RELATING GRAPH THICKNESS TO PLANAR LAYERS AND BEND COMPLEXITY*

STEPHANE DUROCHER† AND DEBAJYOTI MONDAL‡

Abstract. The thickness of a graph $G = (V, E)$ with n vertices is the minimum number of planar subgraphs of G whose union is G. A polyline drawing of G in $\mathbb{R}^2$ is a drawing Γ of G, where each vertex is mapped to a point and each edge is mapped to a polygonal chain. Bend and layer complexities are two important aesthetics of such a drawing. The bend complexity of Γ is the maximum number of bends per edge in Γ, and the layer complexity of Γ is the minimum integer r such that the set of polygonal chains in Γ can be partitioned into r disjoint sets, where each set corresponds to a planar polyline drawing. Let G be a graph of thickness t. By Fáry's theorem, if $t = 1$, then G can be drawn on a single layer with bend complexity 0. A few extensions to higher thickness are known, e.g., if $t = 2$ (resp., $t > 2$), then G can be drawn on t planar layers with bend complexity 2 (resp., $3n + O(1)$).

In this paper we present an elegant extension of Fáry's theorem to draw graphs of thickness $t > 2$. We first prove that thickness-t graphs can be drawn on t planar layers with $2.25n + O(1)$ bends per edge. We then develop another technique to draw thickness-t graphs on t planar layers with bend complexity $O(\sqrt{2}^t \cdot n^{1-(1/\beta)})$, where $\beta = 2^{\lceil (t-2)/2 \rceil}$. Previously, the bend complexity was not known to be sublinear for $t > 2$. Finally, we show that graphs with linear arboricity k can be drawn on k planar layers with bend complexity $\frac{3(k-1)n}{(4k-2)}$.

Key words. graph drawing, geometric thickness, planar graphs, bend complexity

AMS subject classifications. 05C10, 68R10

1. Introduction.

A polyline drawing of a graph $G = (V, E)$ in $\mathbb{R}^2$ maps each vertex of G to a distinct point, and each edge of G to a polygonal chain. Many problems in VLSI layout and software visualization are tackled using algorithms that produce polyline drawings. For a variety of practical purposes, these algorithms often seek to produce drawings that optimize several drawing aesthetics, e.g., minimizing the number of bends, minimizing the number of crossings, etc. In this paper we examine two such parameters: *bend complexity* and *layer complexity*.

The *thickness* of a graph G is the minimum number $\theta(G)$ such that G can be decomposed into $\theta(G)$ planar subgraphs. Let Γ be a polyline drawing of G. Then the *bend complexity* of Γ is the minimum integer b such that each edge in Γ has at most b bends. A set of edges $E' \subseteq E$ is called a *crossing-free edge set in* Γ, if the corresponding polygonal chains correspond to a *planar polyline drawing*, i.e., no two polylines that correspond to a pair of edges in E' intersect, except possibly at their common endpoints. The *layer complexity* of Γ is the minimum integer t such that the edges of Γ can be partitioned into t crossing-free edge sets. Figure 1(a) illustrates a polyline drawing of K_9 on 3 planar layers with bend complexity 1. At first glance the layer complexity of Γ may appear to be related to the thickness of G. However, the layer complexity is a property of the drawing Γ, while thickness is a graph property. The layer complexity of Γ can be arbitrarily large even when G is

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†Department of Computer Science, University of Manitoba, Canada (durocher@cs.umanitoba.ca).

‡David R. Cheriton School of Computer Science, University of Waterloo, Canada (dmondal@uwaterloo.ca).

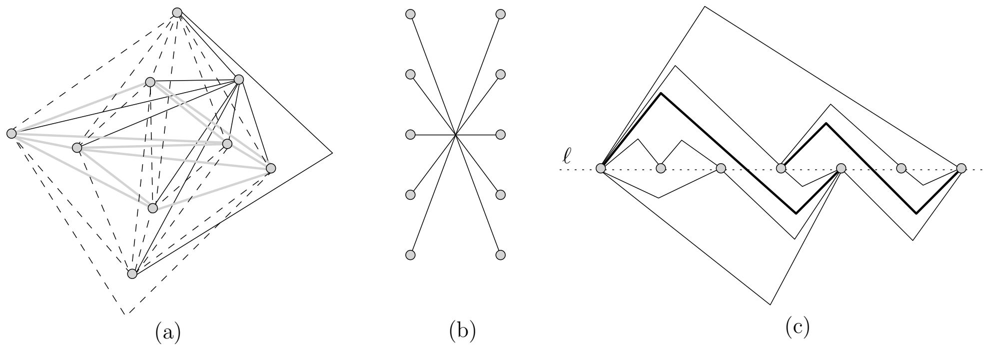

Fig. 1. *(a) A polyline drawing of K_9. (b) A drawing of a matching of size 5. (c) A monotone topological book embedding of some graph. The edges that crosses the spine ℓ are shown in bold.*

planar, e.g., consider the case when G is a matching and Γ is a straight-line drawing, where each edge crosses all the other edges; see Figure 1(b).

The layer complexity of a thickness-t graph G is at least t, and every n-vertex thickness-t graph admits a drawing on t planar layers with bend complexity $O(n)$ [21]. The problem of drawing thickness-t graphs on t planar layers is closely related to the *simultaneous embedding* problem [4], where given a set of planar graphs $G_1, \ldots, G_t$ on a common set of vertices, the task is to compute their planar drawings $D_1, \ldots, D_t$ such that each vertex is mapped to the same point in the plane in each of these drawings. Figure 1(a) can be thought as a simultaneous embedding of three given planar graphs.

1.1. Related Work.

Graphs with low thickness admit polyline drawings on few planar layers with low bend complexity. If $\theta(G) = 1$, then by Fáry's theorem [16], G admits a drawing on a single layer with bend complexity 0. Every pair of planar graphs can be simultaneously embedded using two bends per edge [15, 17]. Therefore, if $\theta(G) = 2$, then G admits a drawing on two planar layers with bend complexity 2. The best known lower bound on the bend complexity of such drawings is one [11]. Wood [22] showed how to construct drawings on $O(\sqrt{m})$ layers with bend complexity 1, where m is the number of edges in G.

Given an n-vertex planar graph G and a point location for each vertex in $\mathbb{R}^2$, Pach and Wenger [21] showed that G admits a planar polyline drawing with the given vertex locations, where each edge has at most $120n$ bends. They also showed that $\Omega(n)$ bends are sometimes necessary. Badent et al. [1] and Gordon [18] independently improved the bend complexity to $3n + O(1)$. Consequently, for $\theta(G) \geq 3$, these constructions can be used to draw G on $\theta(G)$ planar layers with at most $3n + O(1)$ bends per edge.

A rich body of literature [3, 4, 12, 13] examines *geometric thickness*, i.e., the maximum number of planar layers necessary to achieve 0 bend complexity. Two layers suffice for graphs with maximum degree four [10]. Dujmović and Wood [8] proved that $\lceil k/2 \rceil$ layers suffice for graphs of treewidth k. Duncan [9] proved that $O(\log n)$ layers suffice for graphs with arboricity two or outerthickness two, and $O(\sqrt{n})$ layers suffice for thickness-2 graphs. Dillencourt et al. [7] proved that complete graphs with n vertices require at least $\lceil (n/5.646) + 0.342 \rceil$ and at most $\lceil n/4 \rceil$ layers.

1.2. Our Results. The goal of this paper is to extend our understanding of the interplay between the layer complexity and bend complexity in polyline drawings.

We first show that every n-vertex thickness-t graph admits a polyline drawing on t planar layers with bend complexity $2.25n + O(1)$, improving the $3n + O(1)$ upper bound derived from [1, 18]. We then give another drawing algorithm to draw thickness-t graphs on t planar layers with bend complexity $O(\sqrt{2}^t \cdot n^{1-(1/\beta)})$, where $\beta = 2^{\lceil (t-2)/2 \rceil}$. No such sublinear upper bound on the bend complexity was previously known for $t > 2$. Finally, we show that every n-vertex graph with linear arboricity $k \geq 2$ admits a polyline drawing on k planar layers with bend complexity $\frac{3(k-1)n}{(4k-2)}$, where the *linear arboricity* of a graph G is the minimum number of linear forests (i.e., each connected component is a path) whose union is G.

The rest of the paper is organized as follows. We start with some preliminary definitions and results (Section 2). In the subsequent section (Section 3) we present two constructions to draw thickness t graphs on t planar layers. Section 4 presents the results on drawing graphs of bounded arboricity. Finally, Section 5 concludes the paper pointing out the limitations of our results and suggesting directions for future research.

2. Technical Details. In this section we describe some preliminary definitions, and review some known results.

Let $G = (V, E)$ be a planar graph. A *monotone topological book embedding* of G is a planar drawing Γ of G that satisfies the following properties.

- P_1: The vertices of G lie along a horizontal line ℓ in Γ. We refer to ℓ as the *spine* of Γ.
- P_2: Each edge $(u, v) \in E$ is an x-monotone polyline in Γ, where (u, v) either lies on one side of ℓ, or crosses ℓ at most once.
- P_3: Let (u, v) be an edge that crosses ℓ at point d, where u appears before v on ℓ. Let $u, \ldots, d, \ldots, v$ be the corresponding polyline. Then the polyline $u, \ldots, d$ lies above ℓ, and the polyline $d, \ldots, v$ lies below ℓ.

Figure 1(c) illustrates a monotone topological book embedding of a planar graph.

Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs on a common set of vertices. A *simultaneous embedding* Γ of G_1 and G_2 consists of their planar drawings D_1 and D_2, where each vertex is mapped to the same point in the plane in both D_1 and D_2. Erten and Kobourov [15] showed that every pair of planar graphs admit a simultaneous embedding with at most three bends per edge. Giacomo and Liotta [17] observed that by using monotone topological book embeddings Erten and Kobourov's [15] construction can achieve a drawing with two bends per edge. Here we briefly recall this drawing algorithm. Without loss of generality assume that both G_1 and G_2 are triangulations. Let π_i, where $1 \leq i \leq 2$, be a vertex ordering that corresponds to a monotone topological book embedding of G_i. Let P_i be the corresponding *spinal path*, i.e., a path that corresponds to π_i. Note that some of the edges of P_i may not exist in G_i, e.g., edges (a, d) and (b, c) in Figures 2(a) and (b), respectively, and these edges of P_i create edge crossings in G_i. Add a dummy vertex at each such edge crossing. Let $\delta_i(v)$ be the position of vertex v in π_i. Then P_1 and P_2 can be drawn simultaneously on an $O(n) \times O(n)$ grid [5] by placing each vertex at the grid point $(\delta_1(v), \delta_2(v))$; see Figure 2(c). The mapping between the dummy vertices of P_1 and P_2 can be arbitrary, here we map the dummy vertex on (a, d) to the dummy vertex on (b, c). Finally, the edges of G_i that do not belong to P_i are drawn. Let e be such an edge in G_i. If e does not cross the spine, then it is drawn using one bend on one side of P_i according to the book embedding of G_i. Otherwise, let q be a dummy vertex on the edge $e = (u, v)$, which corresponds to the intersection point of e and the spine. The edges (u, q) and (v, q) are drawn on opposite sides of P_i such that the polyline from u to v do not create any bend at q. Since each of (u, q) and (v, q) contains only one bend, e contains only two bends. Finally, the edges of P_i that do not belong to G_i are removed from the drawing; see Figure 2(d).

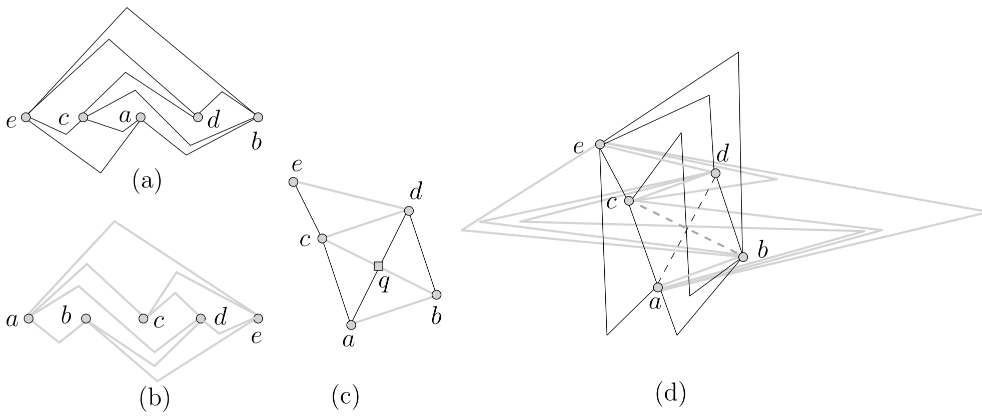

FIG. 2. *(a)–(b) Monotone topological book embeddings of G_1 and G_2. (c)–(d) Simultaneous embedding of G_1 and G_2, where the deleted edges are shown in dashed lines.*

Let Γ be a planar polyline drawing of a path $P = \{v_1, v_2, \ldots, v_n\}$. We call Γ an *uphill* drawing if for any point q on Γ, the upward ray from q does not intersect the path $v_1, \ldots, q$. Note that q may be a vertex location or an interior point of some edge in Γ. Let a and b be two points in $\mathbb{R}^2$. Then a and b are *r-visible* to each other if and only if their exists a polygonal chain of length r with end points a, b that does not intersect Γ at any point except at a, b. A point p *lies between two other points* a, b, if either the inequality $x(a) < x(p) < x(b)$ or $x(b) < x(p) < x(a)$ holds.

A set of points is *monotone* if the polyline connecting them from left to right is monotone, i.e., increasing or decreasing, with respect to y-axis. Let S be a set of n points in general position. By the Erdös-Szekeres theorem [14], S can be partitioned into $O(\sqrt{n})$ disjoint monotone subsets, and such a partition can be computed in $O(n^{1.5})$ time [2].

3. Drawing Thickness-t Graphs on t Layers. In this section we give two separate construction techniques to draw thickness-t graphs on t planar layers. We first present a construction achieving $2.25n + O(1)$ upper bound (Section 3.1), which is simple and intuitive. Although the technique is simple, the idea of the construction will be used frequently in the rest of the paper.

Later, we present a second construction (Section 3.2), which is more involved, and relies on a deep understanding of the geometry of point sets. In this case, the upper bound on the bend complexity will depend on some generalization of Erdös-Szekeres theorem [14], e.g., partitioning a point set into monotone subsequences in higher dimensions (Section 3.2.3).

3.1. A Simple Construction with Bend Complexity $2.25n + O(1)$. Let $G_1, \ldots, G_t$ be the planar subgraphs of the input graph G, and let S be an ordered set of n points on a semicircular arc. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of G. We show that each G_i, where $1 \le i \le t$, admits a polyline drawing with bend complexity $2.25n+O(1)$ such that vertex v_j is mapped to the jth point of S. To draw G_i, we will use the vertex ordering of its monotone topological book embedding. The following lemma will be useful to draw the spinal path P_i of G_i.

LEMMA 3.1. *Let $S = \{p_0, p_1, \ldots, p_{n+1}\}$ be a set of points lying on an x-monotone semicircular arc (e.g., see Figure 3(a)), and let $P = \{v_1, v_2, \ldots, v_n\}$ be a path of n vertices. Assume that p_0 and p_{n+1} are the leftmost and rightmost points of S, respectively, and the points $p_1, \ldots, p_n$ are equally spaced between them in some arbitrary order. Then P admits an uphill drawing Γ with the vertex v_i assigned to p_i, where $1 \le i \le n$, and every point p_i satisfies the following properties:*

A. Both the points p_0 and p_{n+1} are $(3n/4)$-visible to p_i.

B. One can draw an x-monotone polygonal chain from p_0 to p_{n+1} with $3n/4$ bends that intersects Γ only at p_i.

Proof. We prove the lemma by constructing such a drawing Γ for P. The construction assigns a polyline for each edge of P. The resulting drawing may contain edge overlaps, and the bend complexity could be as large as $n-2$. Later we remove these degeneracies and reduce the bend complexity to obtain Γ.

Drawings of Edges: For each point $p_i \in S$, where $1 \le i \le n$, we create an *anchor point* p'_i at $(x(p_i), y(p_i) + \epsilon)$, where $\epsilon > 0$. We choose ϵ small enough such that for any j, where $1 \le i \ne j \le n$, all the points of S between p_i and p_j lie above (p'_i, p'_j). Figure 3(a) illustrates this property for the anchor point p'_1.

We first draw the edge (v_1, v_2) using a straight line segment. For each j from 2 to $n-1$, we now draw the edges (v_j, v_{j+1}) one after another. Assume without loss of generality that $x(p_j) < x(p_{j+1})$. We call a point $p \in S$ between p_j and p_{j+1} a *visited point* if the corresponding vertex v appears in $v_1, \ldots, v_j$, i.e., v has already been placed at p. We draw an x-monotone polygonal chain L that starts at v_j, connects the anchors of the intermediate visited points from left to right, and ends at v_{j+1}. Figure 3(b) illustrates such a construction.

Since the number of bends on L is equal to the number of visited points of S between p_j and p_{j+1}, each edge contains at most α bends, where α is the number of points of S between p_j and p_{j+1}.

Removing Degeneracies: The drawing D_n of the path P constructed above contains edge overlaps, e.g., see the edges (v_3, v_4) and (v_4, v_5) in Figure 3(c). To remove the degeneracies, for each i, we spread the corresponding bend points in the interval $[p_i, p'_i]$, in the order they appear on the path, see Figure 3(d). Consequently, we obtain a planar drawing of P. Let the resulting drawing be D'_n. Since each edge (p_j, p_{j+1}) is drawn as an x-monotone polyline above the path $p_1, \ldots, p_j$, D'_n satisfies the uphill property. Note that D'_n may have bend complexity $n-2$, e.g., see Figure 3(e). We now show how to reduce the bend complexity and satisfy Properties A–B.

Reducing Bend Complexity: A pair of points in S are *consecutive* if they do not contain any other point of S in between. Let e be any edge of P. Let C_e be the corresponding polygonal chain in D'_n. A pair of bends on C_e are called *consecutive bends* if their corresponding points in S are also consecutive. A *bend-interval* of C_e is a maximal sequence of consecutive bends in C_e. Note that we can partition the bends on e into disjoint sets of bend-intervals.

For any bend-interval s, let $l(s)$ and $r(s)$ be the x-coordinates of the left and right endpoints of s, respectively. Let s_1 and s_2 be two bend-intervals lying on two distinct edges e_1 and e_2 in D'_n, respectively, where e_2 appears after e_1 in P. We claim that the intervals $[l(s_1), r(s_1)]$ and $[l(s_2), r(s_2)]$ are either disjoint, or $[l(s_1), r(s_1)] \subseteq$

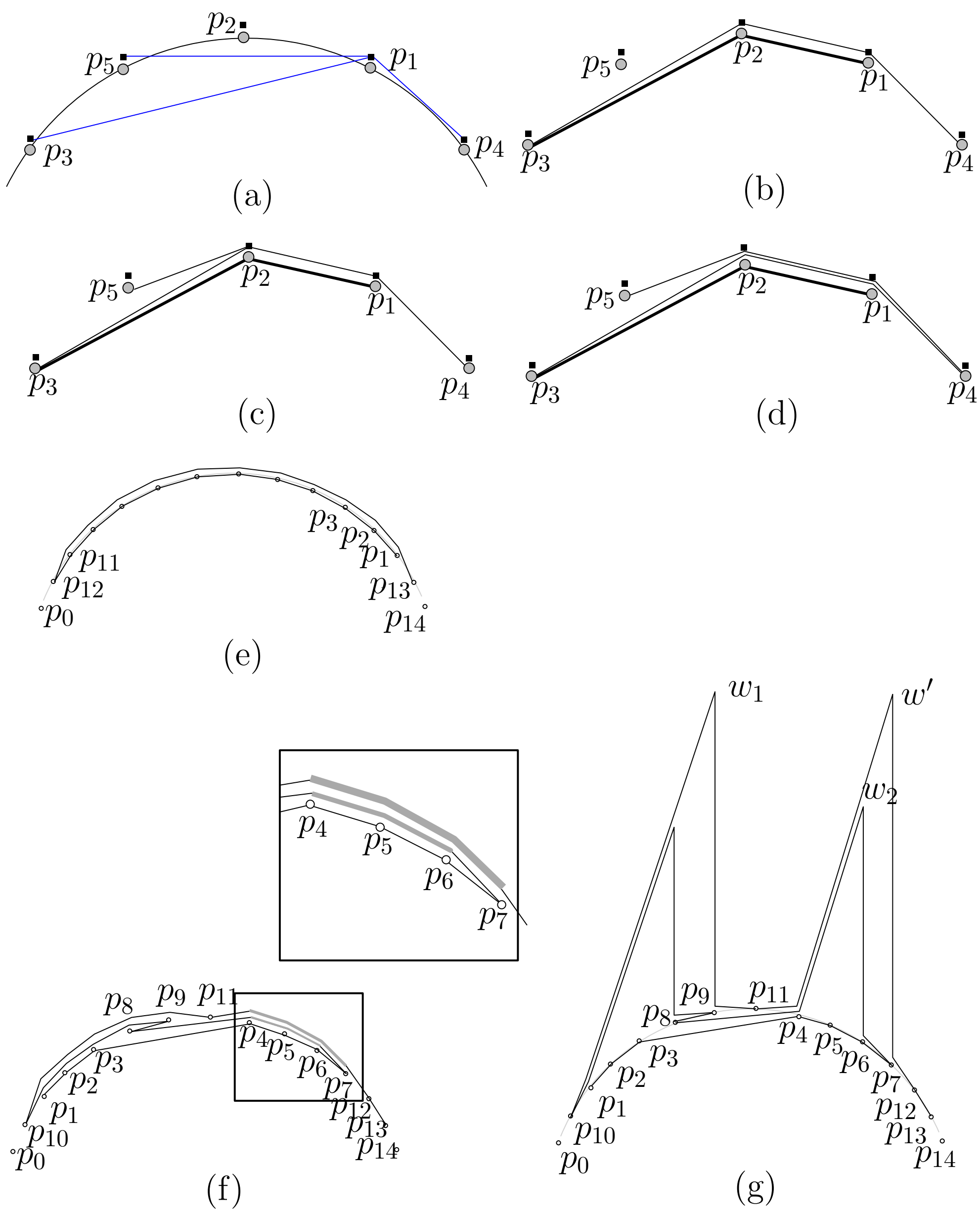

FIG. 3. *Illustration for the proof of Lemma 3.1. (a) Construction of the point set, and the anchor points. The anchor points are shown in black squares. (b)–(d) Construction of D'_n. (e) A scenario when the number of bends may be large. (f)–(g) Reducing bend complexity.*

$[l(s_2), r(s_2)]$. We refer to this property as the *balanced parenthesis property of the bend-intervals*. To verify this property assume that for some s_1, s_2, we have $[l(s_1), r(s_1)] \cap [l(s_2), r(s_2)] \neq \phi$. Since s_2 is a maximal sequence of consecutive bends, the inequalities $l(s_2) \leq l(s_1)$ and $r(s_2) \geq r(s_1)$ hold, i.e., $[l(s_1), r(s_1)] \subseteq [l(s_2), r(s_2)]$. We say that s_1 *is nested by* s_2. Figure 3(f) illustrates such a scenario, where s_1, s_2 are shown in thin and thick gray lines, respectively.

We now consider the edges of P in reverse order, i.e., for each j from n to 2, we modify the drawing of $e = (v_j, v_{j-1})$. For each bend-interval $s = (b_1, b_2 \ldots, b_r)$ of C_e, if s has three or more bends, then we delete the bends $b_2, \ldots, b_{r-1}$, and join b_1 and b_r using a new bend point w. To create w, we consider the two cases of the balanced parenthesis property.

If s is not nested by any other bend-interval in D'_n, then we place w high enough above b_r such that the chain b_1, w, b_r does not introduce any edge crossing, e.g., see the point $w_1 (= w)$ in Figure 3(g). On the other hand, if s is nested by some other bend-interval, then let s' be such a bend-interval immediately above s. Since $s' = (b'_1, b'_2, \ldots, b'_r)$ is already processed, it must have been replaced by some chain b'_1, w', b'_r. Therefore, we can find a location for b inside $\angle b'_1 w' b'_r$ such that the chain b_1, w, b_r does not introduce any edge crossing, e.g., see the points w' and $w_2 (= w)$ in Figure 3(g). Let the resulting drawing of P be Γ.

We now show that the above modification reduces the bend complexity to $3n/4$. Let e be an edge of P that contains α points from S between its endpoints. Let C_e be the corresponding polygonal chain in D'_n. Recall that any bend-interval of length ℓ in C_e contributes to $\min\{\ell, 3\}$ bends on e in Γ. Therefore, if there are at most $\alpha/4$ bend-intervals on C_e, then e can have at most $3\alpha/4$ bends in Γ. Otherwise, if there are more than $\alpha/4$ bend-intervals, then there are at least $\alpha/4$ points[1] of S that do not contribute to bends on C_e. Therefore, in both cases, C_e can have at most $3\alpha/4$ bends in Γ.

Satisfying Properties A–B: Let p_i be any point of $S \setminus \{p_0, p_{n+1}\}$. We first show that p_0 is $(3n/4)$-visible to p_i. Let D_i, where $1 \le i \le n$, be the drawing of the path $v_1, v_2, \ldots, v_i$. Observe that one can insert an edge (p_0, p_i) using an x-monotone polyline L such that the bends on L correspond to the intermediate visited points. Now the drawing of the rest of the path $v_i, v_{i+1}, \ldots, v_n$ can be continued such that it does not cross L. Therefore, if the number of points of S between p_0 and p_i is α, then L has at most α bends. Finally, the process of reducing bend complexity improves the number of bends on L to $3\alpha/4$.

Similarly, we can observe that p_{n+1} is at most $3\alpha'/4$ visible to p_i, where α' is the number of points of S between p_i and p_{n+1}. Since the edges (p_0, p_i) and (p_i, p_{n+1}) are x-monotone, we can draw an x-monotone polygonal chain from p_0 to p_{n+1} with at most $3(\alpha + \alpha')/4 \le (3n/4)$ bends that intersects Γ only at p_i. ◻

We now have the following theorem.

THEOREM 3.2. *Every n-vertex graph of thickness t admits a drawing on t planar layers with bend complexity $2.25n + O(1)$.*

Proof. Let $G_1, \ldots, G_t$ be the planar subgraphs of the input graph G, and let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of G. let $S = \{p_0, p_1, \ldots, p_{n+1}\}$ be a set of $n+2$ points lying on a semicircular arc as defined in Lemma 3.1. Let P_i be spinal path of the monotone topological book embedding of G_i, where $1 \le i \le t$. We first compute an uphill drawing Γ_i of the path P_i. We then draw the edges of G_i that do not belong to P_i. Let $e = (u, v)$ be such an edge, and without loss of generality assume that u appears to the left of v on the spine.

If e lies above (resp., below) the spine, then we draw two x-monotone polygonal chains; one from u to p_0 (resp., p_{n+1}), and the other from v to p_0 (resp., p_{n+1}). By Lemma 3.1, the polygonal chain $u, \ldots, p_0, \ldots, v$ (resp., $u, \ldots, p_{n+1}, \ldots, v$) does not intersect Γ_i except at u and v, and contains at most $2 \cdot (3n/4) = 1.5n$ bends.

[1] Every pair of consecutive bend-intervals contain such a point in between.

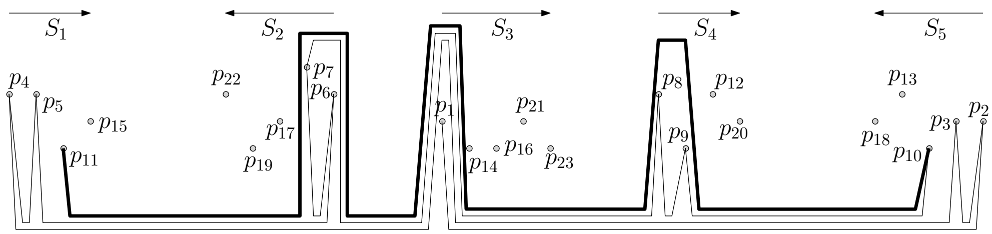

FIG. 4. *Illustration for the proof of Lemma 3.3. The edge (p_{10}, p_{11}) is shown in bold. Passing through each intermediate set requires at most 4 bends.*

If e crosses the spine, then it crosses some edge (w, w') of P_i. Draw the edges (u, w) and (w, v) using the polylines $u, \ldots, p_0, \ldots, w$ and $w, \ldots, p_{n+1}, \ldots, v$, respectively. The polylines $u, \ldots, p_0$ and $p_{n+1}, \ldots, v$ are x-monotone, and have at most $3n/4$ bends each. The polyline $C = (p_0, \ldots, w \ldots, p_{n+1})$ is also x-monotone and has at most $3n/4$ bends. Hence the number of bends is $2.25n$ in total. It is straightforward to avoid the degeneracy at w, by adding a constant number of bends on C.

Note that we still have some edge overlaps at p_0 and p_{n+1}. It is straightforward to remove these degeneracies by adding only a constant number of more bends per edge. ∎

3.2. A Construction for Small Values of *t*. In this section we give another construction to draw thickness-t graphs on t planar layers. We first show that every thickness-t graph, where $t \in \{3, 4\}$, can be drawn on t planar layers with bend complexity $O(\sqrt{n})$, and then show how to extend the technique for larger values of t.

3.2.1. Construction when $t = 3$. Let S be an ordered set of n points, where the ordering is by increasing x-coordinate. A (k, n)-*group* $S_{k,n}$ is a partition of S into k disjoint ordered subsets $\{S_1, \ldots, S_k\}$, each containing contiguous points from S. Label the points of S using a permutation of $p_1, p_2, \ldots, p_n$ such that for each set $S' \in S_{k,n}$, the indices of the points in S' are either increasing or decreasing. If the indices are increasing (resp., decreasing), then we refer S' as a rightward (resp., leftward) set. We will refer to such a labelling as a *smart labelling* of $S_{k,n}$. Figure 4 illustrates a $(5, 23)$-group and a smart labelling of the underlying point set $S_{5,23}$.

Note that for any i, where $1 \le i \le n$, deletion of the points $p_1, \ldots, p_i$ removes the points of the rightward (resp., leftward) sets from their left (resp., right). The *necklace* of $S_{k,n}$ is a path obtained from a smart labelling of $S_{k,n}$ by connecting the points p_i, p_{i+1}, where $1 \le i \le n-1$. The following lemma constructs an uphill drawing of the necklace using $O(k)$ bends per edge.

LEMMA 3.3. *Let S be a set of n points ordered by increasing x-coordinate, and let $S_{k,n} = \{S_1, \ldots, S_k\}$ be a (k, n)-group of S. Label $S_{k,n}$ with a smart labelling. Then the necklace of $S_{k,n}$ admits an uphill drawing with $O(k)$ bends per edge.*

Proof. We construct this uphill drawing incrementally in a similar way as in the proof of Lemma 3.1. Let D_j, where $1 \le j \le n$, be the drawing of the path $p_1, \ldots, p_j$. At each step of the construction, we maintain the invariant that D_j is an uphill drawing.

We first assign v_1 to p_1. Then for each i from 1 to $n-1$, we draw the edge (p_i, p_{i+1}) using an x-monotone polyline L that lies above D_i and below the points $p_{j'}$, where $j' > i+1$. Figure 4 illustrates such a drawing of (p_i, p_{i+1}).

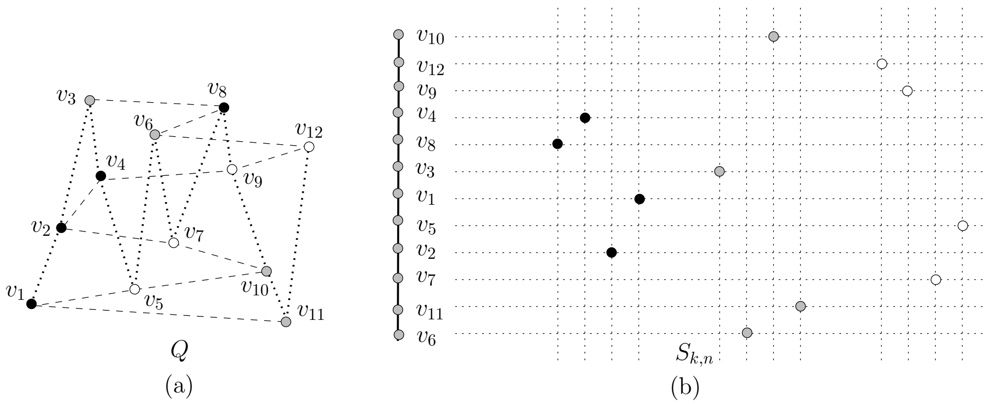

Fig. 5. *Creating vertex locations for drawing thickness-3 graphs, where P_1, P_2 and P_3 are shown in dotted, dashed and thick solid lines, respectively. Illustration for (a) Q, and (b) $S_{k,n}$.*

The crux of the construction is that one can draw such a polyline L using at most $O(k)$ bends. Assume that p_i and p_{i+1} belong to the sets $S_l \in S_{k,n}$ and $S_r \in S_{k,n}$, respectively. If S_l and S_r are identical, then p_i and p_{i+1} are consecutive, and hence it suffices to use at most $O(1)$ bends to draw L. On the other hand, if S_l and S_r are distinct, then there can be at most $k-2$ sets of $S_{k,n}$ between them. Let S_m be such a set. While passing through S_m, we need to keep the points that already belong to the path, below L, and the rest of the points above L. By the property of smart labelling, the points that belong to D_i are consecutive in S_m, and lie to the left or right side of S_m depending on whether S_m is rightward or leftward. Therefore, we need only $O(1)$ bends to pass through S_m. Since there are at most $k-2$ sets between S_l and S_r, $O(k)$ bends suffice to construct L. ◻

We are now ready to describe the main construction. Let G be an n-vertex thickness-3 graph, and let G_1, G_2, G_3 be the planar subgraphs of G. Let P_i be the spinal path of the monotone topological book embedding of G_i, where $1 \le i \le 3$. We first create a set of n points and assign them to the vertices of G. Later we route the edges of G.

Creating Vertex Locations: Assume without loss of generality that $P_1 = (v_1, \ldots, v_n)$. For each i from 1 to n, we place a point at (i, j) in the plane, where j is the position of v_i in P_2. Let the resulting point set be Q. Recall that by Erdös-Szekeres theorem, Q can be partitioned into disjoint monotone subsets $Q_1, \ldots, Q_k$, where $k \in O(\sqrt{n})$ [2, 14]. Figure 5(a) illustrates such a partition.

The sets $Q_1, \ldots, Q_k$ are ordered by the x-coordinate, and the indices of the labels of the points at each set is in increasing order. Therefore, if we place the points of the ith set between the lines $x = 2(i-1)n$ and $x = (2i-1)n$, then the resulting point set Q' would be a (k, n)-group, labelled by a smart labelling. Note that we preserve the monotonicity property of each group. Finally, we adjust the y-coordinates of the points according to the position of the corresponding vertices in P_3. Let the resulting point set be $S_{k,n}$. Figure 5(b) illustrates the vertex locations, where $P_1 = (v_1, v_2, \ldots, v_n)$, $P_2 = (v_{11}, v_1, \ldots, v_3)$, and $P_3 = (v_6, v_{11}, \ldots, v_{10})$.

Edge Routing: It is straightforward to observe that the path P_1 is a necklace for the current labelling of the points of $S_{k,n}$. Therefore, by Lemma 3.3, we can construct an uphill drawing of P_1 on $S_{k,n}$. Observe that for every set $S' \in S_{k,n}$, the corresponding points are monotone in Q, i.e., the points of S' are ordered along the x-axis either in increasing or decreasing order of their y-coordinates in Q. Therefore, relabelling the points according to the increasing order of their y-coordinates in Q will produce another smart labelling of $S_{k,n}$, and the corresponding necklace would be the path P_2. Therefore, we can use Lemma 3.3 to construct an uphill drawing of P_2 on $S_{k,n}$. Since the height of the points of $S_{k,n}$ are adjusted according to the vertex ordering on P_3, connecting the points of $S_{k,n}$ from top to bottom with straight line segments yields a y-monotone drawing of P_3.

We now route the edges of G_i that do not belong to P_i, where $1 \leq i \leq 3$. Since P_3 is drawn as a y-monotone polygonal path, we can use the technique of Erten and Kobourov [15] to draw the remaining edges of G_3. To draw the edges of G_2, we insert two points p_0 and p_{n+1} to the left and right of all the points of $S_{k,n}$, respectively. Then the drawing of the remaining edges of G_1 and G_2 is similar to the edge routing described in the proof of Theorem 3.2. That is, if the edge $e = (u, v)$ lies above (resp., below) the spine, then we draw it using two x-monotone polygonal chains from p_0 (resp., p_{n+1}). Otherwise, if e crosses the spine, then we draw three x-monotone polygonal chains, one from u to p_0, another from p_0 to p_{n+1}, and the third one from v to p_{n+1}. Since $k \in O(\sqrt{n})$, the number of bends on e is $O(\sqrt{n})$. Finally, we remove the degeneracies, which increases the bends per edge by a small constant.

3.2.2. Construction when $t = 4$. We now show that the technique for drawing thickness-3 graphs can be generalized to draw thickness-4 graphs with the same bend complexity.

Let $G_1, \ldots, G_4$ be the planar subgraphs of G, and let $P_1, \ldots, P_4$ be the corresponding spinal paths. While constructing the vertex locations, we use a new y-coordinate assignment for the points of $S_{k,n}$. Instead of placing the points according to the vertex ordering on the path P_3, we create a particular order, by transposing the x- and y-axis, that would help to construct uphill drawings of P_3 and P_4 with bend complexity $O(\sqrt{n})$. That is, we first create a (k', n)-group $S'_{k',n}$ using P_3 and P_4, where $k' \in O(\sqrt{n})$, in a similar way that we created $S_{k,n}$ using P_1 and P_2. We then adjust the y-coordinates of the points of $S_{k,n}$ according to the order these points appear in $S'_{k',n}$. Let S be the resulting point set, and let $P_1, \ldots, P_4$ be the spinal paths of $G_1, \ldots, G_4$, respectively. Figure 6(a) illustrates P_1 and P_2 in black and gray, respectively. Figure 6(b) illustrates P_3 and P_4 in black and gray, respectively. Figure 6(c) depicts the point set S, and Figures 7–8 illustrate the drawings of the spinal paths.

The construction of G_1 and G_2 remains the same as described in the previous section. However, since P_3 and P_4 now admit uphill drawings on S with respect to y-axis, the drawings of G_3 and G_4 are now analogous to the construction of G_1 and G_2.

3.2.3. Construction when $t > 4$. De Bruijn [19] observed that the result of Erdös-Szekeres [14] can be generalized to higher dimensions. Given a sequence ρ of n tuples, each of size κ, one can find a subsequence of at least $n^{1/\lambda}$ tuples, where $\lambda = 2^\kappa$, such that they are monotone (i.e., increasing or decreasing) in every dimension. This result is a repeated application of Erdös-Szekeres result [14] at each dimension. We now show how to partition ρ into few monotone sequences.

We use the partition algorithm of Bar-Yehuda and Sergio Fogel [2] that partitions a given sequence of n numbers into at most $2\sqrt{n}$ monotone subsequences. It is straightforward to restrict the size of the subsequences to $\sqrt{n}$, without increasing the number of subsequences, i.e., by repeatedly extracting a monotone sequence of length exactly $\sqrt{n}$. Consequently, one can partition ρ into $2\sqrt{n}$ subsequences, where each subse-

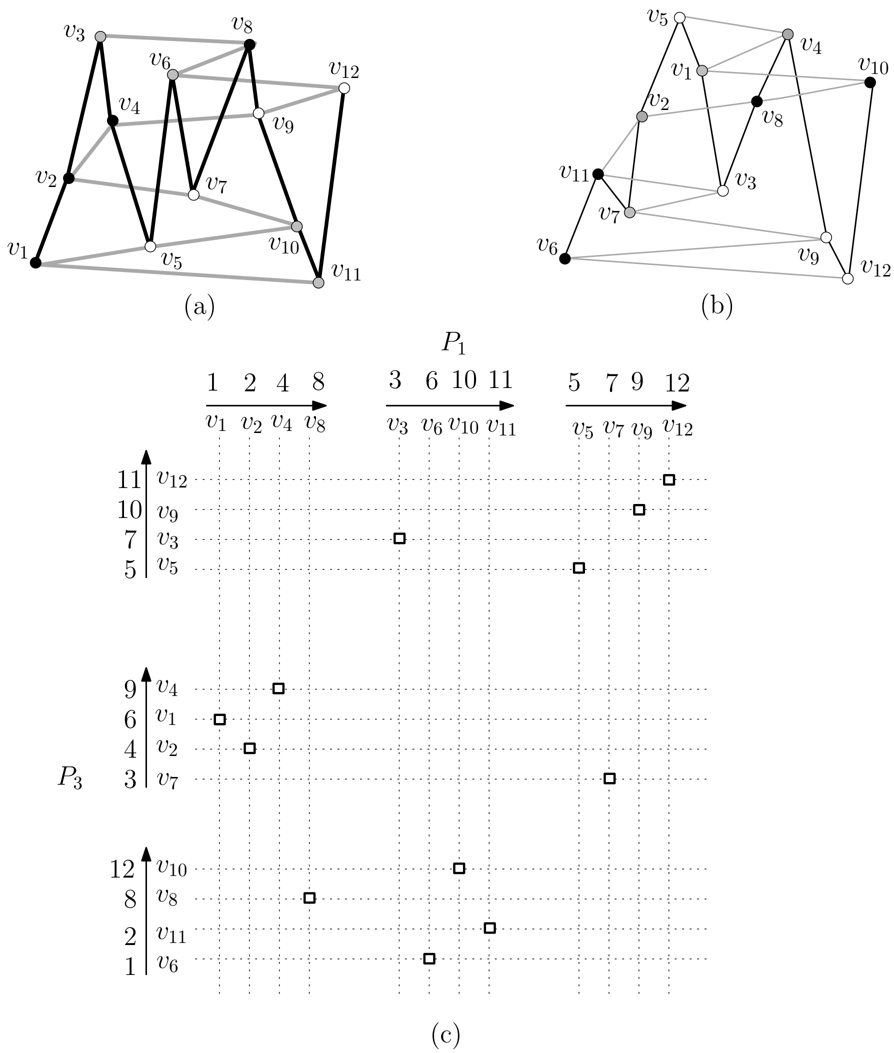

FIG. 6. *(a) A point set, constructed from the paths P_i, where $i \in \{1, 2\}$, by placing each vertex v at $(\delta_1(v), \delta_2(v))$. Here $\delta_i(v)$ is the position of v on P_i. (b) A point set, constructed from the paths P_i, where $i \in \{3, 4\}$, by placing each vertex v at $(\delta_3(v), \delta_4(v))$. (c) The final point set, and the corresponding (k, n)-groups. The numbers denote the vertex positions on the corresponding spinal path. The arrows illustrate whether the corresponding sets are leftward or rightward.*

quence is of length $\sqrt{n}$, and monotone in the first dimension. By applying the partition algorithm on each of these subsequences, we can find $2\sqrt{n}\cdot 2\sqrt{\sqrt{n}}$ subsequence, each of which is of length $\sqrt{\sqrt{n}}$, and monotone in the first and second dimensions. Therefore, after κ steps, we obtain a partition of ρ into $2^{\kappa} \cdot (n^{1/2} \cdot n^{1/4} \cdot \ldots \cdot n^{1/2^{\kappa}}) = 2^{\kappa} \cdot n^{1-(1/\lambda)}$ monotone subsequences, where $\lambda = 2^{\kappa}$. We use this idea to extend our drawing algorithm to higher thickness.

Let $G_1, \ldots, G_t$ be the planar subgraphs of G, and let $P_1, \ldots, P_t$ be the corresponding spinal paths. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. Construct a corresponding sequence $\rho = (\tau_1, \tau_2, \ldots, \tau_n)$ of n tuples, where each tuple is of size t, and the ith element of a tuple τ_j corresponds to the position of the corresponding vertex v_j in

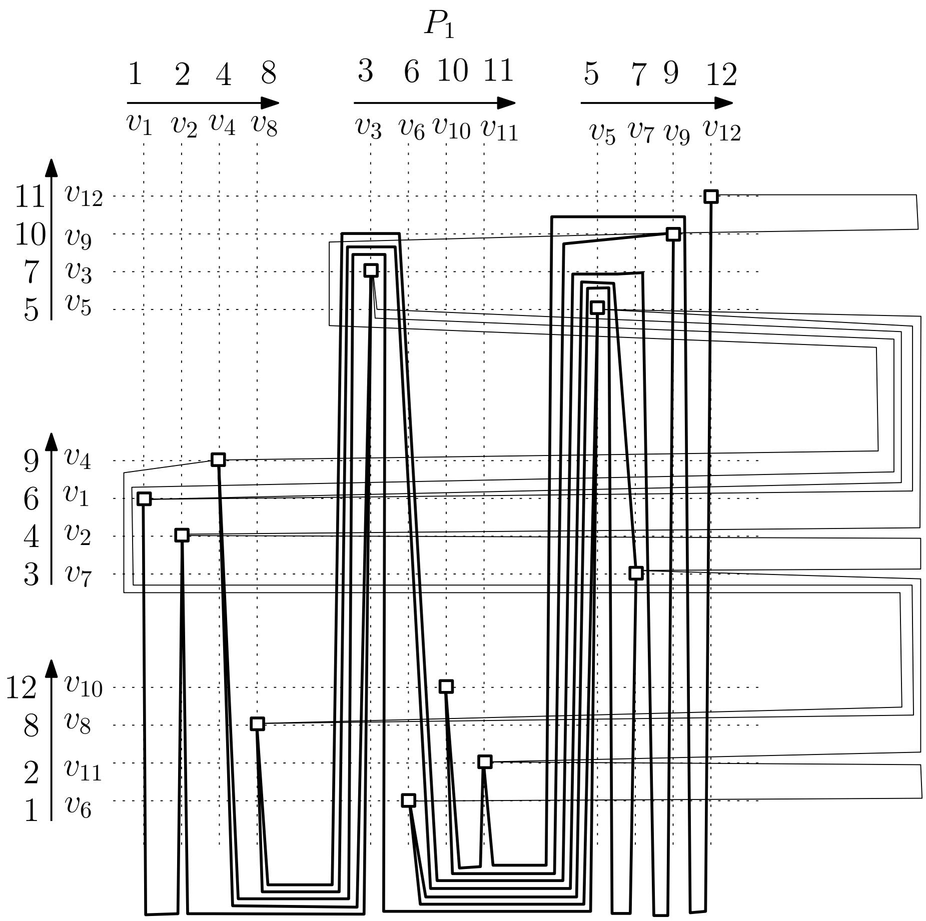

FIG. 7. *Drawings of P_1 and P_3 on the point set of Figure 6(c).*

P_i, where $1 \le i \le t$ and $1 \le j \le n$. We now partition ρ into a set of $2^t \cdot n^{1-(1/\beta)}$ monotone subsequences, where $\beta = 2^t$.

For each of these monotone sequences, we create an ordered set of consecutive points along the x-axis, where the vertex v_j corresponds to the point p_j. It is now straightforward to observe that these sets correspond to a (k, n)-group $S_{k,n}$, where $k \le 2^t \cdot n^{1-(1/\beta)}$. Furthermore, since each group corresponds to a monotone sequence of tuples, for each P_i, the positions of the corresponding vertices are either increasing or decreasing. Hence, every path P_i corresponds to a necklace for some smart labelling of $S_{k,n}$. Therefore, by Lemma 3.3, we can construct an uphill drawing of P_i on S. We now add the remaining edges of G_i following the construction described in Section 3.2.1. Since $k \le 2^t \cdot n^{1-(1/\beta)}$, the number of bends is bounded by $O(2^t \cdot n^{1-(1/\beta)})$.

Observe that all the points in the above construction have the same y-coordinate. Therefore, we can improve the construction by distributing the load equally among the x-axis and y-axis as we did in Section 3.2.2. Specifically, we draw the graphs $G_1, \ldots, G_{\lceil t/2 \rceil}$ using the uphill drawings of their spinal paths with respect to the x-axis, and the remaining graphs using the uphill drawings of their spinal paths with re-

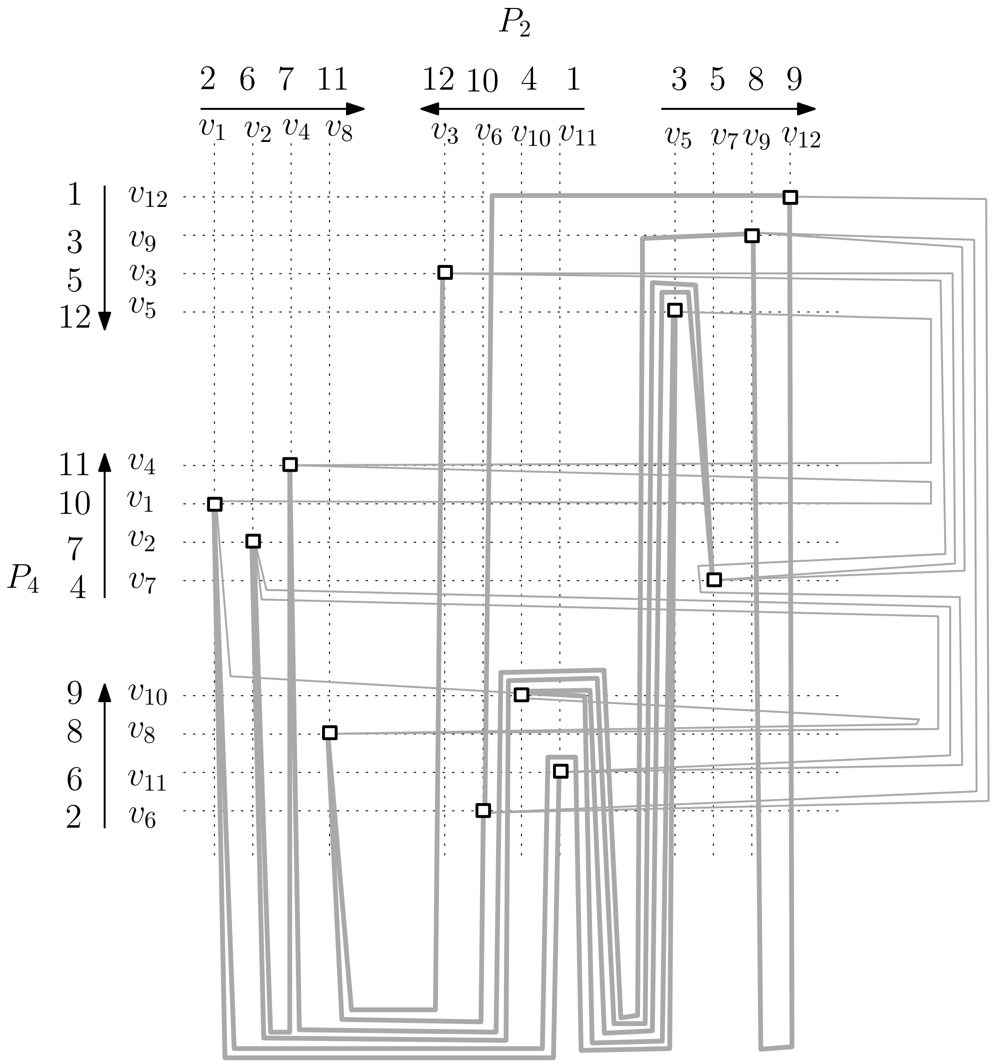

FIG. 8. *Drawings of P_2 and P_4 on the point set of Figure 6(c).*

spect to the y-axis. Consequently, the bend complexity decreases to $O(\sqrt{2}^t \cdot n^{1-(1/\beta')})$, where $\beta' = 2^{\lceil t/2 \rceil}$.

We can improve this bound further by observing that we are free to choose any arbitrary vertex labelling for G while creating the initial sequence of tuples. Instead of using an arbitrary labelling, we could label the vertices according to their ordering on some spinal path, which would reduce the bend complexity to $O(\sqrt{2}^{t-2} \cdot n^{1-(1/\beta'')})$, where $\beta'' = 2^{\lceil (t-2)/2 \rceil}$.

THEOREM 3.4. *Every n-vertex graph G of thickness $t \geq 3$ admits a drawing on t planar layers with bend complexity $O(\sqrt{2}^t \cdot n^{1-(1/\beta)})$, where $\beta = 2^{\lceil (t-2)/2 \rceil}$.*

4. Drawing Graphs of Linear Arboricity *k*. In this section we construct polyline drawings, where the layer number and bend complexities are functions of the linear arboricity of the input graphs. We show that the bandwidth of a graph can be bounded in terms of its linear arboricity and the number of vertices, and then the result follows from an application of Lemma 3.1.

The *bandwidth* of an n-vertex graph $G = (V, E)$ is the minimum integer b such that the vertices can be labelled using distinct integers from 1 to n satisfying the condition that for any edge $(u, v) \in E$, the absolute difference between the labels of u and v is at most b. The following lemma proves an upper bound on the bandwidth of graphs.

LEMMA 4.1. *Given an n-vertex graph $G = (V, E)$ with linear arboricity k, the bandwidth of G is at most $\frac{3(k-1)n}{(4k-2)}$.*

Proof. Without loss of generality assume that G is a union of k spanning paths $P_1, \ldots, P_k$. For any ordered sequence σ, let $\sigma(i)$ be the element at the ith position, and let $|\sigma|$ be the number of elements in σ. We now construct an ordered sequence $\sigma = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_k \circ \sigma_{k+1}$ of the vertices in V, as follows.

σ_1: We initially place the first x vertices of P_1 in the sequence, where the exact value of x is to be determined later.

σ_2: We then place the vertices that are neighbors of σ_1 in P_2, in order, i.e., we first place the neighbors of $\sigma_1(1)$, then the neighbors of $\sigma_1(2)$ that have not been placed yet, and so on.

σ_i: For each $i = 3, \ldots, k$, we place the vertices that are neighbors of σ_1 in P_i in order.

σ_{k+1}: We next place the remaining vertices of P_1 in order.

Figure 9(a) illustrates an example for three paths with $x = 2$. Observe that $|\sigma_1| \leq x$, and $|\sigma_t| \leq 2x$, where $1 < t \leq k$. We now compute an upper bound on the bandwidth of G using the vertex ordering of σ.

For any i, j, where $1 \leq i < j \leq k+1$, let $\sigma_{i,j}$ be the sequence $\sigma_i \circ \ldots \circ \sigma_j$. The edges of P_1 that are in σ_1 have bandwidth 1, and those that are in $\sigma_1(x) \circ (\sigma \setminus \sigma_1)$ have bandwidth at most $(n - x)$, e.g., see Figure 9(b). Now let (v, w) be an edge of G that does not belong to P_1. We compute the bandwidth of (v, w) considering the following cases.

Case 1. If none of v and w belongs to σ_1, then the bandwidth of (v, w) is at most $(n - x)$.

Case 2. If both v and w belong to σ_1, then the bandwidth of (v, w) is at most x.

Case 3. If at most one of v and w belongs to σ_1, then without loss of generality assume that v belongs to σ_1. Since (v, w) does not belong to P_1, we may assume that w belongs to the path P_t, where $1 < t \leq k$. By the construction of σ, w belongs to $\sigma_{1,t}$, e.g., see Figure 9(b). Without loss of generality assume that w belongs to σ_r, where $1 < r \leq t$. Let v be the qth vertex in the sequence σ. Then the position of w cannot be more than $q + 2x \cdot (r-2) + 2q$, where the term $2x \cdot (r-2)$ corresponds to length of $\sigma_2 \circ \ldots \circ \sigma_{r-1}$. Therefore, the bandwidth of the edge (v, w) is at most $2x \cdot (r-2) + q \leq 2x(r-1) \leq 2x(t-1)$.

Observe that the bandwidth of the edges of P_1 is upper bounded by $(n - x)$. The bandwidth of any edge that belongs to P_t, where $1 < t \leq k$ is at most $2x(t-1)$. Consequently, the bandwidth of G is at most $\max\{n - x, 2x(k-1)\} \leq \frac{(2k-2)n}{(2k-1)}$, where $x = \frac{n}{(2k-1)}$. ∎

The following theorem is immediate from the proof of Lemmas 3.1 and 4.1.

THEOREM 4.2. *Every n-vertex graph with linear arboricity k can be drawn on k planar layers with at most $\frac{3(k-1)n}{(4k-2)} < 0.75n$ bends per edge.*

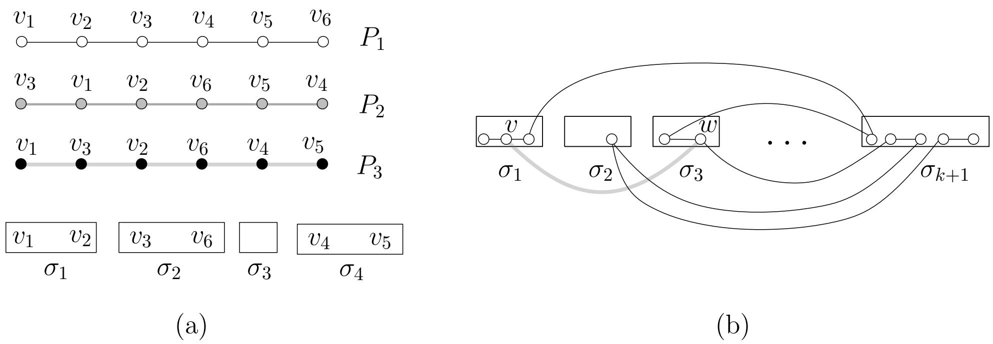

FIG. 9. *(a) Construction of σ. (b) A schematic representation of P_1 and (v, w), where (v, w) belongs to P_3.*

5. Conclusions.

In this paper we have developed algorithms to draw graphs on few planar layers and with low bend complexity. Although our algorithms do not construct drawings with integral coordinates, it is straightforward to see that these drawings can also be constructed on polynomial-size integer grids, where all vertices and bends have integral coordinates. We leave the task of finding compact grid drawings achieving the same upper bounds as a direction for future research.

We believe our upper bounds on bend complexity to be nearly tight, but we require more evidence to support this intuition. The only related lower bound is that of Pach and Wenger [21], who showed that given a planar graph G and a unique location to place each vertex of G, $\Omega(n)$ bends are sometimes necessary to construct a planar polyline drawing of G with the given vertex locations. Therefore, a challenging research direction would be to prove tight lower bounds on the bend complexity while drawing thickness-t graphs on t planar layers.

REFERENCES

[1] M. Badent, E. D. Giacomo, and G. Liotta, *Drawing colored graphs on colored points*, Theoretical Computer Science, 408 (2008), pp. 129–142.

[2] R. Bar-Yehuda and S. Fogel, *Partitioning a sequence into few monotone subsequences*, Acta Informatica, 35 (1998), pp. 421–440.

[3] J. Barát, J. Matoušek, and D. R. Wood, *Bounded-degree graphs have arbitrarily large geometric thickness*, Electronic Journal of Combinatorics, 13 (2006).

[4] T. Bläsius, S. G. Kobourov, and I. Rutter, *Simultaneous embedding of planar graphs*, in Handbook of Graph Drawing and Visualization, R. Tamassia, ed., CRC Press, August 2013, ch. 11, pp. 349–380.

[5] P. Brass, E. Cenek, C. A. Duncan, A. Efrat, C. Erten, D. Ismailescu, S. G. Kobourov, A. Lubiw, and J. S. B. Mitchell, *On simultaneous planar graph embeddings*, Computational Geometry, 36 (2007), pp. 117–130.

[6] I. Chatzigiannakis, M. Mitzenmacher, Y. Rabani, and D. Sangiorgi, eds., *Relating Graph Thickness to Planar Layers and Bend Complexity*, vol. 55 of LIPIcs, Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.

[7] M. B. Dillencourt, D. Eppstein, and D. S. Hirschberg, *Geometric thickness of complete graphs*, Journal of Graph Algorithms and Applications, 4 (2000), pp. 5–17.

[8] V. Dujmović and D. R. Wood, *Graph treewidth and geometric thickness parameters*, Discrete & Computational Geometry, 37 (2007), pp. 641–670.

[9] C. A. Duncan, *On graph thickness, geometric thickness, and separator theorems*, Computational Geometry, 44 (2011), pp. 95–99.

[10] C. A. Duncan, D. Eppstein, and S. G. Kobourov, *The geometric thickness of low degree graphs*, in Proceedings of the 20th ACM Symposium on Computational Geometry (SoCG),

ACM, 2004, pp. 340–346.

[11] S. Durocher, E. Gethner, and D. Mondal, *Thickness and colorability of geometric graphs*, Computational Geometry: Theory and Applications, 56 (2016), pp. 1–18.

[12] H. Enomoto and M. S. Miyauchi, *Embedding graphs into a three page book with $O(m \log n)$ crossings of edges over the spine*, SIAM Journal on Discrete Mathematics, 12 (1999), pp. 337–341.

[13] D. Eppstein, *Separating thickness from geometric thickness*, in Towards a Theory of Geometric Graphs, J. Pach, ed., American Mathematical Society, 2004.

[14] P. Erdös and G. Szekeres, *A combinatorial theorem in geometry*, Compositio Math., 2 (1935), pp. 463–470.

[15] C. Erten and S. G. Kobourov, *Simultaneous embedding of planar graphs with few bends*, Journal of Graph Algorithms and Applications, 9 (2005), pp. 347–364.

[16] I. Fáry, *On straight-line representation of planar graphs*, Acta Sci. Math. (Szeged), 11 (1948), pp. 229–233.

[17] E. D. Giacomo and G. Liotta, *Simultaneous embedding of outerplanar graphs, paths, and cycles*, International Journal of Computational Geometry & Applications, 17 (2007), pp. 139–160.

[18] T. Gordon, *Simultaneous embeddings with vertices mapping to pre-specified points*, in Proceedings of the 18th Annual International Conference on Computing and Combinatorics (COCOON), vol. 7434 of LNCS, Springer, 2012, pp. 299–310.

[19] J. B. Kruskal, *Monotonic subsequences*, Proceedings of the American Mathematical Society, 4 (1953), pp. 264–274.

[20] D. Mondal, *Visualizing graphs: optimization and trade-offs*, PhD thesis, University of Manitoba, Winnipeg, Canada, 2016.

[21] J. Pach and R. Wenger, *Embedding planar graphs at fixed vertex locations*, Graphs & Combinatorics, 17 (2001), pp. 717–728.

[22] D. R. Wood, *Geometric thickness in a grid*, Discrete Mathematics, 273 (2003), pp. 221–234.