Searching on a Line:
A Complete Characterization of the Optimal Solution[⋆]

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Abstract

We revisit the problem of searching for a target at an unknown location on a line when given upper and lower bounds on the distance D that separates the initial position of the searcher from the target. Prior to this work, only asymptotic bounds were known for the optimal competitive ratio achievable by any search strategy in the worst case. We present the first tight bounds on the exact optimal competitive ratio achievable, parameterized in terms of the given bounds on D, along with an optimal search strategy that achieves this competitive ratio. We prove that this optimal strategy is unique. We characterize the conditions under which an optimal strategy can be computed exactly and, when it cannot, we explain how numerical methods can be used efficiently. In addition, we answer several related open questions, including the *maximal reach problem*, and we discuss how to generalize these results to m rays, for any $m \geq 2$.

Keywords: Computational Geometry, Online Algorithms, Optimization, Fibonacci polynomials

1. Introduction

Search problems are broadly studied within computer science. A fundamental search problem, which is the focus of this paper, is to specify how a searcher should move to find an immobile target at an unknown location on a line such that the total relative distance travelled by the searcher is minimized in the worst case [4, 12, 15]. The searcher is required to move continuously on the line, i.e., discontinuous jumps, such as random access in an array, are not possible. Thus, a search corresponds to a sequence of alternating left and right displacements by the searcher. This class of geometric search problems was introduced by Bellman [5] who first formulated the problem of searching for the boundary of a region from an unknown random point within its interior. Since then, many variants of the line search problem have been studied, including multiple rays sharing a common endpoint (as opposed to a line, which corresponds

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to two rays), multiple targets, multiple searchers, moving targets, and randomized search strategies (e.g., [1, 2, 3, 4, 6, 8, 9, 10, 11, 14, 15, 16]).

For any given search strategy f and any given target location, we consider the ratio A/D, where A denotes the total length of the search path travelled by a searcher before reaching the target by applying strategy f, and D corresponds to the minimum travel distance necessary to reach the target. That is, the searcher and target initially lie a distance D from each other on a line, but the searcher knows neither the value D nor whether the target lies to its left or right. The *competitive ratio* of a search strategy f, denoted $CR(f)$, is measured by the supremum of the ratios achieved over all possible target locations. Observe that $CR(f)$ is unbounded if D can be assigned any arbitrary real value; specifically, the searcher must know a lower bound $\lambda \leq D$. Thus, it is natural to consider scenarios where the searcher has additional information about the distance to the target. In particular, in many instances the searcher can estimate good lower and upper bounds on D. Given a lower bound $D \geq \lambda$, Baeza-Yates et al. [4] show that any optimal strategy achieves a competitive ratio of 9. They describe such a strategy, which we call the *Power of Two* strategy. Furthermore, they observe that when D is known to the searcher, it suffices to travel a distance of $3D$ in the worst case, achieving a competitive ratio of 3.

We represent a search strategy by a function $f : \mathbb{N} \to \mathbb{R}^+$. Given such a function, a searcher travels a distance of $f(0)$ in one direction from the origin (say, to the right), returns to the origin, travels a distance of $f(1)$ in the opposite direction (to the left), returns to the origin, and so on, until reaching the target. We refer to $f(i)$ as the distance the searcher travels from the origin during the i-th iteration. The corresponding function for the Power of Two strategy of Baeza-Yates et al. is $f(i) = 2^i\lambda$. Showing that every optimal strategy achieves a competitive ratio of exactly 9 relies on the fact that no upper bound on D is specified [4]. Therefore, it is natural to ask whether a search strategy can achieve a better competitive ratio when provided both lower and upper bounds $\lambda \leq D \leq \Lambda$. In what follows, we use the scale invariant $\rho = \Lambda/\lambda$ to indicate how good the upper bound is. When $\rho = 1$, then $\Lambda = \lambda$. This corresponds to the case where the searcher knows D. Moreover, if we let $\rho \to \infty$, this represents the case where the searcher is not given any upper bound.

Given R, the *maximal reach problem*, examined by Hipke et al. [12], is to identify the largest bound Λ such that there exists a search strategy that finds any target within distance $D \leq \Lambda$ with competitive ratio at most R. López-Ortiz and Schuierer [15] study the maximal reach problem on m rays, from which they deduce that the competitive ratio $CR(f_{opt})$ of any optimal strategy f_{opt} is at least

$$1 + 2\frac{m^m}{(m-1)^{m-1}} - O\left(\frac{1}{\log^2 \rho}\right).$$

When $m = 2$, the corresponding lower bound becomes

$$9 - O\left(\frac{1}{\log^2 \rho}\right).$$

They also provide a general strategy that achieves this asymptotic behaviour on m concurrent rays, given by

$$f(i) = \sqrt{1 + \frac{i}{m}} \left(\frac{m}{m-1} \right)^i \lambda.$$

Again, for $m = 2$ this is

$$f(i) = \sqrt{1 + \frac{i}{2}} \, 2^i \lambda.$$

Surprisingly, this general strategy is independent of ρ. In essence, it ignores any upper bound on D, regardless of how tight it is. Thus, we examine whether there exists a better search strategy that depends on ρ, thereby using both the upper and lower bounds on D. Furthermore, previous lower bounds on $CR(f_{opt})$ have an asymptotic dependence on ρ applying only to large values of ρ, corresponding to having only coarse bounds on D. Can we express tight bounds on $CR(f_{opt})$ in terms of ρ?

Let $f_{opt}(i) = a_i \lambda$ denote an optimal strategy for given values λ and Λ. Since f_{opt} is optimal and $D \geq \lambda$, we must have $f_{opt}(i) \geq \lambda$ for all $i \geq 0$. Therefore, $a_i \geq 1$ for all $i \geq 0$. Moreover, for any possible position of the target, the strategy f_{opt} must eventually reach it. Hence, there must be two integers i and i' of different parities such that $f_{opt}(i) \geq \Lambda$ and $f_{opt}(i') \geq \Lambda$, which implies that $a_i \geq \rho$ and $a_{i'} \geq \rho$. Since f_{opt} is optimal, we have $f_{opt}(i) = f_{opt}(i') = \Lambda$. Moreover, let n be the smallest integer such that $f_{opt}(n) = \Lambda$ (equivalently, $a_n = \rho$). Since f_{opt} is optimal, we have $f_{opt}(n+1) = \Lambda$ (equivalently, $a_{n+1} = \rho$). Consequently, $n+2$ is the number of iterations necessary to reach the target with strategy f_{opt} in the worst case (recall that the sequence starts at $i = 0$). The question is now to determine the sequence $\{a_i\}_{i=0}^n$ that defines f_{opt}. López-Ortiz and Schuierer [15] provide an algorithm to compute the maximal reach for a given competitive ratio together with a strategy corresponding to this maximal reach. They state that the value n and the sequence $\{a_i\}_{i=0}^n$ can be computed using binary search, which increases the running time proportionally to $\log \rho$. Can we find a faster algorithm for computing f_{opt}? Since in general, a_0 is the root of a polynomial equation of unbounded degree (see Theorem 1), a binary search is equivalent to the bisection method for solving polynomial equations. However, the bisection method is a slowly converging numerical method. Can the computational efficiency be improved? Moreover, given ε, can we bound the number of steps necessary for a root-finding algorithm to identify a solution within tolerance ε of the exact value?

1.1. Overview of Results

We address all of the questions raised above in Section 2. We characterize f_{opt} by computing the sequence $\{a_i\}_{i=0}^n$ for the optimal strategy. We do this by computing the number of iterations $n+2$ needed to find the target in the worst case. We can compute n in $O(1)$ time since we prove that $n \in \{\lfloor \log_2 \rho \rfloor - 1, \lfloor \log_2 \rho \rfloor\}$, where $\rho = \Lambda/\lambda$. Then, we define a family of $n+1$ polynomials $p_0, \ldots, p_n$, where p_i has degree $i+1$. We show that a_0 is the largest real solution to the polynomial equation $p_n(x) = \rho$. Each of the remaining elements in the sequence $\{a_i\}_{i=0}^n$ can be computed in $O(1)$ time since we prove that $a_1 = a_0(a_0 - 1)$ and

$a_i = a_0(a_{i-1} - a_{i-2})$ for $2 \le i \le n$. This also shows that the optimal strategy is unique. However, as we show in Proposition 3, when no upper bound is known, there exist infinitely many optimal strategies for any $m \ge 2$.

We give an exact characterization of f_{opt} and show that $CR(f_{opt}) = 2a_0 + 1$. This allows us to establish the following bounds on the competitive ratio of an optimal strategy in terms of ρ:

$$8\cos^2\left(\frac{\pi}{\lceil \log_2 \rho \rceil + 2}\right) + 1 \le CR(f_{opt}) < 8\cos^2\left(\frac{\pi}{\lfloor \log_2 \rho \rfloor + 4}\right) + 1.$$

López-Ortiz and Schuierer [15] show that $CR(f_{opt}) \to 9$ as $\rho \to \infty$. We show that $f_{opt} \to f_\infty$ as $\rho \to \infty$, where $f_\infty(i) = (2i+4)2^i\lambda$ has a competitive ratio of 9. Notice that f_∞ is different than the power of two strategy. We thereby obtain an alternate proof of the result of Baeza-Yates et al. [4]. The strategy f_∞ is a member of the infinite family of optimal strategies in the unbounded case which we describe in Proposition 3.

We assume the Real RAM model of computation, including k-th roots, logarithms, exponentiation, and trigonometric functions [19]. The computation of each term a_i in the sequence defining f_{opt} involves computing the largest real root of a polynomial equation of degree $n+1$. We prove that $n+1 \le 4$ if and only if $\rho \le 32\cos^5(\pi/7) \approx 18.99761$. In this case the root can be expressed exactly using only the operations $+$, $-$, $\times$, $\div$, $\sqrt{\cdot}$ and $\sqrt[3]{\cdot}$. This implies that if $\Lambda \le 32\cos^5(\pi/7)\lambda$, then f_{opt} can be computed exactly in $O(1)$ time ($O(1)$ time per a_i for $0 \le i \le n < 4$). In general, when $n+1 \ge 5$, Galois theory implies that the equation $p_n(x) = \rho$ cannot be solved by radicals. Since the corresponding polynomials have unbounded degree, we are required to consider approximate solutions when $\rho > 32\cos^5(\pi/7)$. Therefore, we explain how to find a solution f^*_{opt}, such that $CR(f^*_{opt}) \le CR(f_{opt}) + \varepsilon$ for a given tolerance ε. If $n \ge 7\varepsilon^{-1/3} - 4$, we give an explicit formula for a_0. Hence, an ε-approximation can be computed in $O(n) = O(\log \rho)$ time ($O(1)$ time per a_i for $0 \le i \le n$). Otherwise, if $32\cos^5(\pi/7) < n < 7\varepsilon^{-1/3} - 4$, we show that a_0 lies in an interval of length at most $7^3\,(n+4)^{-3}$. Moreover, we prove that the polynomial is strictly increasing on this interval. Hence, usual root-finding algorithms work well. Given a_0, the remaining elements of the sequence $\{a_1, \ldots, a_n\}$ can be computed in $O(n)$ time ($O(1)$ time per a_i for $1 \le i \le n$ since $a_i = a_0(a_{i-1} - a_{i-2})$).

These results are summarized in Theorems 1—5 (see Section 2) and Algorithm 1 (see below). The input to the algorithm is an interval $[\lambda, \Lambda]$ that contains D and a tolerance $\varepsilon > 0$. In Line 1, we set the values of ρ and a_{-1}. The variable a_{-1} allows us to simplify the code for the computation of the a_i's ($1 \le i \le n$). The number of iterations $(n+2)$ for the optimal strategy is computed in Lines 2 to 5. The correctness of the code for these lines is a consequence of (8) (see Section 2) and Theorem 2. We also get

$$\alpha_{n+1} = 4\cos^2\left(\frac{\pi}{n+3}\right) \le a_0 < 4\cos^2\left(\frac{\pi}{n+4}\right) = \alpha_{n+2} \tag{1}$$

from (8) and Theorem 2. The numbers α_{n+1} and α_{n+2} are computed in Line 6. In Lines 7 to 16, we compute a_0, which is the more challenging step in the computation of an optimal strategy. In Section 2, we define

Algorithm 1 Optimal Strategy for Searching on a Line

Input: An interval $[\lambda, \Lambda]$ that contains D and a tolerance $\varepsilon > 0$.

Output: A sequence $\{a_i\}$ defining a search strategy f and its corresponding competitive ratio, where f is an exact optimal strategy if $\rho = \Lambda/\lambda \leq 32\cos^5(\pi/7) \approx 18.99761$, and f has a competitive ratio of at most an additive constant ε more than optimal otherwise.

1: $\rho = \Lambda/\lambda$; $a_{-1} = 1$
2: $n = \lfloor \log_2 \rho \rfloor$; $\gamma = 2\cos\left(\frac{\pi}{n+3}\right)$
3: **if** $n + 1 > \log_\gamma \rho$ **then**
4: $\quad n = n - 1$
5: **end if**
6: $\alpha_{n+1} = 4\cos^2\left(\frac{\pi}{n+3}\right)$; $\alpha_{n+2} = 4\cos^2\left(\frac{\pi}{n+4}\right)$
7: **if** $n \leq 3$ **then**
8: $\quad$/* Find an exact solution for a_0 in $O(1)$ time. */
9: $\quad$Find $a_0 \in [\alpha_{n+1}, \alpha_{n+2}]$ such that $p_n(a_0) = \rho$.
10: **else if** $n \geq 7\varepsilon^{-1/3} - 4$ **then**
11: $\quad$/* Find an approximate solution for a_0 in $O(1)$ time. */
12: $\quad a_0 = \alpha_{n+2}$
13: **else**
14: $\quad$/* Find an approximate solution for a_0 using numerical methods. */
15: $\quad$Find $a_0 \in [\alpha_{n+1}, \alpha_{n+2}]$ such that $|p_n(a_0) - \rho| < \varepsilon$, using numerical methods.
16: **end if**
17: **for** $1 \leq i \leq n$ **do**
18: $\quad a_i = a_0(a_{i-1} - a_{i-2})$
19: **end for**
20: **return** $\{a_0\lambda, a_1\lambda, ..., a_n\lambda, \Lambda\}$ and $2a_0 + 1$

the following family of polynomials:

$$p_0(x) = x,$$
$$p_1(x) = x(x-1),$$
$$p_i(x) = x\left(p_{i-1}(x) - p_{i-2}(x)\right) \qquad (i \geq 2).$$

Then we prove in Theorem 1 that $p_n(a_0) = \rho$. Together with (1), this tells us that, to find a_0, we need to solve $p_n(a_0) = \rho$ on the interval $[\alpha_{n+1}, \alpha_{n+2}]$. Since p_n has degree $n+1$, if $n \leq 3$, then $p_n(a_0) = \rho$ can be solved exactly on $[\alpha_{n+1}, \alpha_{n+2}]$. We address this case in Lines 7 to 9. In Theorem 3, we prove that if $n \geq 7\varepsilon^{-1/3} - 4$ and we let $a_0 = \alpha_{n+2}$, we get a strategy whose competitive ratio is at most an additive constant ε more than optimal. This case corresponds to Lines 10 to 12. Otherwise, we need to use numerical methods to solve $p_n(a_0) = \rho$ on $[\alpha_{n+1}, \alpha_{n+2}]$ (see Lines 13 to 16). However, we show in Theorem 3 that p_n is strictly increasing on $[\alpha_{n+1}, \alpha_{n+2}]$ and that $|\alpha_{n+2} - \alpha_{n+1}| \leq 7^3\,(n+4)^{-3}/2$. Therefore, usual root-finding algorithms behave well on this problem. It remains to compute the a_i's for $1 \leq i \leq n$. This is done in Lines 17 to 19. The formula for the a_i's comes from (5) (refer to Section 2) and Theorem 1. In Line 20, we return a strategy f and its cost. By Theorem 5, if $\rho = \Lambda/\lambda \leq 32\cos^5(\pi/7) \approx 18.99761$, f is an exact optimal strategy. Otherwise, f has a competitive ratio of at most an additive constant ε more than optimal.

The crux of the problem is to realize that the optimal strategy can be characterized using the polynomials p_i's and that these polynomials are *generalized Fibonacci polynomials* (refer to the proof of Proposition 1). The proofs of all the theorems in Section 2 rely on properties of this family of polynomials. With a result by Hoggatt and Long [13] on generalized Fibonacci polynomials as a starting point, we prove several properties of this family of polynomials in Propositions 1 and 2, and Lemma 1. Then we prove the theorems in Section 2.

In Section 3, we explain how to solve the maximal reach problem using the results of Section 2. Given a competitive ratio R and a lower bound λ on D, the *maximal reach problem* is to identify the largest bound Λ such that there exists a search strategy that finds any target within distance $\lambda \leq D \leq \Lambda$ with competitive ratio at most R. We prove that $\Lambda = p_n((R-1)/2)\lambda$, where

$$n = \left\lfloor \frac{\pi}{\arccos\left(\frac{\sqrt{R-1}}{2\sqrt{2}}\right)} \right\rfloor - 3.$$

In Section 4, we address the problem of searching on m rays ($m \geq 2$). In Proposition 3, we characterize an infinite family of optimal solutions for the unbounded case. Then we apply our technique for searching on a bounded line to the problem of searching on m bounded rays. This leads to partial results. When $m = 2$, the characterization of an optimal solution relies on the computation of the first number a_0. In general, the characterization of an optimal solution relies on the computation of the first $m-1$ numbers a_i ($0 \leq i \leq m-2$). Let $\overline{a} = (a_0, a_1, ..., a_{m-2})$. In Theorem 6, we characterize a family of polynomials $p_i(\overline{x})$ such that $a_i = p_i(\overline{a})$ and $\overline{a}$ is a solution to the system of equations

$$p_n(\overline{x}) = \rho, \quad p_{n+1}(\overline{x}) = \rho, \quad \ldots, \quad p_{n+(m-2)}(\overline{x}) = \rho. \tag{2}$$

Moreover, the competitive ratio of the optimal strategy has a cost of $1 + 2|\overline{a}|$. This corresponds to a generalization of Theorem 1, except that we could not provide a domain that contain a unique solution $\overline{a}$ to the system (2) (in Theorem 1 for $m = 2$, we prove that $a_0 > \alpha_n$). This does not improve upon the general result of López-Ortiz and Schuierer [15]. However, we believe it is a promising direction to follow if one wants to obtain a complete characterization for $m > 2$.

We conclude in Section 5 with a discussion on open problems for the case $m > 2$. We conjecture that the main parts of Theorems 2, 3 and 4 can be be generalized to the case $m > 2$.

2. Searching on a Bounded Line

López-Ortiz and Schuierer [15] showed that there always exists an optimal strategy for searching on a line segment that is *periodic* and *monotone*[3]. Let the rays on each side of the starting point be labelled 0

[3]They actually showed that there always exists an optimal strategy for searching on m bounded rays ($m \geq 2$) that is periodic and monotone.

and 1, respectively. A strategy is periodic if after visiting the ray k (for any $k \in \{0,1\}$), it visits the ray $(k+1) \pmod 2$. A strategy is monotone if the values in the sequence $\{a_i\}_{i=0}^n$ are non-decreasing: $a_i \leq a_{i+1}$ for all $0 \leq i \leq n-1$.

As we explained in Section 1, we are looking for a sequence of numbers $\{a_i\}_{i=0}^n$ that defines an optimal strategy (recall that we have $a_n = \rho$, where $\rho = \Lambda/\lambda$)

$$f_{opt}(i) = \begin{cases} a_i\lambda & \text{if } 0 \leq i < n, \\ \rho\lambda & \text{if } i \geq n. \end{cases}$$

Let

$$\phi(f, D) = 2 \sum_{i=0}^{f^{-1}(D)} f(i) + D$$

denote the cost incurred by a strategy f to find a target at distance D in the worst case and $f^{-1}(D)$ be the smallest integer j such that $f(j) \geq D$. Our goal is to identify a sequence of numbers $\{a_i\}_{i=0}^n$ that defines a periodic and monotone strategy f_{opt} which minimizes

$$CR(f_{opt}) = \sup_{D\in[\lambda,\Lambda]} \frac{\phi(f_{opt}, D)}{D}.$$

At first, for each $n \geq 0$, we find the optimal strategy f_n that takes $n+2$ iterations in the worst case for a given ρ (refer to Theorem 1). Then, we explain how to compute the optimal number of iterations $n_{opt}+2$ for a given ρ (refer to Theorem 2), from which $f_{opt} = f_{n_{opt}}$. We first focus on the cases $n = 0$, $n = 1$ and $n = 2$. Then we characterize the optimal strategy f_n for a general n.

If $n = 0$, then we must have $f_0(i) = \Lambda$. The competitive ratio of f_0 is then

$$CR(f_0) = \sup_{D\in[\lambda,\Lambda]} \frac{\phi(f_0, D)}{D} = \sup_{D\in[\lambda,\Lambda]} \frac{2\Lambda + D}{D} = 2\rho + 1.$$

Observe that f_0 is optimal when $\rho = 1$, i.e., when D is known. When $n \geq 1$, finding a sequence of numbers $\{a_i\}_{i=0}^n$ corresponds to partitioning the interval $[\lambda, \Lambda]$ into $n+1$ subintervals. At each iteration, the corresponding subinterval represents a set of candidate locations for the target.

If $n = 1$, this corresponds to cutting $[\lambda, \Lambda]$ once at a point $\lambda \leq a_0\lambda \leq \rho\lambda = \Lambda$. Namely, we search a sequence of two intervals, $[\lambda, a_0\lambda]$ and $(a_0\lambda, \rho\lambda] = (a_0\lambda, \Lambda]$, from which we define

$$f_1(i) = \begin{cases} a_0\lambda & \text{if } 0 \leq i < 1, \\ \rho\lambda & \text{if } i \geq 1. \end{cases}$$

Therefore, a_0 needs to be chosen such that $CR(f_1)$ is minimized. We have

$$\sup_{D\in[\lambda,a_0\lambda]} \frac{\phi(f_1, D)}{D} = \sup_{D\in[\lambda,a_0\lambda]} \frac{2a_0\lambda + D}{D} = 2a_0 + 1,$$
$$\sup_{D\in(a_0\lambda,\rho\lambda]} \frac{\phi(f_1, D)}{D} = \sup_{D\in[a_0\lambda,\rho\lambda]} \frac{2a_0\lambda + 2\rho\lambda + D}{D} = 3 + 2\frac{\rho}{a_0}.$$

Hence, to minimize $CR(f_1)$, we must select a_0, where $1 \leq a_0 \leq \rho$, such that

$$2a_0 + 1 = 3 + 2\frac{\rho}{a_0}.$$

Therefore, $a_0 = (1 + \sqrt{1+4\rho})/2$ and $CR(f_1) = 2 + \sqrt{1+4\rho}$, which implies that $CR(f_0) \leq CR(f_1)$ if and only if $1 \leq \rho \leq 2$.

If $n = 2$, this corresponds to cutting $[\lambda, \Lambda]$ twice at points $\lambda \leq a_0\lambda \leq a_1\lambda \leq \rho\lambda = \Lambda$. Namely, we are searching for a sequence of three intervals, $[\lambda, a_0\lambda]$, $(a_0\lambda, a_1\lambda]$ and $(a_1\lambda, \Lambda]$, from which we define

$$f_2(i) = \begin{cases} a_i\lambda & \text{if } 0 \leq i < 2, \\ \rho\lambda & \text{if } i \geq 2. \end{cases}$$

Therefore, a_0 and a_1 need to be chosen such that $CR(f_2)$ is minimized. We have

$$\begin{aligned}
\sup_{D\in[\lambda,a_0\lambda]} \frac{\phi(f_2,D)}{D} &= \sup_{D\in[\lambda,a_0\lambda]} \frac{2a_0\lambda + D}{D} = 2a_0 + 1, \\
\sup_{D\in[a_0\lambda,a_1\lambda]} \frac{\phi(f_2,D)}{D} &= \sup_{D\in[a_0\lambda,a_1\lambda]} \frac{2a_0\lambda + 2a_1\lambda + D}{D} = 3 + 2\frac{a_1}{a_0}, \\
\sup_{D\in(a_1\lambda,\rho\lambda]} \frac{\phi(f_2,D)}{D} &= \sup_{D\in[a_1\lambda,\rho\lambda]} \frac{2a_0\lambda + 2a_1\lambda + 2\rho\lambda + D}{D} = 3 + 2\frac{a_0}{a_1} + 2\frac{\rho}{a_1}.
\end{aligned}$$

Hence, to minimize $CR(f_2)$, we must select a_0 and a_1, where $1 \leq a_0 \leq a_1 \leq \rho$, such that

$$\begin{aligned}
2a_0 + 1 &= 3 + 2\frac{a_1}{a_0}, \\
2a_0 + 1 &= 3 + 2\frac{a_0}{a_1} + 2\frac{\rho}{a_1}.
\end{aligned}$$

Therefore, $a_0^3 - 2a_0^2 = \rho$, from which

$$\begin{aligned}
a_0 &= \frac{1}{3}\left(2 + \frac{4}{\left(8 + \frac{27\rho}{2} + \frac{3}{2}\sqrt{3}\sqrt{\rho(32+27\rho)}\right)^{1/3}} + \left(8 + \frac{27\rho}{2} + \frac{3}{2}\sqrt{3}\sqrt{\rho(32+27\rho)}\right)^{1/3}\right), \\
a_1 &= a_0(a_0 - 1)
\end{aligned}$$

and $CR(f_2) = 2a_0 + 1$. We have that $CR(f_1) \leq CR(f_2)$ if and only if $1 \leq \rho \leq 2 + \sqrt{5}$.

In general, we can partition the interval $[\lambda, \rho\lambda]$ into $n+1$ subintervals whose endpoints correspond to the sequence $\lambda, a_0\lambda, \ldots, a_{n-1}\lambda, \rho\lambda$, from which we define

$$f_n(i) = \begin{cases} a_i\lambda & \text{if } 0 \leq i < n, \\ \rho\lambda & \text{if } i \geq n. \end{cases}$$

Therefore, we must select $a_0, \ldots, a_{n-1}$, where $1 \leq a_0 \leq a_1 \leq \ldots \leq a_{n-1} \leq \rho$, such that $CR(f_n)$ is minimized.

We have

$$\sup_{D\in[\lambda,a_0\lambda]} \frac{\phi(f_n,D)}{D} = 2a_0+1,$$
$$\sup_{D\in(a_i\lambda,a_{i+1}\lambda]} \frac{\phi(f_n,D)}{D} = 1+2\sum_{k=0}^{i+1}\frac{a_k}{a_i} \quad (0\le i\le n-2),$$
$$\sup_{D\in(a_{n-1}\lambda,\rho\lambda]} \frac{\phi(f_n,D)}{D} = 1+2\sum_{k=0}^{n-1}\frac{a_k}{a_{n-1}}+2\frac{\rho}{a_{n-1}}.$$

Hence, the values a_i are solutions to the following system of equations:

$$1+2\sum_{k=0}^{i+1}\frac{a_k}{a_i} = 2a_0+1 \quad (0\le i\le n-2), \tag{3}$$
$$1+2\sum_{k=0}^{n-1}\frac{a_k}{a_{n-1}}+2\frac{\rho}{a_{n-1}} = 2a_0+1. \tag{4}$$

In Theorem 1, we explain how to calculate the values a_i. We prove that the solution to this system of equations can be obtained using the following family of polynomials:

$$p_0(x) = x,$$
$$p_1(x) = x(x-1),$$
$$p_i(x) = x\left(p_{i-1}(x)-p_{i-2}(x)\right) \qquad (i\ge 2). \tag{5}$$

We apply (5) without explicitly referring to it when we manipulate the polynomials p_i. Let α_i denote the largest real root of p_i for each i.

Theorem 1. *The following statement is true for all $n\in\mathbb{N}$.*

(i) *For all $0\le i<n$, the values a_i that define f_n satisfy $a_i = p_i(a_0)$.*
(ii) *The number a_0 is the unique solution to the equation $p_n(x)=\rho$ where $a_0>\alpha_n$.*
(iii) *$a_n = p_n(a_0) = \rho$.*
(iv) *$CR(f_n) = 2a_0+1$.*

To prove Theorem 1, we need the following proposition and lemma (see Proposition 1 and Lemma 1). They yield technical properties of the p_i's. For instance, the polynomial p_{n+1} is defined in terms of p_n and p_{n-1} (refer to (5)). Equation (6) in Proposition 1 provides a way of expressing p_{n+1} in terms of the p_i's $(0\le i\le n)$, which is useful for proving Theorem 1(i). This is also used in the proof of $p_n(a_0)=\rho$ (refer to Theorem 1(ii)). To prove that $a_0>\alpha_n$, and hence, that a_0 is unique, we must understand how the roots of the p_i's behave. This is the purpose of (7) in Proposition 1, and of Lemma 1. We need (8) in Proposition 1 to prove Lemma 1 and further results.

Proposition 1. *The following equalities are true for all $n \in \mathbb{N}$.*

$$p_{n+1}(x) = xp_n(x) - \sum_{i=0}^{n} p_i(x) \tag{6}$$

$$p_n(x) = x^{\lfloor (n+1)/2 \rfloor} \prod_{k=1}^{\lfloor (n+2)/2 \rfloor} \left(x - 4\cos^2\left(\frac{k\pi}{n+2} \right) \right) \tag{7}$$

$$\alpha_n = 4\cos^2\left(\frac{\pi}{n+2} \right) \tag{8}$$

Proof. We prove (6) by induction on n. If $n = 0$,

$$\begin{aligned} xp_0(x) - \sum_{i=0}^{0} p_i(x) &= xp_0(x) - p_0(x) \\ &= x^2 - x \\ &= p_1(x). \end{aligned}$$

Suppose that the proposition is true for $n = \ell - 1$, we now prove it for $n = \ell$.

$$\begin{aligned} & xp_\ell(x) - \sum_{i=0}^{\ell} p_i(x) \\ &= xp_\ell(x) - p_\ell(x) - \sum_{i=0}^{\ell-1} p_i(x) \\ &= xp_\ell(x) - p_\ell(x) + (p_\ell(x) - xp_{\ell-1}(x)) && \text{by the induction hypothesis,} \\ &= x(p_\ell(x) - p_{\ell-1}(x)) \\ &= p_{\ell+1}(x) \end{aligned}$$

Equation (7) is a consequence of Corollary 10 in [13]. Hoggatt and Long [13] studied a family of polynomials $u_n(x, y)$ called *generalized Fibonacci polynomials*. They are defined recursively in the following way:

$$\begin{aligned} u_0(x, y) &= 0, \\ u_1(x, y) &= 1, \\ u_{n+2}(x, y) &= x\, u_{n+1}(x, y) + y\, u_n(x, y) && (n \geq 0). \end{aligned}$$

From Corollary 10 in [13] and (5), we get

$$
\begin{aligned}
p_n(x) &= u_{n+2}(x,-x) \\
&= \begin{cases} x \prod_{k=1}^{\frac{n}{2}} \left(x^2 - 4x\cos^2\left(\frac{k\pi}{n+2}\right)\right) & (n \text{ even}) \\ \prod_{k=1}^{\frac{n+1}{2}} \left(x^2 - 4x\cos^2\left(\frac{k\pi}{n+2}\right)\right) & (n \text{ odd}) \end{cases} \\
&= \begin{cases} x^{\frac{n+2}{2}} \prod_{k=1}^{\frac{n}{2}} \left(x - 4\cos^2\left(\frac{k\pi}{n+2}\right)\right) & (n \text{ even}) \\ x^{\frac{n+1}{2}} \prod_{k=1}^{\frac{n+1}{2}} \left(x - 4\cos^2\left(\frac{k\pi}{n+2}\right)\right) & (n \text{ odd}) \end{cases} \\
&= \begin{cases} x^{\frac{n}{2}} \prod_{k=1}^{\frac{n+2}{2}} \left(x - 4\cos^2\left(\frac{k\pi}{n+2}\right)\right) & (n \text{ even}) \\ x^{\frac{n+1}{2}} \prod_{k=1}^{\frac{n+1}{2}} \left(x - 4\cos^2\left(\frac{k\pi}{n+2}\right)\right) & (n \text{ odd}) \end{cases} \\
&= x^{\lfloor (n+1)/2 \rfloor} \prod_{k=1}^{\lfloor (n+2)/2 \rfloor} \left(x - 4\cos^2\left(\frac{k\pi}{n+2}\right)\right).
\end{aligned}
$$

Since by definition, α_n is the largest real root of p_n, (8) is a direct consequence of (7). □

Lemma 1 is used to prove Theorem 1(ii). Intuitively, it says that if we chose an a_0 satisfying $p_n(a_0) = \rho$ but such that $a_0 < \alpha_n$, there would exist an $0 \le i < n$ such that $a_i < 0$. Therefore, we must reject such an a_0.

Lemma 1. *The following statement is true for all $n \in \mathbb{N}$. For all $t \in \mathbb{R}$ such that $0 < t < \alpha_n$ and $p_n(t) > 0$, there exists an $i \in \mathbb{N}$ such that $0 \le i < n$ and $p_i(t) < 0$.*

Proof. We consider six cases: (1) $n = 0$, (2) $n = 1$, (3) $n = 2$, (4) $n = 3$, (5) $n = 4$ and (6) $n \ge 5$.

1. Since $\alpha_0 = 0$ by (8), then there does not exist a t such that $0 < t < \alpha_0$. Hence, the statement is vacuously true.
2. Since $\alpha_1 = 1$ by (8) and $p_1(x) < 0$ for any $0 < x < 1$, then there does not exist a t such that $0 < t < \alpha_1$ and $p_1(t) > 0$. Hence, the statement is vacuously true.
3. Since $\alpha_2 = 2$ by (8) and $p_2(x) < 0$ for any $0 < x < 2$, then there does not exist a t such that $0 < t < \alpha_2$ and $p_2(t) > 0$. Hence, the statement is vacuously true.
4. By (7), $p_3(x) = x^2\left(x - (3-\sqrt{5})/2\right)\left(x - (3+\sqrt{5})/2\right)$. Hence, t is such that $0 < t < \frac{3-\sqrt{5}}{2} < 1$. Hence, $p_1(t) = t(t-1) < 0$ so that we can take $i = 1$.
5. By (7), $p_4(x) = x^3(x-1)(x-3)$. Hence, t is such that $0 < t < 1$. Hence, $p_1(t) = t(t-1) < 0$ so that we can take $i = 1$.
6. From (7), t is such that
$$4\cos^2\left(\frac{(2\ell+1)\pi}{n+2}\right) < t < 4\cos^2\left(\frac{2\ell\pi}{n+2}\right),$$
for an integer ℓ satisfying $1 \le \ell \le \lfloor n/2 \rfloor / 2$. We prove that the i we need to pick is any integer in the interval
$$I = \left(\frac{n+2-4\ell}{2\ell}, \frac{2n+2-4\ell}{2\ell+1}\right).$$

Notice that, from elementary calculus, we have

$$\min_{\substack{n\geq 5\\ 1\leq \ell\leq \frac{1}{2}\lfloor \frac{n}{2}\rfloor}} \left(\frac{2n+2-4\ell}{2\ell+1}-\frac{n+2-4\ell}{2\ell}\right)=\frac{7}{6}.$$

Hence, if $n\geq 5$ and $1\leq \ell\leq \lfloor n/2\rfloor/2$, then there always exists an integer in I. Also notice that for any integer $i\in I$, then $0\leq i<n$. Hence, for any $i\in I$,

$$\begin{gathered}
\frac{n+2-4\ell}{2\ell}<i<\frac{2n+2-4\ell}{2\ell+1},\\
\frac{n-4\ell}{2\ell+1}<\frac{n+2-4\ell}{2\ell}<i<\frac{2n+2-4\ell}{2\ell+1}<\frac{2n+4-4\ell}{2\ell},\\
\frac{n-4\ell}{2\ell+1}<i<\frac{2n+2-4\ell}{2\ell+1}\quad\text{and}\quad\frac{n+2-4\ell}{2\ell}<i<\frac{2n+4-4\ell}{2\ell},\\
\frac{\pi}{i+2}<\frac{(2\ell+1)\pi}{n+2}<\frac{2\pi}{i+2}\quad\text{and}\quad\frac{\pi}{i+2}<\frac{2\ell\pi}{n+2}<\frac{2\pi}{i+2},
\end{gathered}$$

from which

$$\frac{\pi}{i+2}<\frac{2\ell\pi}{n+2}<\frac{(2\ell+1)\pi}{n+2}<\frac{2\pi}{i+2},$$

and hence,

$$4\cos^2\left(\frac{2\pi}{i+2}\right)<4\cos^2\left(\frac{(2\ell+1)\pi}{n+2}\right)<t<4\cos^2\left(\frac{2\ell\pi}{n+2}\right)<4\cos^2\left(\frac{\pi}{i+2}\right).$$

Consequently, $p_i(t)<0$ by (7).

□

We now prove Theorem 1.

Theorem 1.

(i) We prove this theorem by induction on i. If $i=0$, then $p_0(a_0)=a_0$.
Suppose the statement is true for any i such that $0<i<\ell<n$, we now prove it for $i=\ell$. From (3), we have

$$1+2\sum_{k=0}^{\ell}\frac{a_k}{a_{\ell-1}}=2a_0+1.$$

Therefore,

$$\begin{aligned}
a_\ell &= a_0a_{\ell-1}-\sum_{k=0}^{\ell-1}a_k\\
&= a_0p_{\ell-1}(a_0)-\sum_{k=0}^{\ell-1}p_k(a_0) &&\text{by the induction hypothesis,}\\
&= p_\ell(a_0) &&\text{by (6).}
\end{aligned}$$

(ii) From (4), we have

$$\sum_{k=0}^{n-1}a_k+\rho=a_0a_{n-1}.$$

Therefore,

$$\begin{aligned}
\rho &= a_0 a_{n-1} - \sum_{k=0}^{n-1} a_k \\
&= a_0 p_{n-1}(a_0) - \sum_{k=0}^{n-1} p_k(a_0) && \text{by Theorem 1(i),} \\
&= p_n(a_0) && \text{by (6).}
\end{aligned}$$

We show $a_0 \geq \alpha_n$ by contradiction. Suppose $a_0 < \alpha_n$. Then, by Lemma 1, there exists an $i \in \mathbb{N}$ such that $0 \leq i < n$ and $p_i(a_0) < 0$. Hence, $a_i = p_i(a_0) < 0$ by Theorem 1(i). This is impossible since all the a_i's are such that $1 \leq a_i \leq \rho$. Therefore, $a_0 \geq \alpha_n$. Moreover, $a_0 \neq \alpha_n$ since $p_n(a_0) = \rho \geq 1$, whereas $p_n(\alpha_n) = 0$ by the definition of α_n. Finally, this solution is unique since α_n is the largest real root of p_n, and the leading coefficient of p_n is positive.

(iii) As we explained in the introduction, $a_n = \rho$. Also, by Theorem 1(ii), $p_n(a_0) = \rho$.

(iv) This follows directly from the discussion preceding Theorem 1.

□

From Theorem 1, the optimal strategy f_n is uniquely defined for each n. However, this still leaves an infinite number of possibilities for the optimal strategy (one for each n). We aim to find, for a given ρ, what value of n leads to the optimal strategy. Theorem 2 gives a criterion for the optimal n in terms of ρ together with a formula that enables us to compute this optimal n in $O(1)$ time.

Theorem 2.

(i) *For a given ρ, if $n \in \mathbb{N}$ is such that*

$$p_n(\alpha_{n+1}) \leq \rho < p_n(\alpha_{n+2}), \tag{9}$$

then f_n is the optimal strategy and $\alpha_{n+1} \leq a_0 < \alpha_{n+2}$.

(ii) *For all $n \in \mathbb{N}$,*

$$2^n \leq p_n(\alpha_{n+1}) \leq \rho < p_n(\alpha_{n+2}) \leq 2^{n+2}. \tag{10}$$

Notice that the criterion in Theorem 2(i) covers all possible values of ρ since $p_0(\alpha_1) = 1$ by (8) and $p_n(\alpha_{n+2}) = p_{n+1}(\alpha_{n+2})$ by Proposition 2(i) (see below).

To prove Theorem 2, we need the following proposition. It provides another set of technical properties satisfied by the p_i's. Proposition 2(i) is used to prove Proposition 2(v) and further results. Propositions 2(ii), 2(iii) and 2(iv) describe how the α_i's compare to each other as well as how the p_i's compare to each other. They are used in the proof of Theorem 2(i). The purpose of Proposition 2(v) is to simplify the proof of Theorem 2(ii).

Proposition 2. *The following properties are true for all $n \in \mathbb{N}$.*

(i) *$p_{n+1}(x) = p_n(x)$ if and only if $p_{n+2}(x) = 0$.*

(ii) *$0 \leq \alpha_n < \alpha_{n+1} < 4$.*

(iii) *If $x \geq \alpha_{n+2}$, then $p_{n+1}(x) \geq p_n(x)$.*

(iv) *For all $x \in \mathbb{R}$, if $\alpha_{n+1} < x < \alpha_{n+2}$, then $p_{n+1}(x) < p_n(x)$.*

(v)

$$p_n(\alpha_{n+1}) = \alpha_{n+1}^{(n+1)/2} \tag{11}$$
$$p_n(\alpha_{n+2}) = \alpha_{n+2}^{(n+2)/2} \tag{12}$$

Proof.

(i)

$$p_{n+1}(x) = p_n(x) \tag{13}$$
$$x(p_{n+1}(x) - p_n(x)) = 0 \tag{14}$$
$$p_{n+2}(x) = 0 \tag{15}$$

(ii) This follows directly from (8).

(iii) Since α_{n+2} is the largest real root of p_{n+2}, p_{n+2} is strictly increasing on $[\alpha_{n+2}, \infty)$. Moreover, since $x \geq \alpha_{n+2}$, we have

$$p_{n+2}(x) \geq 0 \tag{16}$$
$$x(p_{n+1}(x) - p_n(x)) \geq 0 \tag{17}$$
$$p_{n+1}(x) \geq p_n(x) \tag{18}$$

(iv) By (8), if $\alpha_{n+1} < x < \alpha_{n+2}$, then

$$4\cos^2\left(\frac{\pi}{n+3}\right) < x < 4\cos^2\left(\frac{\pi}{n+4}\right).$$

Hence, since $\frac{\pi}{n+3} < \frac{2\pi}{n+4}$, we have

$$4\cos^2\left(\frac{2\pi}{(n+2)+2}\right) < 4\cos^2\left(\frac{\pi}{n+3}\right) < x < 4\cos^2\left(\frac{\pi}{(n+2)+2}\right).$$

Together with (7), this implies that $p_{n+2}(x) < 0$. Thus,

$$\begin{aligned} p_{n+2}(x) &< 0 \\ x(p_{n+1}(x) - p_n(x)) &< 0 \\ p_{n+1}(x) &< p_n(x). \end{aligned}$$

(v) Notice that (11) is a direct consequence of (12) since $p_n(\alpha_{n+2}) = p_{n+1}(\alpha_{n+2})$ by Proposition 2(i). We now show (12) by proving

$$p_n(\alpha_{n+2}) = \alpha_{n+2}^{i+1} \frac{p_{n-i}(\alpha_{n+2})}{p_i(\alpha_{n+2})} \qquad (0 \leq i \leq n) \tag{19}$$

by induction on i. If $i = 0$, then

$$p_n(\alpha_{n+2}) = \alpha_{n+2}^{0+1} \frac{p_{n-0}(\alpha_{n+2})}{p_0(\alpha_{n+2})}.$$

If $i = 1$, then

$$\begin{aligned} p_n(\alpha_{n+2}) &= p_{n+1}(\alpha_{n+2}) && \text{by Proposition 2(i),} \\ &= \alpha_{n+2}(p_n(\alpha_{n+2}) - p_{n-1}(\alpha_{n+2})). \end{aligned}$$

Therefore,

$$\begin{aligned}
p_n(\alpha_{n+2}) &= \alpha_{n+2}\frac{p_{n-1}(\alpha_{n+2})}{\alpha_{n+2}-1} \\
&= \alpha_{n+2}\frac{p_{n-1}(\alpha_{n+2})}{\frac{1}{\alpha_{n+2}}p_1(\alpha_{n+2})} \\
&= \alpha_{n+2}^{1+1}\frac{p_{n-1}(\alpha_{n+2})}{p_1(\alpha_{n+2})}.
\end{aligned}$$

Suppose that (19) is true for $i = \ell - 1$ and $i = \ell < n$, we now prove it for $i = \ell + 1$.

$$\begin{aligned}
p_{n-(\ell-1)}(\alpha_{n+2}) &= \alpha_{n+2}(p_{n-\ell}(\alpha_{n+2}) - p_{n-(\ell+1)}(\alpha_{n+2})) \\
\frac{p_{\ell-1}(\alpha_{n+2})}{\alpha_{n+2}^{(\ell-1)+1}}p_n(\alpha_{n+2}) &= \alpha_{n+2}\left(\frac{p_\ell(\alpha_{n+2})}{\alpha_{n+2}^{\ell+1}}p_n(\alpha_{n+2}) - p_{n-(\ell+1)}(\alpha_{n+2})\right) \\
(p_\ell(\alpha_{n+2}) - p_{\ell-1}(\alpha_{n+2}))p_n(\alpha_{n+2}) &= \alpha_{n+2}^{\ell+1}p_{n-(\ell+1)}(\alpha_{n+2}) \\
\frac{p_{\ell+1}(\alpha_{n+2})}{\alpha_{n+2}}p_n(\alpha_{n+2}) &= \alpha_{n+2}^{\ell+1}p_{n-(\ell+1)}(\alpha_{n+2}) \\
p_n(\alpha_{n+2}) &= \alpha_{n+2}^{(\ell+1)+1}\frac{p_{n-(\ell+1)}(\alpha_{n+2})}{p_{\ell+1}(\alpha_{n+2})},
\end{aligned}$$

where the second equality comes from the induction hypothesis. That completes the proof of 19. Now, by taking $i = n$ in 19, we find

$$\begin{aligned}
p_n(\alpha_{n+2}) &= \alpha_{n+2}^{n+1}\frac{p_{n-n}(\alpha_{n+2})}{p_n(\alpha_{n+2})}, \\
p_n^2(\alpha_{n+2}) &= \alpha_{n+2}^{n+1}\,p_0(\alpha_{n+2}), \\
p_n^2(\alpha_{n+2}) &= \alpha_{n+2}^{n+2}, \\
p_n(\alpha_{n+2}) &= \alpha_{n+2}^{\frac{n+2}{2}}.
\end{aligned}$$

□

We now prove Theorem 2.

Theorem 2.

(i) Consider the strategy f_n. Since α_n is the largest real root of p_n, p_n is strictly increasing on $[\alpha_n, \infty)$. Moreover, by Proposition 2(ii), $\alpha_n < \alpha_{n+1}$. Therefore, by Theorem 1(ii) and since $p_n(\alpha_{n+1}) \leq \rho < p_n(\alpha_{n+2})$ by the hypothesis, we have

$$\alpha_{n+1} \leq a_0 < \alpha_{n+2} \tag{20}$$

We first prove that $CR(f_n) \leq CR(f_m)$ for all $m < n$ by contradiction. Suppose that there exists an $m < n$ such that $CR(f_m) < CR(f_n)$. By Theorem 1(ii), there exists an a_0' such that $a_0' > \alpha_m$ and $p_m(a_0') = \rho$. Moreover, since $CR(f_m) < CR(f_n)$ by the hypothesis, then $2a_0' + 1 < 2a_0 + 1$ by Theorem 1(iv). Therefore,

$$\alpha_m < a_0' < a_0. \tag{21}$$

Also, since $m < n$, then $m + 2 \leq n + 1$. Thus, by (20) and since the α_n's are increasing with respect to n (see Proposition 2(ii)), $\alpha_i \leq \alpha_{n+1} \leq a_0$ for all $m + 2 \leq i \leq n + 1$. Hence, by repeated applications of Proposition 2(iii), we find

$$p_m(a_0) \leq p_{m+1}(a_0) \leq p_{m+2}(a_0) \leq \ldots \leq p_{n-1}(a_0) \leq p_n(a_0). \tag{22}$$

But then,

$$\begin{aligned}
\rho &= p_m(a_0') \\
&< p_m(a_0) && \text{by (21) and since } p_m \text{ is increasing on } [\alpha_m, \infty), \\
&\leq p_n(a_0) && \text{by (22),} \\
&= \rho,
\end{aligned}$$

which is a contradiction. Consequently, $CR(f_n) < CR(f_m)$ for all $m < n$.
We now prove that $CR(f_n) \leq CR(f_{n'})$ for all $n' > n$ by contradiction. Suppose that there exists an $n' > n$ such that $CR(f_{n'}) < CR(f_n)$. By Theorem 1(ii), there exists an a_0' such that $a_0' > \alpha_{n'}$ and $p_{n'}(a_0') = \rho$. Moreover, since $CR(f_{n'}) < CR(f_n)$ by the hypothesis, then $2a_0' + 1 < 2a_0 + 1$ by Theorem 1(iv). Therefore, by (20) and since the α_n's are strictly increasing with respect to n (see Proposition 2(ii)),

$$\alpha_n < \alpha_{n'} < a_0' < a_0 < \alpha_{n+2}. \tag{23}$$

Moreover, since $n' > n$, (23) implies $n' = n + 1$. But then,

$$\begin{aligned}
\rho &= p_{n'}(a_0') \\
&= p_{n+1}(a_0') \\
&< p_n(a_0') && \text{by (23) and Proposition 2(iv),} \\
&< p_n(a_0) && \text{by (23) and since } p_n \text{ is increasing on } [\alpha_n, \infty), \\
&= \rho,
\end{aligned}$$

which is a contradiction. Consequently, $CR(f_n) < CR(f_{n'})$ for all $n' > n$.

(ii) We first prove that

$$2\cos^{n+1}\left(\frac{\pi}{n+3}\right) \geq 1 \tag{24}$$

for all $n \geq 0$. We can easily verify (24) for $n = 0$ and $n = 1$. We provide a general proof for $n \geq 2$. By elementary calculus, we have $\cos(x) \geq 1 - x^2/2$ for all $0 \leq x \leq \pi/4$. We have $0 \leq \pi/(n+3) \leq \pi/4$ because $n \geq 2$. Hence, if we can prove

$$g(n) = 2\left(1 - \frac{1}{2}\left(\frac{\pi}{n+3}\right)^2\right)^{n+1} \geq 1 \tag{25}$$

for $n \geq 2$, then we are done. We have $g'(n) = g(n)h(n)$, where

$$h(n) = \log\left(1 - \frac{1}{2}\left(\frac{\pi}{n+3}\right)^2\right) + \frac{(n+1)\pi^2}{(n+3)^3\left(1 - \frac{1}{2}\left(\frac{\pi}{n+3}\right)^2\right)}.$$

The function g is positive for $n \geq 2$. We can prove by elementary calculus that h also is positive for $n \geq 2$. Therefore, we conclude that g' is strictly positive, and hence, that g is strictly increasing for $n \geq 2$. Thus, since $g(2) > 1$, then (25) is true for $n \geq 2$ and the proof of (24) is complete.

We now prove (10).

$$\begin{aligned}
2^n &\le 2^{n+1}\cos^{n+1}(\pi/(n+3)) && \text{by (24),}\\
&= \alpha_{n+1}^{(n+1)/2} && \text{by (7),}\\
&= p_n(\alpha_{n+1}) && \text{by (11),}\\
&\le \rho && \text{by (9),}\\
&< p_n(\alpha_{n+2}) && \text{by (9),}\\
&= \alpha_{n+2}^{(n+2)/2} && \text{by (12),}\\
&= 2^{n+2}\cos^{n+2}(\pi/(n+4)) && \text{by (7),}\\
&\le 2^{n+2} && \text{since } 0 < \cos(\pi/(n+4)) < 1.
\end{aligned}$$

□

From (9), there is only one possible optimal value for n. From Theorem 1, once we are given a ρ and an n, there is only one possible optimal strategy. Therefore, we have the following corollary.

Corollary 1. *For a given ρ, there exists a unique optimal strategy that is monotonic and periodic.*

By (10), it suffices to examine two values to find the optimal n, namely $\lfloor \log_2 \rho \rfloor - 1$ and $\lfloor \log_2 \rho \rfloor$. To compute the optimal n, let $n = \lfloor \log_2 \rho \rfloor$ and let $\gamma = 2\cos\left(\frac{\pi}{n+3}\right)$. If $n+1 \le \log_\gamma \rho$, then n is optimal. Otherwise, take $n = \lfloor \log_2 \rho \rfloor - 1$. By Theorem 2, this gives us the optimal n in $O(1)$ time.

Now that we know the optimal n, we need to compute a_i for each $0 \le i < n$. Suppose that we know a_0. By (5), and Theorems 1(i) and 1(iii), $a_1 = p_1(a_0) = a_0(a_0 - 1)$ and $a_i = a_0(p_{i-1}(a_0) - p_{i-2}(a_0)) = a_0(a_{i-1} - a_{i-2})$ for $2 \le i \le n$. Therefore, given a_0, each a_i can be computed in $O(1)$ time for $1 \le i \le n$. It remains to show how to compute a_0 efficiently. Since f_n is defined by n values, $\Omega(n) = \Omega(\log \rho)$ time is necessary to compute f_n. Hence, if we can compute a_0 in $O(1)$ time, then our algorithm is optimal.

By Theorem 1(ii), for a given n, we need to solve a polynomial equation of degree $n+1$ to find the value of a_0. By Galois theory, this cannot be done by radicals if $n+1 > 4$. Moreover, the degree of the p_n's is unbounded, so a_0 cannot be computed exactly in general. Theorem 3 explains how and why numerical methods can be used efficiently to address this issue.

Theorem 3. *Take ρ and n such that f_n is optimal for ρ.*

(i) *Let $a_0^* \in \mathbb{R}$ be such that $\alpha_{n+1} \le a_0 < a_0^* \le \alpha_{n+2}$ and define f_n^* by*

$$f_n^*(i) = \begin{cases} p_i(a_0^*)\lambda & \text{if } 0 \le i < n,\\ \rho\lambda & \text{if } i \ge n. \end{cases}$$

Then $|CR(f_n) - CR(f_n^*)| \le 7^3\,(n+4)^{-3}$.

(ii) *The polynomial p_n is strictly increasing on $[\alpha_{n+1}, \alpha_{n+2})$ and $|\alpha_{n+2} - \alpha_{n+1}| \le 7^3\,(n+4)^{-3}/2$.*

Proof.

(i) Let $a_i^* = p_i(a_0^*)$ for $0 \le i < n$. We first prove that $CR(f_n^*) = 2a_0^* + 1$. By Theorems 1 and 2(i), there is a $\rho^* \in \mathbb{R}$ such that $p_n(\alpha_{n+1}) \le \rho < \rho^* \le p_n(\alpha_{n+2})$, $p_n(a_0^*) = \rho^*$ and f_n^* is optimal for ρ^*. By Theorem 1 and the discussion preceding it, we have

$$\begin{aligned}
\sup_{D \in [\lambda, a_0^* \lambda]} \frac{1}{D}\phi(f_n^*, D) &= 2a_0^* + 1, \\
\sup_{D \in (a_i^* \lambda, a_{i+1}^* \lambda]} \frac{1}{D}\phi(f_n^*, D) &= 1 + 2\sum_{k=0}^{i+1} \frac{a_k^*}{a_i^*} && (0 \le i \le n-2) \\
&= 2a_0^* + 1 && (0 \le i \le n-2), \\
\sup_{D \in (a_{n-1}^* \lambda, \rho\lambda]} \frac{1}{D}\phi(f_n^*, D) &= 1 + 2\sum_{k=0}^{n-1} \frac{a_k^*}{a_{n-1}^*} + 2\frac{\rho}{a_{n-1}^*} \\
&< 1 + 2\sum_{k=0}^{n-1} \frac{a_k^*}{a_{n-1}^*} + 2\frac{\rho^*}{a_{n-1}^*} \\
&= 2a_0^* + 1.
\end{aligned}$$

This establishes that $CR(f_n^*) = 2a_0^* + 1$. Therefore,

$$\begin{aligned}
&|CR(f_n) - CR(f_n^*)| \\
&= |(2a_0 + 1) - (2a_0^* + 1)| && \text{by Theorem 1(iv) and since } CR(f_n^*) = 2a_0^* + 1, \\
&= 2(a_0^* - a_0) \\
&\le 2(\alpha_{n+2} - \alpha_{n+1}) && \text{by the hypothesis and Theorem 2(i),} \\
&= 8\left(\cos^2\left(\frac{\pi}{n+4}\right) - \cos^2\left(\frac{\pi}{n+3}\right)\right) && \text{by (7).} \\
&\le 7^3\,(n+4)^{-3} && \text{by elementary calculus.}
\end{aligned}$$

(ii) Since α_n is the largest real root of p_n, p_n is strictly increasing on $[\alpha_n, \infty)$. Since the α_n's are strictly increasing with respect to n (see Proposition 2(ii)), p_n is strictly increasing on $[\alpha_{n+1}, \alpha_{n+2}) \subset [\alpha_n, \infty)$. The inequality $|\alpha_{n+2} - \alpha_{n+1}| \le 7^3\,(n+4)^{-3}/2$ follows directly from the proof of Theorem 3(i)

□

We now explain how to compute a_0. We know what is the optimal n for a given ρ. From (7) and Theorem 2(i), n satisfies $n + 1 \le 4$ if and only if $\rho \le 32\cos^5(\pi/7) \approx 18.99761$. In this case, $p_n(x) = \rho$ is a polynomial equation of degree at most 4. Hence, by Theorem 1(ii) and elementary algebra, a_0 can be computed exactly and in $O(1)$ time. Otherwise, let $\varepsilon > 0$ be a given tolerance. We explain how to find a solution f_{opt}^*, such that $CR(f_{opt}^*) \le CR(f_{opt}) + \varepsilon$.

If $n \ge 7\varepsilon^{-1/3} - 4$, then by Theorem 3, it suffices to take $a_0 = \alpha_{n+2}$ to compute an ε-approximation of the optimal strategy. But $\alpha_{n+2} = 4\cos^2(\pi/(n+4))$ by (7). Hence, a_0 can be computed in $O(1)$ time and thus, an ε-approximation of the optimal strategy can be computed in $\Theta(n) = \Theta(\log\rho)$ time. Otherwise, if $4 \le n < 7\varepsilon^{-1/3} - 4$, then we have to use numerical methods to find the value of a_0. By Theorem 2(i), we need to solve $p_n(x) = \rho$ for $x \in [\alpha_{n+1}, \alpha_{n+2})$. However, by Theorem 3(ii), $|\alpha_{n+2} - \alpha_{n+1}| < 7^3(n+4)^{-3}/2$ and p_n is strictly increasing on this interval. Hence, usual root-finding algorithms behave well on this problem.

Hence, if $n < 4$ or $n \geq 7\varepsilon^{-1/3} - 4$, then our algorithm is optimal. When $4 \leq n < 7\varepsilon^{-1/3} - 4$, then our algorithm's computation time is as fast as the fastest root-finding algorithm (refer to [17, 18]).

What remains to be proven are bounds on $CR(f_n)$ for an optimal n; we present exact bounds in Theorem 4.

Theorem 4.

(i) *The strategy f_0 is optimal for a given ρ if and only if $1 \leq \rho < 2$. In this case, $CR(f_0) = 2\rho + 1$. Otherwise, if f_n is optimal for a given ρ ($n \geq 1$), then*

$$8\cos^2\left(\frac{\pi}{\lceil \log_2 \rho \rceil + 2}\right) + 1 \leq CR(f_n) \leq 8\cos^2\left(\frac{\pi}{\lfloor \log_2 \rho \rfloor + 4}\right) + 1. \tag{26}$$

(ii) *For a fixed λ, when $\Lambda \to \infty$, $f_{opt} \to f_\infty$, where $f_\infty(i) = (2i+4)2^i\lambda$ ($i \geq 0$) and $CR(f_\infty) = 9$.*

Proof.

(i) The first statement is a direct consequence of Theorem 2(i), (8) and Theorem 1(iv). Otherwise, if f_n is optimal ($n \geq 1$), then $2^n \leq \rho < 2^{n+2}$ by (10). Therefore,

$$\log_2(\rho) - 2 < n \leq \log_2(\rho),$$
$$\lceil \log_2(\rho) \rceil - 2 \leq n \leq \lfloor \log_2(\rho) \rfloor,$$

since n is an integer. Moreover, $CR(f_n) = 2a_0 + 1$ by Theorem 1(iv). Hence, by Theorem 2(i), (8) and the previous derivation, we get

$$2\alpha_{n+1} + 1 \leq CR(f_n) \leq 2\alpha_{n+2} + 1,$$
$$8\cos^2\left(\frac{\pi}{\lceil \log_2 \rho \rceil + 1}\right) + 1 \leq CR(f_{opt}) < 8\cos^2\left(\frac{\pi}{\lfloor \log_2 \rho \rfloor + 4}\right) + 1.$$

(ii) Let f_n be the optimal strategy for ρ and suppose that λ is fixed. When $\Lambda \to \infty$, then $\rho \to \infty$ and then, $n \to \infty$ by (10). Hence, by Theorem 2(i) and (7),

$$4 = \lim_{n\to\infty} \alpha_{n+1} \leq \lim_{n\to\infty} a_0 \leq \lim_{n\to\infty} \alpha_{n+2} = 4.$$

Let us now prove that

$$p_i(4) = (2i+4)2^i$$

for all $i \geq 0$. We proceed by induction on i. For the base case, notice that $p_0(4) = 4 = (2(0)+4)2^0$ and $p_1(4) = 4(4-1) = (2(1)+4)2^1$. Suppose that

$$p_{i-1}(4) = (2(i-1)+4)2^{i-1},$$
$$p_i(4) = (2i+4)2^i,$$

(for an $i \geq 1$) and let us prove that $p_{i+1}(4) = (2(i+1)+4)2^{i+1}$.

$$\begin{aligned} p_{i+1}(4) &= 4(p_i(4) - p_{i-1}(4)) \\ &= 4\left((2i+4)2^i - (2(i-1)+4)2^{i-1}\right) && \text{by the induction hypothesis,} \\ &= (2(i+1)+4)2^{i+1} \quad . \end{aligned}$$

Thus, when $\Lambda \to \infty$, $a_i = p_i(a_0) = p_i(4) = (2i+4)2^i$ by Theorem 1(i) and since p_i is continuous. Hence, $f_n \to f_\infty$.

□

$CR(f_{opt})$	3	4	5	6	7	8	9
ρ	1	1.5	2	3.75	9	69.671875	∞
Optimal n	0	0	1	1	3	5	∞

Table 1: Exact values of ρ together with optimal n such that $CR(f_{opt}) = 3$, 4, 5, 6, 7 and 8. We also include the limit case where $\rho \to \infty$.

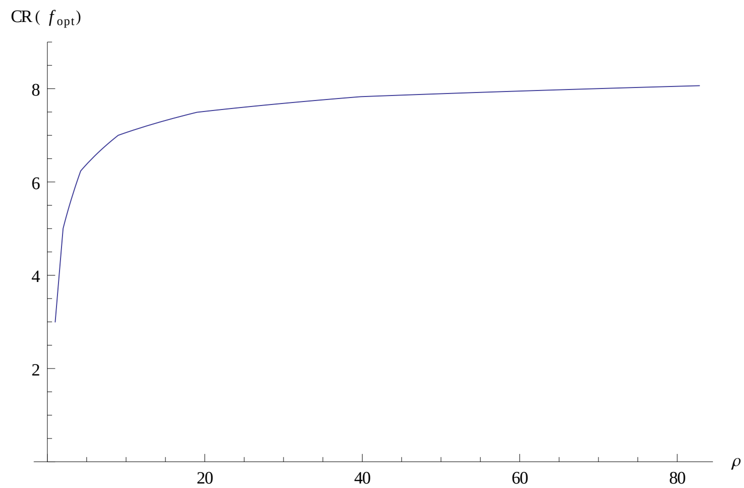

Figure 1: Graph of $CR(f_{opt})$ in terms of ρ, for $1 = p_0(\alpha_1) \le \rho \le p_5(\alpha_7) \approx 82.81532$.

The competitive cost of the optimal strategy is $2a_0 + 1$ by Theorem 1(iv). Theorem 4(i) gives nearly tight bounds on $2a_0 + 1$. Notice that when $\rho = 1$, i.e., when D is known, then $2a_0 + 1 = 3$ which corresponds to the optimal strategy in this case. From the Taylor series expansion of $\cos^2(\cdot)$ and Theorem 4(i), we have $CR(f_n) = 9 - O(1/\log^2 \rho)$ for an optimal n. This is consistent with López-Ortiz and Schuierer' result (see [15]), although our result (26) is exact. Refer to Table 1 and Figure 1 to see how fast $CR(f_{opt})$ increases in terms of ρ.

Letting $\rho \to \infty$ corresponds to not knowing any upper bound on D. Thus, Theorem 4(ii) provides an alternate proof to the competitive ratio of 9 shown by Baeza-Yates et al. [4]. From Theorems 2 and 4, the optimal solution for a given ρ is unique. This optimal solution tends towards f_∞, suggesting that f_∞ is the canonical optimal strategy when no upper bound is given (rather than the power of two strategy).

In this section, we proved the following theorem.

Theorem 5. *Let* λ, Λ *and* $\varepsilon > 0$ *be given, where* $0 < \lambda \le D \le \Lambda$.

- *If* $\rho = \Lambda/\lambda \le 32\cos^5(\pi/7) \approx 18.99761$, *Algorithm 1 computes, in* $O(1)$ *time, the exact optimal strategy* f_{opt}.
- *Otherwise, Algorithm 1 computes a strategy* f^*_{opt} *such that* $CR(f^*_{opt}) \le CR(f_{opt}) + \varepsilon$.
  - *If* $n \ge 7\varepsilon^{-1/3} - 4$, *Algorithm 1 computes* f^*_{opt} *in* $O(n) = O(\log \rho)$ *time.*
  - *Otherwise, the time needed for Algorithm 1 to compute* f^*_{opt} *is equal to the time needed for the fastest root-finding algorithm to solve* $p_n(x) = \rho$ *on* $[\alpha_{n+1}, \alpha_{n+2})$.

3. Maximal Reach Problem

In this section, we explain how to solve the maximal reach problem using the result from Section 2. Given a competitive ratio R and a lower bound λ on D, the *maximal reach problem* is to identify the largest bound Λ such that there exists a search strategy that finds any target within distance $\lambda \leq D \leq \Lambda$ with competitive ratio at most R.

We have $a_0 = (R-1)/2$ by Theorem 1(iv). Therefore, by Theorem 2(i) and (8),

$$\begin{gathered}
\alpha_{n+1} \leq a_0 < \alpha_{n+2}, \\
4\cos^2\left(\frac{\pi}{n+3}\right) \leq a_0 < 4\cos^2\left(\frac{\pi}{n+4}\right), \\
\cos\left(\frac{\pi}{n+3}\right) \leq \frac{\sqrt{a_0}}{2} < \cos\left(\frac{\pi}{n+4}\right), \\
\frac{\pi}{\arccos\left(\frac{\sqrt{a_0}}{2}\right)} - 4 < n \leq \frac{\pi}{\arccos\left(\frac{\sqrt{a_0}}{2}\right)} - 3,
\end{gathered}$$

from which we find

$$n = \left\lfloor \frac{\pi}{\arccos\left(\frac{\sqrt{a_0}}{2}\right)} \right\rfloor - 3 = \left\lfloor \frac{\pi}{\arccos\left(\frac{\sqrt{R-1}}{2\sqrt{2}}\right)} \right\rfloor - 3$$

since n is an integer.

Consequently,

$$\Lambda = p_n((R-1)/2)\lambda,$$

which can be computed in $O(n)$ by the definition of the p_n's.

4. Searching on m Bounded Concurrent Rays

For $m \geq 2$, if we know D, then the optimal strategy has a competitive cost of $1 + 2(m-1)$. Indeed, in the worst case, we have to walk $2D$ on the first $m-1$ rays and then D on the m-th ray. When no upper bound is known, Baeza-Yates et al. [4] proved that the optimal strategy has a competitive cost of

$$1 + 2\frac{m^m}{(m-1)^{m-1}}.$$

There exist infinitely many strategies that achieve this optimal cost.

Proposition 3. *All the strategies in the following family are optimal:*

$$f_{a,b}(i) = (ai+b)\left(\frac{m}{m-1}\right)^i \lambda,$$

where $a \geq 0$ *and*

$$\max\{1, ma\} \leq b \leq \frac{\left(\frac{m^m}{(m-1)^{m-1}} - m^2\right)a + \frac{m}{m-1}\,\frac{m^m}{(m-1)^{m-1}}}{\frac{m^m}{(m-1)^{m-1}} - m}. \tag{27}$$

Notice that for $m = 2$, when a and b are respectively equal to their smallest allowed value, then $f_{a,b}$ corresponds to the power of two strategy of Baeza-Yates et al. (refer to [4]). Moreover, when a is equal to its largest allowed value, i.e. $a = 2$, then $b = 4$ and $f_{a,b} = f_\infty$ (refer to Theorem 4(ii)).

Proof. Without loss of generality, let $\lambda = 1$. We first explain why we need $a \geq 0$ and (27).

For any $a \geq 0$ and any $b \geq 0$, the function $f_{a,b}$ is strictly increasing on $[0, \infty)$. However, to have $f_{a,b}(i) \geq 1$ for all $i \geq 0$, we also need $b \geq 1$. Indeed, $f_{a,b}(0) = b$ and $f_{a,b}(i) \geq 1$ for all $i \geq 1$ implies

$$a \geq \frac{1}{i\left(\frac{m}{m-1}\right)^i} - \frac{b}{i}$$

for all $i \geq 1$. Therefore, we must have $a \geq 0$, which we already knew.

We must keep control on the competitive cost of $f_{a,b}$. The following two inequalities must be satisfied for all D:

$$1 + 2(m-1) \leq \frac{\phi(f_{a,b}, D)}{D} \leq 1 + 2\frac{m^m}{(m-1)^{m-1}}. \tag{28}$$

The first inequality ensures that $f_{a,b}$ does not do better than the optimal strategy for the case where we know D (refer to the discussion at the beginning of this section). The second inequality ensures that $f_{a,b}$ is optimal for all D. We consider the case where $1 = \lambda \leq D < f_{a,b}(0)$ separately. In this case, we have

$$\begin{aligned}
\frac{\phi(f_{a,b}, D)}{D} &= \frac{2\sum_{i=0}^{m-2} f_{a,b}(i) + D}{D} \\
&= \frac{1}{mD}\left(2(m-1)\left((b-a)\frac{m^m}{(m-1)^{m-1}} - m(b-am)\right) + D\right).
\end{aligned}$$

Therefore, from (28), we must have

$$1 + 2(m-1) \leq \lim_{D \to f_{a,b}(0)} \frac{\phi(f_{a,b}, D)}{D} \leq \frac{\phi(f_{a,b}, D)}{D} \leq \lim_{D \to 1} \frac{\phi(f_{a,b}, D)}{D} \leq 1 + 2\frac{m^m}{(m-1)^{m-1}},$$

from which we get

$$\frac{2(m-1)}{mb}\left((b-a)\frac{m^m}{(m-1)^{m-1}} - m(b-am)\right) + 1 \geq 1 + 2(m-1), \tag{29}$$

$$\frac{2(m-1)}{m}\left((b-a)\frac{m^m}{(m-1)^{m-1}} - m(b-am)\right) + 1 \leq 1 + 2\frac{m^m}{(m-1)^{m-1}}. \tag{30}$$

Equation (29) leads to

$$b \geq \frac{\frac{m^m}{(m-1)^{m-1}} - m^2}{\frac{m^m}{(m-1)^{m-1}} - 2m}\, a.$$

Since

$$\frac{\frac{m^m}{(m-1)^{m-1}} - m^2}{\frac{m^m}{(m-1)^{m-1}} - 2m} \leq 0$$

and $b \geq 1$, that condition is already satisfied. Equation (30) leads to

$$b \leq \frac{\left(\frac{m^m}{(m-1)^{m-1}} - m^2\right)a + \frac{m}{m-1}\,\frac{m^m}{(m-1)^{m-1}}}{\frac{m^m}{(m-1)^{m-1}} - m}.$$

We now consider the general case where $f_{a,b}(j) \le D < f_{a,b}(j+1)$ for a $j \ge 1$. After simplification, we get

$$\begin{aligned}
\frac{\phi(f_{a,b},D)}{D} &= \frac{2\sum_{i=0}^{(j+1)+(m-2)} f_{a,b}(i) + D}{D} \\
&= \frac{2(m-1)\left((aj+b)\left(\frac{m}{m-1}\right)^{m+j} - (b-am)\right) + D}{D}.
\end{aligned}$$

Therefore, from (28), we must have

$$1 + 2(m-1) \le \lim_{D\to f_{a,b}(j+1)} \frac{\phi(f_{a,b},D)}{D} \le \frac{\phi(f_{a,b},D)}{D} \le \lim_{D\to f_{a,b}(j)} \frac{\phi(f_{a,b},D)}{D} \le 1 + 2\frac{m^m}{(m-1)^{m-1}},$$

from which we get

$$2\frac{(\frac{m}{m-1})^j\left((aj+b)\frac{m^m}{(m-1)^{m-1}} - m(a(j+1)+b)\right) - (m-1)(b-am)}{(a(j+1)+b)\left(\frac{m}{m-1}\right)^{j+1}} \ge 1 + 2(m-1), \tag{31}$$

$$1 + 2\frac{m^m}{(m-1)^{m-1}} - \frac{2(m-1)}{aj+b}\left(\frac{m}{m-1}\right)^{-j}(b-am) \le 1 + 2\frac{m^m}{(m-1)^{m-1}}. \tag{32}$$

Equation (31) leads to

$$b \ge -\frac{\left(\frac{m}{m-1}\right)^j\left(j\frac{m^m}{(m-1)^{m-1}} - (j+1)m\right) + m(m-1)}{\left(\frac{m}{m-1}\right)^j\left(\frac{m^m}{(m-1)^{m-1}} - m\right) - (m-1)}\, a.$$

Since

$$-\frac{\left(\frac{m}{m-1}\right)^j\left(j\frac{m^m}{(m-1)^{m-1}} - (j+1)m\right) + m(m-1)}{\left(\frac{m}{m-1}\right)^j\left(\frac{m^m}{(m-1)^{m-1}} - m\right) - (m-1)} \le 0$$

and $b \ge 1$, that condition is already satisfied. Equation (32) leads to $b \ge ma$.

Finally,

$$\begin{aligned}
CR(f_{a,b}) &= \sup_{D\ge\lambda} \frac{\phi(f_{a,b},D)}{D} \\
&= \lim_{j\to\infty} \lim_{D\to f_{a,b}(j)} \frac{2(m-1)\left((aj+b)\left(\frac{m}{m-1}\right)^{m+j} - (b-am)\right) + D}{D} \\
&= 9.
\end{aligned}$$

□

When we are given an upper bound $\Lambda \ge D$, the solution presented in Section 2 partially applies to the problem of searching on m concurrent bounded rays. In this setting, we start at the crossroads and we know that the target is on one of the m rays at a distance D such that $\lambda \le D \le \Lambda$. Given a strategy $f(i)$, we walk a distance of $f(i)$ on the $(i \bmod m)$-th ray and go back to the crossroads. We repeat for all $i \ge 0$ until we find the target. As in the case where $m = 2$, we can suppose that is the solution is periodic and monotone (refer to Section 2 or see Lemmas 2.1 and 2.2 in [15]).

Unfortunately, we have not managed to push the analysis as far as in the case where $m = 2$ because the expressions in the general case do not simplify as easily. We get the following system of equations by applying similar techniques as in Section 2

$$1 + 2\sum_{k=0}^{i+m-1} \frac{a_k}{a_i} = 1 + 2\sum_{k=0}^{m-2} a_k \qquad (0 \leq i \leq n-m),$$

$$1 + 2\sum_{k=0}^{n-1} \frac{a_k}{a_i} + \frac{(i-(n-m))\rho}{a_i} = 1 + 2\sum_{k=0}^{m-2} a_k \qquad (n-m+1 \leq i \leq n-1),$$

for f_n, where

$$f_n(i) = \begin{cases} a_i\lambda & \text{if } 0 \leq i < n, \\ \rho\lambda & \text{if } i \geq n. \end{cases}$$

We prove in Theorem 6 that the solution to this system of equations can be obtained using the following family of polynomials in $m-1$ variables, where $\overline{x} = (x_0, x_1, ..., x_{m-2})$ and $|\overline{x}| = x_0 + x_1 + ... + x_{m-2}$.

$$\begin{aligned} p_i(\overline{x}) &= x_i && (0 \leq i \leq m-2) \\ p_{m-1}(\overline{x}) &= |\overline{x}|(x_0 - 1) && \\ p_i(\overline{x}) &= |\overline{x}|(p_{i-(m-1)}(\overline{x}) - p_{i-m}(\overline{x})) && (i \geq m) \end{aligned}$$

In the rest of this section, for all $i \in \mathbb{N}$, we let $\overline{\alpha}_i = (\alpha_{i,0}, \alpha_{i,1}, ..., \alpha_{i,m-2})$ be the (real) solution to the system

$$p_i(\overline{x}) = 0, \quad p_{i+1}(\overline{x}) = 0, \quad \ldots, \quad p_{i+m-2}(\overline{x}) = 0$$

such that

$$0 \leq \alpha_{i,0} \leq \alpha_{i,1} \leq \cdots \leq \alpha_{i,m-2} \tag{33}$$

and $|\overline{\alpha}_i|$ is maximized (refer to Table 2 for examples with $2 \leq m \leq 5$ and $0 \leq i \leq 6$). Notice that $\overline{\alpha}_i$ exists for any $i \in \mathbb{N}$ since $(0, 0, ..., 0)$ is a solution for any $i \in \mathbb{N}$ by the definition of the p_i's. The proof of the following theorem is similar to those of (6) and Theorem 1.

Theorem 6.

(i) *For all $n \in \mathbb{N}$, the values a_i $(0 \leq i < n)$ that define f_n satisfy the following properties.*
  (a) $a_i = p_i(\overline{a})$.
  (b) $\overline{a}$ *is a solution to the system of equations*

$$p_n(\overline{x}) = \rho, \quad p_{n+1}(\overline{x}) = \rho, \quad \ldots, \quad p_{n+(m-2)}(\overline{x}) = \rho.$$

  (c) $CR(f_n) = 1 + 2|\overline{a}|$.

(ii) *The strategy f_0 is optimal if and only if $1 \leq \rho \leq \frac{m}{m-1}$. In this case, $CR(f_0) = 2(m-1)\rho + 1$.*

$m=2$	$m=3$
$\alpha_0 = 0$	$\overline{\alpha}_0 = (0,0)$
$\alpha_1 = 1$	$\overline{\alpha}_1 = (0,0)$
$\alpha_2 = 2$	$\overline{\alpha}_2 = (1,1)$
$\alpha_3 = \frac{3+\sqrt{5}}{2}$	$\overline{\alpha}_3 = \left(\frac{3}{2}, \frac{3}{2}\right)$
$\alpha_4 = 3$	$\overline{\alpha}_4 = \left(\frac{3+\sqrt{3}}{3}, \frac{3+2\sqrt{3}}{3}\right)$
$\alpha_5 = \frac{1}{3}\left(5 + \frac{7^{2/3}}{\left(\frac{1}{2}(1+3i\sqrt{3})\right)^{1/3}} + \left(\frac{7}{2}\left(1+3i\sqrt{3}\right)\right)^{1/3}\right)$	$\overline{\alpha}_5 = \left(\frac{7+\sqrt{13}}{6}, \frac{4+\sqrt{13}}{3}\right)$
$\alpha_6 = 2+\sqrt{2}$	$\overline{\alpha}_6 = \left(\frac{15+3\sqrt{3}}{11}, \frac{18+8\sqrt{3}}{11}\right)$
$m=4$	$m=5$
$\overline{\alpha}_0 = (0,0,0)$	$\overline{\alpha}_0 = (0,0,0,0)$
$\overline{\alpha}_1 = (0,0,0)$	$\overline{\alpha}_1 = (0,0,0,0)$
$\overline{\alpha}_2 = (0,0,0)$	$\overline{\alpha}_2 = (0,0,0,0)$
$\overline{\alpha}_3 = (1,1,1)$	$\overline{\alpha}_3 = (0,0,0,0)$
$\overline{\alpha}_4 = \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$	$\overline{\alpha}_4 = (1,1,1,1)$
$\overline{\alpha}_5 = \left(\frac{9+\sqrt{21}}{10}, \frac{4+\sqrt{21}}{5}, \frac{4+\sqrt{21}}{5}\right)$	$\overline{\alpha}_5 = \left(\frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}\right)$
$\overline{\alpha}_6 = \left(\frac{6+\sqrt{6}}{6}, \frac{3+\sqrt{6}}{3}, \frac{2+\sqrt{6}}{2}\right)$	$\overline{\alpha}_6 = \left(\frac{6+2\sqrt{2}}{7}, \frac{5+4\sqrt{2}}{7}, \frac{5+4\sqrt{2}}{7}, \frac{5+4\sqrt{2}}{7}\right)$

Table 2: Values of $\overline{\alpha}_i$ for $0 \le i \le 6$ and $2 \le m \le 5$.

(iii) *For all* $n \in \mathbb{N}$,

$$p_{n+m-1}(x) = p_n(\overline{x}) \sum_{i=0}^{m-2} x_i - \sum_{i=0}^{n+m-2} p_i(\overline{x}).$$

(iv) *For all* $n \in \mathbb{N}$, $p_n\left(f_\infty(0), f_\infty(1), ..., f_\infty(m-2)\right) = f_\infty(n)$.

(v) *For all* $0 \le n \le m-2$, $\overline{\alpha}_n = (0,0,...,0)$. *Moreover,* $\overline{\alpha}_{m-1} = (1,1,...,1)$ *and*

$$\overline{\alpha}_m = \left(\frac{m}{m-1}, \frac{m}{m-1}, ..., \frac{m}{m-1}\right).$$

5. Conclusion

We have generalized many of our results for searching on a line to the problem of searching on m rays for any $m \ge 2$. Even though we could not extend the analysis of the polynomials p_n as far as was possible for the case where $m = 2$, we believe this to be a promising direction for future research. By approaching the problem directly instead of studying the inverse problem (maximal reach), we were able to provide exact characterizations of f_{opt} and $CR(f_{opt})$. Moreover, the sequence of implications in the proofs of Section 2 all depend on (7), where (7) is an exact general expression for all roots of all equations p_n. As some readers may have observed, exact values of the roots of the equation p_n are not required to prove the results in Section 2; we need disjoint and sufficiently tight lower and upper bounds on each of the roots of p_n. In the case where $m > 2$, finding a factorization similar to (7) appears highly unlikely. We believe, however, that establishing good bounds for each of the roots of the p_n should be possible. Equipped with such bounds, the general problem could be solved exactly on $m > 2$ concurrent rays. We conclude with the following conjecture. It states that the strategy f_n is uniquely defined for each n, it gives a criterion for the optimal n in terms of ρ (and m) and gives the limit of f_n when $\Lambda \to \infty$.

Conjecture 1.

1. *For all $n \in \mathbb{N}$, the system of equations of Theorem 6*(ib) *has a unique solution $\overline{a}^* = (a_0^*, a_1^*, ..., a_{m-2}^*)$ satisfying* (33) *and such that $|\overline{a}^*| > |\overline{\alpha}_n|$. Moreover, there is a unique choice of $\overline{a}$ for f_n and this choice is $\overline{a} = \overline{a}^*$.*
2. *For a given ρ, if $p_n(\overline{\alpha}_{n+m-1}) \le \rho < p_n(\overline{\alpha}_{n+m})$, then f_n is the best strategy and $|\overline{\alpha}_{n+m-1}| \le |\overline{a}| < |\overline{\alpha}_{n+m}|$.*
3. *When $\Lambda \to \infty$, then the optimal strategy tends toward f_∞.*
4. *For all $n \in \mathbb{N}$,*
$$0 \le |\overline{\alpha}_n| \le |\overline{\alpha}_{n+1}| < \frac{m^m}{(m-1)^{m-1}}$$
with equality if and only if $0 \le n \le m-3$.

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