Therefore, assume C has two or more diametric pairs. Choose any two diametric pairs, $\{a_1, b_1\}$ and $\{a_2, b_2\}$. Without loss of generality, assume

$$d(a_1, a_2) \leq d(a_1, b_2). \tag{21}$$

Let $\{\dot{\Xi}_1, \dot{\Xi}_2\}$ denote the equidistant 2-centre defined in terms of $\{a_1, b_1\}$ and let $\{\ddot{\Xi}_1, \ddot{\Xi}_2\}$ denote the equidistant 2-centre defined in terms of $\{a_2, b_2\}$. Without loss of generality, assume $d(a_1, \dot{\Xi}_1) \leq d(a_1, \dot{\Xi}_2)$ and $d(a_2, \ddot{\Xi}_1) \leq d(a_2, \ddot{\Xi}_2)$. Let ρ denote the 2-radius of C.

If $a_1 = a_2$, then $\dot{\Xi}_1 = \ddot{\Xi}_1$ by Definition 3. Therefore, assume $a_1 \neq a_2$. By (3) and the triangle inequality, $\min\{d(a_1, a_2), d(a_1, b_2)\} \leq 2\rho$. By (21), $d(a_1, a_2) \leq 2\rho$. Let v denote the vertex of T that joins the branches containing a_1, a_2, and Ξ, respectively. Since a_1 and a_2 are diametric clients, $d(a_1, \Xi) = d(a_2, \Xi)$ and, therefore, $d(a_1, v) = d(a_2, v)$. Consequently, $d(a_1, v) = d(a_2, v) \leq \rho$. The point that lies at a distance ρ from a_1 on the path between a_1 and Ξ coincides with the point that lies at a distance ρ from a_2 on the path between a_2 and Ξ. That is, $\dot{\Xi}_1 = \ddot{\Xi}_1$.

Since $\{a_1, b_1\}$ and $\{a_2, b_2\}$ are diametric pairs and by (21), the path in T from a_1 to a_2 need not pass through Ξ whereas the path from a_1 to b_2 must pass through Ξ. Consequently, $d(b_1, b_2) \leq d(b_1, a_2)$. Therefore, an analogous argument can be used to show $\dot{\Xi}_2 = \ddot{\Xi}_2$. $\square$

Corollary 16 *$\dot{\Xi}_1(C)$ and $\dot{\Xi}_2(C)$ lie in the intersection of all diametric paths of C.*

Lemma 17 *The equidistant 2-centre of C is a 2-centre of C.*

Proof. Choose any client $c \in C$. Let $\{a, b\}$ be a diametric pair of C. Let v denote the point in U_T that joins the branch containing c to the path between a and b (c may coincide with v). Let $\{C_1, C_2\}$ be a diametric partition of C induced by a and b such that $a \in C_1$. Without loss of generality, assume $c \in C_1$ and $d(a, \dot{\Xi}_1) \leq d(a, \dot{\Xi}_2)$. Let ρ denote the 2-radius of C. By Corollary 16, $\dot{\Xi}_1$ and v lie on the path between a and b. By Definition 3, $d(a, \dot{\Xi}_1) = \rho$. If v lies between $\dot{\Xi}_1$ and a, then $d(\dot{\Xi}_1, c) \leq d(\dot{\Xi}_1, a) = \rho$, otherwise a is not a diametric client. Therefore, assume $\dot{\Xi}_1$ lies between a and v. Since $\{a, c\} \subseteq C_1$, $d(a, c) \leq 2\rho$. Furthermore, since $d(a, c) = d(a, \dot{\Xi}_1) + d(\dot{\Xi}_1, c) = \rho + d(\dot{\Xi}_1, c)$, therefore, $d(\dot{\Xi}_1, c) \leq \rho$. $\square$

Lemma 18 *The relative rate of change of the 2-radius is at most one on trees.*

Proof. We show

$$\forall t_1, t_2 \in I,\ |\rho(t_1) - \rho(t_2)| \leq |t_1 - t_2|, \tag{22}$$

where $\rho(t_i)$ denotes the 2-radius of $C(t_i)$. Choose any $t_1, t_2 \in I$. Let $\delta = |t_1 - t_2|$. Let a_i and b_i be clients in C such that $\{a_i(t_i), b_i(t_i)\}$ is a diametric pair of $C(t_i)$, for $i \in \{1, 2\}$. Since a local 1-centre is the midpoint of a local diametric path, the 2-radius of $C(t_i)$ can be expressed as,

$$\rho(t_i) = \frac{1}{2} \max_{c \in C} \min\{d(c(t_i), a_i(t_i)), d(c(t_i), b_i(t_i))\}. \tag{23}$$

Since clients move with at most unit velocity,

$$\forall \{c, e\} \subseteq C,\ |d(c(t_1), e(t_1)) - d(c(t_2), e(t_2))| \leq 2\delta. \tag{24}$$

Let $\{A_2(t_2), B_2(t_2)\}$ denote the diametric partition of $C(t_2)$ induced by $\{a_2(t_2), b_2(t_2)\}$ such that $a_2(t_2) \in A_2(t_2)$ and $b_2(t_2) \in B_2(t_2)$.

Case 1. Suppose $a_1(t_2)$ and $b_1(t_2)$ are in different diametric subsets of $\{A_2(t_2), B_2(t_2)\}$. Without loss of generality, assume $a_1(t_2) \in A_2(t_2)$ and $b_1(t_2) \in B_2(t_2)$.

$$\begin{aligned}
\rho(t_1) &= \frac{1}{2} \max_{c \in C} \min \Big\{ d(c(t_1), a_1(t_1)), d(c(t_1), b_1(t_1)) \Big\}, && \text{by (23)}, \\
&\leq \frac{1}{2} \max_{c \in C} \min \Big\{ d(c(t_2), a_1(t_2)), d(c(t_2), b_1(t_2)) \Big\} + \delta, && \text{by (24)}, \\
&\leq \frac{1}{2} \max_{c \in C} \min \Big\{ d(c(t_2), a_2(t_2)), d(c(t_2), b_2(t_2)) \Big\} + \delta,
\end{aligned}$$

since $\{a_2(t_2), b_2(t_2)\}$ is a diametric pair of $C(t_2)$, $\{a_1(t_2), a_2(t_2)\} \subseteq A_2(t_2)$, and $\{b_1(t_2), b_2(t_2)\} \subseteq B_2(t_2)$,

$$= \rho(t_2) + \delta, \qquad \text{by (23)}.$$

Case 2. Suppose $a_1(t_2)$ and $b_1(t_2)$ are in the same diametric subset of $\{A_2(t_2), B_2(t_2)\}$. Let $r(t_1)$ denote the radius of $C(t_1)$. Since $\{a_1(t_1), b_1(t_1)\}$ is a diametric pair of $C(t_1)$,

$$\begin{aligned}
\rho(t_1) &\leq r(t_1) \\
&= \frac{1}{2} d(a_1(t_1), b_1(t_1)) \\
&\leq \frac{1}{2} d(a_1(t_2), b_1(t_2)) + \delta, && \text{by (24)}, \\
&\leq \rho(t_2) + \delta,
\end{aligned}$$

since $a_1(t_2)$ and $b_1(t_2)$ are in the same diametric subset of $C(t_2)$.

Therefore, $\rho(t_1) \leq \rho(t_2) + \delta$ in both cases. An analogous argument can be used to show that $\rho(t_2) \leq \rho(t_1) + \delta$, proving (22). □

Theorem 19 *Each facility in the mobile equidistant 2-centre has relative velocity at most two.*

In brief, the velocity of each facility is bounded by the sum of the rate of change of the 2-radius plus the maximum velocity of the corresponding diametric client.

Proof. Choose any $t_1, t_2 \in I$. Let $\delta = |t_1 - t_2|$.

Case 1. Suppose a pair of clients $\{a, b\}$ remains diametric during the time interval $[t_1, t_2]$. Without loss of generality, assume $d(\dot{\Xi}_1(t), a(t)) \leq d(\dot{\Xi}_2(t), a(t))$ for all $t \in [t_1, t_2]$. By Lemma 18 and Definition 3,

$$\begin{aligned}
d(\dot{\Xi}_1(t_1), \dot{\Xi}_1(t_2)) &\leq d(a(t_1), a(t_2)) + |\rho(t_1) - \rho(t_2)| \\
&\leq 2\delta. \qquad (25)
\end{aligned}$$

An analogous argument shows that $d(\dot{\Xi}_2(t_1), \dot{\Xi}_2(t_2)) \leq 2\delta$.

Case 2. Suppose no pair of clients remains diametric for the duration of time interval $[t_1, t_2]$. Changes in the set of diametric clients are discrete events that occur instantaneously. Let $t \in [t_1, t_2]$ denote such an instant. Since clients move continuously, any client that is diametric during the interval $[t - \epsilon, t)$ must remain diametric at time t, for any $\epsilon > 0$. The same holds for interval $(t, t + \epsilon]$. As shown in Case 1, both facilities of the equidistant 2-centre have relative velocity at most two during the intervals $[t - \epsilon, t]$ and $[t, t + \epsilon]$. By Lemma 15, the equidistant 2-centre is uniquely defined at time t. Consequently, the relative velocity remains at most two for the duration of the interval $[t - \epsilon, t + \epsilon]$. □

Since no mobile 2-centre can guarantee relative velocity less than two in one dimension [13], it follows that the maximum relative velocity of the equidistant 2-centre is optimal.

Corollary 20 *Each facility in the mobile equidistant 2-centre is continuous.*

5.3 Complexity of the Motion of the 2-Centre

When clients move with linear motion, we derive combinatorial bounds of $O(n^2\alpha(n))$ on the complexity of the motion of the equidistant 2-centre and $\Omega(n^2)$ on the worst-case complexity of the motion of any 2-centre.

Theorem 21 *When each client in C moves with linear motion along a path on T, the motion of each facility in the equidistant 2-centre of C is piecewise linear and is composed of $O(n^2\alpha(n))$ linear segments, where $n = |C|$.*

Proof. By Theorem 11, there exists a sequence of diametric pairs of C, denoted $\{a_1, b_1\}, \ldots, \{a_m, b_m\}$, and a corresponding partition of the time interval I, denoted $I_1, \ldots, I_m$, such that $m \in O(n)$. It suffices to show that for every i, the motion of each facility in the equidistant 2-centre of C is piecewise linear and is composed of $O(n\alpha(n))$ linear segments during I_i.

Choose any $i \in \{1, \ldots, m\}$ and consider the motion of clients in C during I_i. For every t, let $C_1(t)$ and $C_2(t)$ be a diametric partition induced by $a_i(t)$ and $b_i(t)$. By Lemma 14, $a_i(t)$ is in a local diametric pair of $C_1(t)$ for all t. The second client opposite $a_i(t)$ in the local diametric pair corresponds to a furthest client from $a_i(t)$ in $C_1(t)$. For any client $c \in C$, $d(c(t), a_i(t))$ and $d(c(t), b_i(t))$ are piecewise-linear functions composed of at most three linear segments; consequently, c changes partitions $O(1)$ times. Within C_1, therefore, the function $d(c(t), a_i(t))$, may be partially defined, with $O(1)$ intervals over which it is undefined. Finding the furthest client from a_i in C_1 corresponds to finding the upper envelope of the $n-2$ distance functions $d(c(t), a_i(t))$ for all clients $c \in C \setminus \{a_i, b_i\}$. Since the functions are partially defined, the upper envelope consists of $O(n\alpha(n))$ linear segments [3]. This function corresponds to the local diameter of $C_1(t)$. The maximum of the two local diameters determines the 2-radius; therefore, the 2-radius of C also consists of $O(n\alpha(n))$ linear segments during I_i. Since a_i and b_i have linear motion, the result follows by Definition 3. $\square$

Theorem 22 *There exists a set of mobile clients C, each moving with linear motion on the real line, such that the motion of some facility in any 2-centre of C whose motion is piecewise linear is composed of $\Omega(n^2)$ linear segments, where $n = |C|$.*

In brief, the worst case is realized when $\Theta(n)$ clients are positioned at distinct points near the middle of the set and the 1-centre sweeps back and forth across each of these $\Theta(n)$ times. The motion of the 1-centre results from $\Theta(n)$

additional clients that realize an alternating sequence of diametric clients.

Proof. We define a set of n mobile clients on a line segment for any even $n \geq 2$. For each $i \in \{0, \ldots, n/2-1\}$, let client c_i have position $c_i(t) = (-1)^i(2n^2 - 2i^2 + it)$. Observe that client c_i has velocity $(-1)^i i$ relative to $-\infty$. Let the remaining $n/2$ clients have velocity zero and be positioned at distinct points in $(-1, 1)$. See Fig. A.3.

For all $i \in \{2, \ldots, n/2-1\}$, client c_i passes client c_{i-2} at time $t = 4(i-1)$. Consequently, clients $c_0, c_2, \ldots, c_{n/2-2}$ realize a sequence of diametric clients to the right of the origin while clients $c_1, c_3, \ldots, c_{n/2-1}$ realize a sequence of diametric clients to the left of the origin. Furthermore, $\Xi(t) = 1$ when $t \bmod 8 = 0$ and $\Xi(t) = -1$ when $t \bmod 8 = 4$ for all $t \in [0, 2n-4]$. Therefore, the 1-centre traverses the interval $[-1, 1]$ $n/2 - 1$ times, crossing each of the $n/2$ static clients on each traversal.

The diametric partition of C defined by (20) is unique whenever $\Xi(t)$ does not coincide with any client in C. The 2-radius is uniquely determined by the partition of larger local radius. Furthermore, any 2-centre of C must include one facility whose position is uniquely determined by the midpoint of the local diametric path of the partition with larger local radius. The motion of this facility changes $\Omega(n)$ times between each change to the motion of $\Xi(t)$, resulting in $\Omega(n^2)$ changes in total. $\square$

5.4 Kinetic Maintenance of the Mobile 2-Centre

Capitalizing on our 1-centre results, we describe an algorithm for identifying local 1-centres and the equidistant 2-centre of a set of mobile clients, each moving with linear motion on a tree.

By Theorem 22, even under linear motion of clients in C, the motion of any 2-centre of C has complexity $\Omega(n^2)$ in the worst case. It follows that any algorithm that enumerates the components of the trajectories of a mobile 2-centre of C requires $\Omega(n^2)$ time in the worst case.

Algorithm Description. We first run our 1-centre algorithm to find a sequence of diametric pairs of C, denoted $\{a_1, b_1\}, \ldots, \{a_m, b_m\}$, and a corresponding partition of the time interval I, denoted $I_1, \ldots, I_m$, such that $m \in O(n)$. For each time interval I_i, determine when each client c is closer to a_i and when it is closer to b_i. This determines the sets $C_1(t)$ and $C_2(t)$ for all $t \in I_i$. Consider C_1 (an analogous algorithm applies to C_2). A diametric pair of $C_1(t)$ is given by $a_i(t)$ and a furthest client from $a_i(t)$ in $C_1(t)$. Each local diametric pair determines the motion of the corresponding local 1-centre and the local radius, from which the motion of the equidistant 2-centre is straightforward to calculate.

Time Complexity. For a client $c \in C$, the functions $d(c(t), a_i(t))$ and $d(c(t), b_i(t))$ are piecewise linear, each composed of at most four linear segments. Therefore, c changes partitions $O(1)$ times during interval I_i and calculating the interval for which c resides in either partition is achieved in constant time. Finding a furthest client from $a_i(t)$ for all $t \in I_i$ corresponds to finding the upper envelope of $n-2$ partially-defined, piecewise-linear functions, which can be done in $O(n \log n)$ time using Hershberger's [30] algorithm. Since there are $O(n)$ time intervals, the total runtime is $O(n^2 \log n)$.

KDS Implementation. We describe a KDS that maintains the equidistant 2-centre over time along with a set of certificates that validates its identity.

Theorem 23 *Given a tree T and a set of mobile clients C, each moving with linear motion on a path of T, there exists a KDS to maintain the mobile equidistant 2-centre of C that is compact and has responsiveness $O(n)$, locality $O(n)$, and efficiency $O(n^2 \log n)$.*

Proof. We augment the 1-centre KDS described in Sec. 4.3. We require one additional certificate per client c to verify whether c is in C_1 or C_2. We require a maximum KDS for C_1 (and a second one for C_2) that maintains the furthest client from a_i (respectively, b_i). The kinetic tournament KDS described by Basch et al. [5] allows for clients to be inserted and deleted from the set (recall that each client changes sets $O(1)$ times between changes to the diametric pair). This latter KDS has efficiency, compactness, locality, and responsiveness that is comparable to our 1-centre KDS.

In terms of performance, the worst case occurs whenever the diametric pair changes and $O(n)$ certificates must be updated. Therefore, this KDS has responsiveness $O(n)$. The total number of certificate failures is $O(n \log n)$ between changes to the diametric pair, or $O(n^2 \log n)$ in total. By Theorem 22, the number of external events is $\Omega(n^2)$ in the worst case; therefore, the KDS has good efficiency (but possibly not optimal). The total number of certificates remains $O(n)$; therefore, the KDS is compact. Finally, $O(n)$ certificates are associated with each diametric client; therefore, the KDS has locality $O(n)$. $\square$

6 The Mobile k-Centre on Graphs

In this section we briefly examine properties of mobile 1-centres and 2-centres on graphs. Unlike the 1-centre which is always continuous on trees and the 2-centre for which there always exists a pair of continuous trajectories on trees, neither the 1-centre nor the 2-centre is continuous on graphs in general. Consequently, no upper bound on velocity can be guaranteed.

Proposition 24 *For any graph G that contains a cycle, there exists a set of mobile clients C on G such that any mobile 1-centre of C is discontinuous.*

Proof. Let $G' \subseteq G$ denote a cycle of minimum diameter in G. Let a denote the diameter of G'. Let p_1 and p_2 be points in $U_{G'}$ such that $d(p_1, p_2) = a$. Let $C = \{c_1, c_2\}$ denote a set of mobile clients such that $c_1(0) = c_2(0) = p_1$, $c_1(a) = c_2(a) = p_2$, and c_1 and c_2 move in opposite directions with unit velocity. Since G' has minimum diameter over all cycles in G, while $t \in [0, a/2)$, the unique 1-centre of C lies at $\Xi(t) = p_1$. Similarly, while $t \in (a/2, a]$, the unique 1-centre of C lies at $\Xi(t) = p_2$. Therefore,

$$\forall \delta \in (0, a/2],\ d(\Xi(a/2 - \delta), \Xi(a/2 + \delta)) = a.$$

Since $\Xi(t)$ is uniquely defined at $t = a/2-\delta$ and $t = a/2+\delta$ for any $\delta \in (0, a/2)$, it follows that $\Xi(t)$ is discontinuous at time $t = a/2$. $\square$

Proposition 25 *For any graph G that contains a cycle, there exists a set of mobile clients C on G such that any mobile 2-centre of C is discontinuous.*

Proof. The proof is analogous to the proof of Proposition 24 with the addition of two more mobile clients. Add clients c_3 and c_4 such that $c_3(0) = c_4(0) = p_2$, $c_3(a) = c_4(a) = p_1$, and c_3 and c_4 move in opposite directions with unit velocity. Discontinuities occur at times $t = a/4$ and $t = 3a/4$. $\square$

Continuity and a finite upper bound on velocity impose natural constraints on any physical moving object. It follows that there exist sets of mobile clients C moving on a graph such that the position of any mobile facility that moves with bounded velocity on the graph must differ from the mobile 1-centre of C. Consequently, one may consider bounded-velocity approximations of the 1-centre.

Following Bespamyatnikh et al. [9] who describe a similar strategy for approximating the rectilinear 1-centre in $\mathbb{R}^2$, a simple unit-velocity 2-approximation of the 1-centre on graphs is achieved by selecting an arbitrary client $c_0 \in C$ and setting the position of the facility to coincide with $c_0(t)$. The distance from c_0 to any other client in C is at most the diameter of C, that is, at most twice the radius of C. Perhaps surprisingly, we show that this simple strategy is optimal:

Theorem 26 *No continuous mobile facility can guarantee a $(2-\epsilon)$-approximation of the mobile 1-centre of a set of mobile clients on a graph for any $\epsilon > 0$.*

Proof. Let G', $C = \{c_1, c_2\}$, p_1, p_2, and a be as defined in the proof of Proposition 24. Let $r(t)$ denote the radius of $C(t)$. Choose any $\epsilon > 0$. Assume mobile facility f guarantees an approximation factor of $2 - \epsilon$ of the mobile 1-centre of C; this implies $\epsilon \in (0, 1]$. While $t \in [0, a/2)$, the unique 1-centre of C lies at p_1 and the corresponding radius of C is t. Therefore,

$$\begin{aligned}
\forall t \in [0, a/2),\ d(p_2, f(t)) &= d(p_1, p_2) - d(p_1, f(t)) \\
&= d(p_1, p_2) - \max_{c \in \{c_1, c_2\}} \Big[d(f(t), c(t)) - d(c(t), p_1)\Big] \\
&= d(p_1, p_2) + r(t) - \max_{c \in \{c_1, c_2\}} d(f(t), c(t)) \\
&\geq d(p_1, p_2) + r(t) - (2 - \epsilon) r(t) \\
&= a - (1 - \epsilon) t \\
&> \frac{a}{2}(1 + \epsilon). \qquad (26)
\end{aligned}$$

Similarly, while $t \in (a/2, a]$, the unique 1-centre of C lies at p_2 and the corresponding radius of C is $a - t$. Consequently,

$$\begin{aligned}
\forall t \in (a/2, a],\ d(p_2, f(t)) &= \max_{c \in \{c_1, c_2\}} \Big[d(f(t), c(t)) - d(c(t), p_2)\Big] \\
&= \max_{c \in \{c_1, c_2\}} d(f(t), c(t)) - r(t) \\
&\leq (2 - \epsilon) r(t) - r(t) \\
&= (1 - \epsilon)(a - t) \\
&< \frac{a}{2}(1 + \epsilon). \qquad (27)
\end{aligned}$$

Regardless of its velocity, the motion of f is discontinuous at time $t = a/2$ by (26) and (27). □

7 Directions for Future Research

Discrete k-Centre. The mobile discrete k-centre (when each facility is a client in C) is discontinuous. Maintaining a sequence of clients that realize a discrete k-centre of a set of mobile clients on a tree is an open problem. Maintaining a discrete 1-centre of C corresponds to maintaining the identity of a client in C that is closest to $\Xi(t)$. For the discrete 2-centre, however, the problem is complicated by the fact that a diametric partition does not necessarily correspond to a discrete diametric partition; that is, (19) and (20) are not necessarily equivalent in the discrete case.

k-Centre on Graphs. Proposition 24 states that the mobile 1-centre is discontinuous on any graph containing a cycle. This motivates the search for bounded-velocity approximations of k-centres. In Section 6 we describe a simple 2-approximation to the mobile 1-centre which we show is optimal. It is unknown whether any bounded-velocity approximation exists for mobile 2-centres on graphs. As shown by Durocher [13], even in one dimension, no bounded-velocity approximation of the mobile 3-centre is possible; the corresponding result holds on graphs since any edge is a one-dimensional interval.

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A Figures

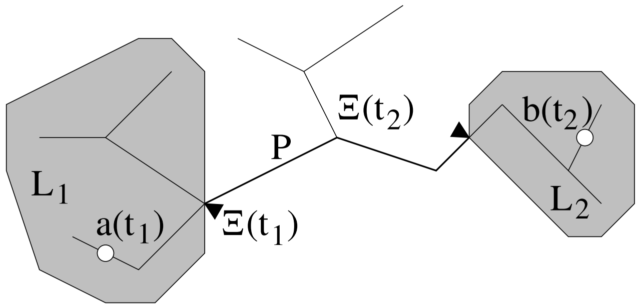

Fig. A.1. Illustration in support of Theorem 1

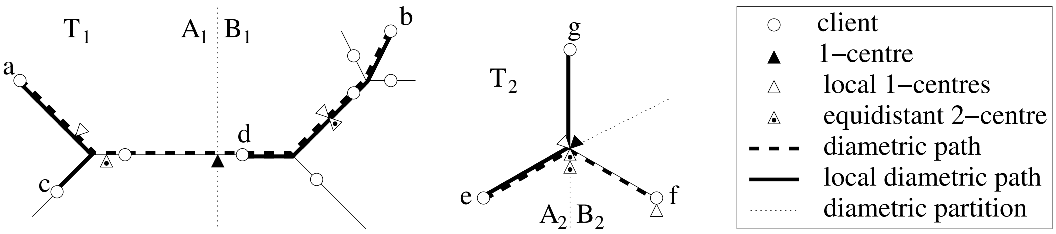

Fig. A.2. **Equidistant 2-centre example.** In tree T_1, $\{a, b\}$ is the unique diametric pair. In tree T_2, any two clients form a diametric pair; the diametric path between e and f is displayed. The corresponding diametric subsets are A_1 and B_1 in T_1 (unique) and A_2 and B_2 in T_2 (not unique). In tree T_1, the 2-radius is realized by the local diametric pair $\{b, d\}$. Consequently, the local 1-centre of B_1 coincides with the equidistant 2-centre in B_1. In tree T_2, the 2-radius is equal to the radius. Consequently, the local 1-centre of A_2 coincides with the 1-centre and with both equidistant 2-centres. The local 1-centre of B_2 coincides with client f.

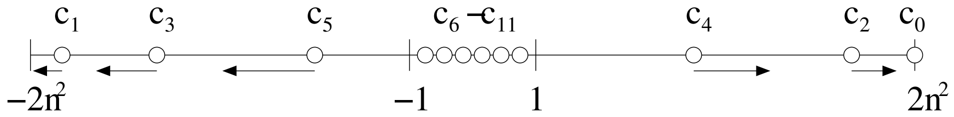

Fig. A.3. The initial configuration of clients described in the proof of Theorem 22 when $n = 12$

References

[1] Pankaj K. Agarwal, Leonidas J. Guibas, John Hershberger, and Eric Veach. Maintaining the extent of a moving point set. *Discrete and Computational Geometry*, 26:353–374, 2001.

[2] Pankaj K. Agarwal and Sariel Har-Peled. Maintaining approximate extent measures of moving points. In *Proceedings of the Symposium on Discrete Algorithms*, pages 148–157. ACM Press, 2001.

[3] Pankaj K. Agarwal and Micha Sharir. *Davenport-Schinzel Sequences and Their Geometric Applications.* Cambridge University Press, 1995.

[4] Julien Basch. *Kinetic Data Structures.* PhD thesis, Stanford University, 1999.

[5] Julien Basch, Leonidas Guibas, and John Hershberger. Data structures for mobile data. *Journal of Algorithms*, 31(1):1–28, 1999.

[6] Julien Basch, Leonidas J. Guibas, Craig Silverstein, and Li Zhang. A practical evaluation of kinetic data structures. In *Proceedings of the ACM Symposium on Computational Geometry*, pages 388–390, 1997.

[7] Michael A. Bender, Martin Farach-Colton, Giridhar Pemmasani, Steven Skiena, and Pavel Sumazin. Lowest common ancestors in trees and directed acyclic graphs. *Journal of Algorithms*, 57(2):75–94, 2005.

[8] Sergey Bereg, Binay Bhattacharya, David Kirkpatrick, and Michael Segal. Competitive algorithms for mobile centers. *Mobile Networks and Applications*, 11(2):177–186, 2006.

[9] Sergei Bespamyatnikh, Binay Bhattacharya, David Kirkpatrick, and Michael Segal. Mobile facility location. In *Proceedings of the International ACM Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications*, volume 4, pages 46–53, 2000.

[10] Giuseppe Bruno, Gianpaolo Ghiani, and Gennaro Improta. Dynamic positioning of idle automated guided vehicles. *Journal of Intelligent Manufacturing*, 11(2):209–215, 2000.

[11] Gary Chartrand and Ping Zhang. h-convex graphs. *Mathematica Bohemica*, 126(1):209–220, 2001.

[12] P. M. Dearing and Richard L. Francis. A minimax location problem on a network. *Transportation Science*, 8:333–343, 1974.

[13] Stephane Durocher. *Geometric Facility Location under Continuous Motion.* PhD thesis, University of British Columbia, 2006.

[14] Stephane Durocher and David Kirkpatrick. The Steiner centre: Stability, eccentricity, and applications to mobile facility location. *International Journal of Computational Geometry and Applications*, 16(4):345–371, 2006.

[15] Stephane Durocher and David Kirkpatrick. Bounded-velocity approximations of mobile Euclidean 2-centres. *International Journal of Computational Geometry and Applications*, 18(3):161–183, 2008.

[16] Stephane Durocher and Christophe Paul. Kinetic maintenance of mobile k-centres on trees. In *Proceedings of the International Symposium on Algorithms and Computation*, volume 4835 of *Lecture Notes in Computer Science*, pages 341–352. Springer-Verlag, 2007.

[17] Dov Dvir and Gabriel Y. Handler. The absolute center of a network. *Networks*, 43(2):109–118, 2004.

[18] Herbert Edelsbrunner, Leonidas J. Guibas, and Micha Sharir. The upper envelope of piecewise linear functions: Algorithms and applications. *Discrete and Computational Geometry*, 4:311–336, 1989.

[19] Greg N. Frederickson. Parametric search and locating supply centers in trees. In *Proceedings of the Workshop on Algorithms and Data Structures*, volume 519 of *Lecture Notes in Computer Science*, pages 299–319. Springer-Verlag, 1991.

[20] Jie Gao, Leonidas J. Guibas, John Hershberger, Li Zhang, and An Zhu. Discrete mobile centers. *Discrete and Computational Geometry*, 30(1):45–65, 2003.

[21] Jie Gao, Leonidas J. Guibas, and An Nguyen. Deformable spanners and applications. *Computational Geometry: Theory and Applications*, 35:2–19, 2006.

[22] Leonidas J. Guibas. Kinetic data structures: a state of the art report. In *Proceedings of the Workshop on the Algorithmic Foundations of Robotics*, pages 191–209. A. K. Peters, Ltd., 1998.

[23] Leonidas J. Guibas. Kinetic data structures. In Dinesh P. Mehta and Sartaj Sahni, editors, *Handbook of Data Structures and Applications*, pages 23.1–23.18. Chapman and Hall/CRC, 2004.

[24] Jonathan Halpern and Oded Maimon. Algorithms for the m-center problems: A survey. *European Journal of Operational Research*, 10:90–99, 1982.

[25] Gabriel Y. Handler. Minimax location of a facility in an undirected tree graph. *Transportation Science*, 7:287–293, 1973.

[26] Gabriel Y. Handler. Finding two-centers of a tree: The continuous case. *Transportation Science*, 12:93–106, 1978.

[27] Gabriel Y. Handler and Pitu B. Mirchandani. *Location on Networks: Theory and Algorithms*. MIT Press, Cambridge, Massachusetts, 1979.

[28] Pierre Hansen, Martine Labbé, Dominique Peeters, and Jacques-François Thisse. Single facility location on networks. *Annals of Discrete Mathematics*, 31:113–146, 1987.

[29] Sergiu Hart and Micha Sharir. Nonlinearity of Davenport-Schinzel sequences and of a generalized path compression scheme. In *Proceedings of Foundations of Computer Science*, volume 25, pages 313–319, 1984.

[30] John Hershberger. Finding the upper envelope of n line segments in $O(n \log n)$ time. *Information Processing Letters*, 33(4):169–174, 1989.

[31] John Hershberger. Smooth kinetic maintenance of clusters. In *Proceedings of the ACM Symposium on Computational Geometry*, pages 48–57. ACM Press, 2003.

[32] Oded Kariv and S. Louis Hakimi. An algorithmic approach to network location problems. I: The p-centers. *SIAM Journal of Applied Mathematics*, 37(3):513–538, 1979.

[33] Arie Tamir. Improved complexity bounds for center location problems on networks by using dynamic data structures. *SIAM Journal on Discrete Mathematics*, 1(3):377–396, 1988.

[34] Barbaros C. Tansel, Richard L. Francis, and Timothy J. Lowe. Location on networks: A survey. part I: The p-center and p-median problems. *Management Science*, 29(4):482–497, 1983.

[35] Barbaros C. Tansel, Richard L. Francis, and Timothy J. Lowe. Location on networks: A survey. part II: Exploiting tree network structure. *Management Science*, 29(4):498–511, 1983.

[36] Ady Wiernik and Micha Sharir. Planar realizations of nonlinear Davenport-Schinzel sequences by segments. *Discrete and Computational Geometry*, 3(1):15–47, 1988.