Computing Conforming Partitions of Orthogonal Polygons with Minimum Stabbing Number

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Abstract

Let P be an orthogonal polygon with n vertices. A partition of P into rectangles is called *conforming* if it results from cutting P along a set of interior-disjoint line segments, each having both endpoints on the boundary of P. The stabbing number of a partition of P into rectangles is the maximum number of rectangles stabbed by any orthogonal line segment inside P. In this paper, we consider the problem of finding a conforming partition of P with minimum stabbing number. We first give an $O(n \log n)$-time algorithm to solve the problem when P is a histogram. For an arbitrary orthogonal polygon (even with holes), we give an integer programming formulation of the problem and show that a simple rounding results in a 2-approximation algorithm for the problem. Finally, we show that the problem is NP-hard if P is allowed to have holes.

Keywords: Orthogonal polygons, Conforming partitions, Stabbing number, Approximation algorithms

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1. Introduction

The problem of partitioning a polygonal shape into simpler components is a well-studied problem in computational geometry, with many applications in other areas of research including VLSI layout design [3, 4], chip manufacturing [5], geoinformatics [6], image processing [7], and pattern recognition [8, 9]. Previous related research in this area was focused on "convexity"; that is, partitioning polygons into convex regions so as to minimize the number of convex components [10, 11, 12, 13, 14]. Another optimality criterion studied in the literature is to minimize the total length of partition segments [15, 16, 17, 18, 19]. Another line of research focused on restricting the shape of the input polygon, among which orthogonal polygons were frequently studied as natural polygonal shapes. For instance, in their seminal paper, Lingas et al. [15] showed that minimizing the total length of partition segments on a simple orthogonal polygon is polynomial-time solvable, while the problem becomes NP-hard if the polygon is allowed to have holes [15]. Moreover, Gonzalez and Zheng [20, 21] studied the approximability of the same problem exclusively on orthogonal polygons with additional constraint that the partition segments must pass through a given set of points in the polygon (see also [22]).

Preliminaries and Definitions. A polygon P is orthogonal if all of its edges are either vertical or horizontal. A *rectangular partition* of an orthogonal polygon P is a set of interior-disjoint rectangles whose union is P. Let R be a rectangular partition of an orthogonal polygon P. Given a line segment ℓ inside P, we say that ℓ *stabs* a rectangle of R if ℓ passes through the interior of the rectangle. The *orthogonal stabbing number* of R is the maximum number of rectangles of R stabbed by any orthogonal line segment inside P. We define the *vertical* (resp., *horizontal*) stabbing number of R as the maximum number of rectangles stabbed by any vertical (resp., horizontal) line segment inside P. For the rest of this paper, "stabbing" is assumed to be orthogonal stabbing, unless noted otherwise. A rectangular partition of P is called *conforming* if it corresponds to the faces of the arrangement of a set of line segments in P, such that each line segment has both endpoints on the boundary of P, and no two line segments intersect, except possibly at their endpoints on the boundary of P. In this paper, we study the *Optimal Conforming Partition* problem: given an orthogonal polygon, the objective is to compute a conforming partition of the polygon whose stabbing number is minimum over all such partitions of the polygon.

Let R be a conforming partition of P. We refer to an edge of a rectangle of R that is not a subset of an edge of P a *partition edge*. That is, the partition edges of R correspond to the "cuts" that divide P into rectangles. A vertex u of P is a *reflex* vertex if the angle at u interior to P is $3\pi/2$. We denote the set of reflex vertices of P by $\texttt{reflexV}(P)$. For each reflex vertex $u \in \texttt{reflexV}(P)$, we denote the maximal horizontal (resp., vertical) line segment contained in the interior of P with one endpoint at u by H_u (resp., V_u) and refer to it as the *horizontal line segment* (resp., *vertical line segment*) of u. Observe that for every reflex vertex u of P, at least one of H_u and V_u must be present in R. The following observation allows us to consider only a discrete subset of the set of all possible rectangular partitions of P to find an optimal conforming partition:

Observation 1. Any orthogonal polygon P has an optimal conforming partition in which every partition edge is either H_u or V_u for some $u \in \texttt{reflexV}(P)$.

Related Work. It is shown by de Berg and van Kreveld [23] that every n-vertex orthogonal polygon has a rectangular (not necessarily conforming) partition with stabbing number $O(\log n)$. They show that this bound is asymptotically tight, as the stabbing number of any rectangular partition of a staircase polygon with n vertices is $\Omega(\log n)$. Independently, de Berg and van Kreveld [23] and Hershberger and Suri [24] gave polynomial-time algorithms that compute partitions with stabbing number $O(\log n)$. Recently, Abam et al. [25] considered the problem of computing an optimal rectangular partition of a simple orthogonal polygon; that is, a rectangular partition (not restricted to being conforming) whose stabbing number is minimum over all such partitions of the polygon. By finding an optimal partition for histograms in $O(n^7 \log n \log \log n)$ time, they obtained a 3-approximation algorithm for this problem. The complexity of finding an optimal partition for an arbitrary orthogonal polygon remains open.

Minimizing the stabbing number of partitions of other inputs are also studied. For instance, de Berg et al. [26] studied the problem of partitioning a given set of n points in $\mathbb{R}^d$ into sets of cardinality between $n/2r$ and $2n/r$ for a given r, where each set is represented by its bounding box, such that the stabbing number is minimized. Here, the stabbing number is defined as the maximum number of bounding boxes intersected by any axis-parallel hyperplane. They showed that the problem is NP-hard in $\mathbb{R}^2$. They also gave an exact $O(n^{4dr+3/2}\log^2 n)$-time algorithm in $\mathbb{R}^d$ as well as an $O(n^{3/2}\log^2 n)$-time 2-approximation algorithm in $\mathbb{R}^2$ when r is constant. Fekete et al. [27] proved that the problem of finding a perfect matching with minimum stabbing number for a given point set is NP-hard, where the stabbing number of a matching is the maximum number of edges of the matching intersected by any axis-parallel line. They also showed that the problems of finding a spanning tree or a triangulation of a given point set with minimum stabbing number are NP-hard.

Our Results. This paper examines the problem of finding an optimal conforming partition of an orthogonal polygon. First, we give an $O(n \log n)$-time algorithm for computing an optimal partition when the input polygon is a histogram with n vertices (Section 2). Next, we give a polynomial-time 2-approximation algorithm for the problem on arbitrary orthogonal polygons, even with holes (Section 3). Finally, we show the NP-hardness of the optimal conforming partition problem on orthogonal polygons with holes in Section 4. To the authors' knowledge, this is the first complexity result related to determining the minimum stabbing number of a rectangular partition of an orthogonal polygon. We conclude the paper with a discussion on open problems in Section 5.

2. Histograms

In this section, we give an $O(n \log n)$-time algorithm for the optimal conforming problem on a histogram with n vertices. A *histogram* (polygon) H is

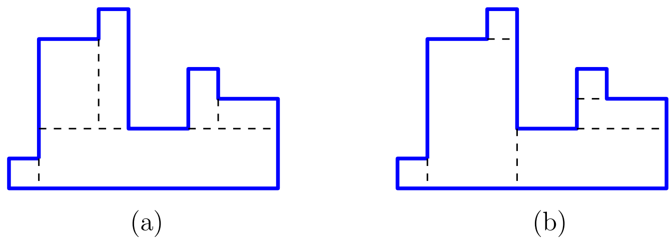

Figure 1: A vertical histogram H. (a) An optimal rectangular partition of H with stabbing number 2. (b) Any conforming partition of H has stabbing number at least 3.

a simple orthogonal polygon that has one edge e that can see every point in P. More formally, H is a vertical (resp., horizontal) histogram if it is monotone with respect to some horizontal (resp., vertical) edge e on the boundary of P [28, 29]; i.e., e spans the width (resp., height) of P. We call e the *base* of H. For the rest of this section, we assume that H is a vertical histogram with n vertices.

We note that Abam et al. [25] gave a polynomial-time algorithm for computing an optimal rectangular partition of a histogram; their algorithm may not necessarily produce a conforming partition. Figure 1 shows a histogram whose optimal rectangular partition has stabbing number 2, while any conforming partition of this histogram has stabbing number at least 3.

Let H^- denote the set of horizontal edges of H. Recall by Observation 1 that every conforming partition of H must include at least one of the edges H_u or V_u for every reflex vertex u in H. The algorithm begins with an initial partition of H, consisting of all horizontal partition edges, that will be modified to produce an optimal conforming partition of H by greedily replacing horizontal edges with vertical edges. The initial partition of H is obtained by adding the edge H_u for each reflex vertex u.

Observation 2. For any conforming partition of any vertical histogram H and any reflex vertex u in H, the vertical partition edge V_u may be included at u if and only if no horizontal partition edge is included directly below u (otherwise it would intersect V_u).

Constructing a Tree. Observation 2 suggests a hierarchical tree structure that determines a partial order in which each horizontal partition edge can be removed and replaced by a vertical partition edge, provided it does not intersect any horizontal partition edge below it. Thus, we construct a forest (initially a single tree denoted T_0) associated with the partition; the algorithm proceeds to update the forest and, in doing so, modifies the associated partition as horizontal partition edges are replaced by vertical ones. Define a tree node for each edge in $H^- \cup S$, where $S = \{H_u \mid u \in \texttt{reflexV}(H)\}$. Add an edge between two vertices u and v if some vertical line segment intersects both edges associated with u and v, but no other edge of $H^- \cup S$. When the polygon H is a histogram, the resulting graph, T_0, is a tree. See the example in Figure 2(a). We now describe how to construct T_0 in $O(n \log n)$ time. Note that the set S need not be known before construction.

Each edge in H^- is adjacent to two vertical edges on the boundary of H, which we call its left and right neighbours, respectively. Sort the edges of H^- lexicographically, first by y-coordinates and then by x-coordinates. The algorithm sweeps a horizontal line ℓ across H from bottom to top. Initially, ℓ coincides with the base of H; root the tree T_0 at a node u that corresponds to the base of H. The construction refers to a separate balanced search tree [28] that archives the set of vertical edges of H on or below the sweepline, indexed by x-coordinates. Initially, only the leftmost and rightmost vertical edges of H are in the search tree, i.e., the base's neighbours. The construction of the tree T_0 proceeds recursively on u as follows.

Suppose the next edges of H^- encountered by the sweepline ℓ are $e_1, \ldots, e_k$, each of which has equal y-coordinate. Add the respective left and right neighbours of $e_1, \ldots, e_k$ to the search tree. Let l_1 and r_1 denote the x-coordinates of the respective left and right endpoints of edge e_1. Add a node representing e_1 to T_0 as a child of u. Check whether the left neighbour of e_1 (indexed by l_1) lies below ℓ. If not, then find the predecessor of l_1 in the search tree and let x^* denote its x-coordinate. Let u' denote the line segment on line ℓ with respective endpoints at the x-coordinates x^* and l_1. Check whether there is a node in T_0

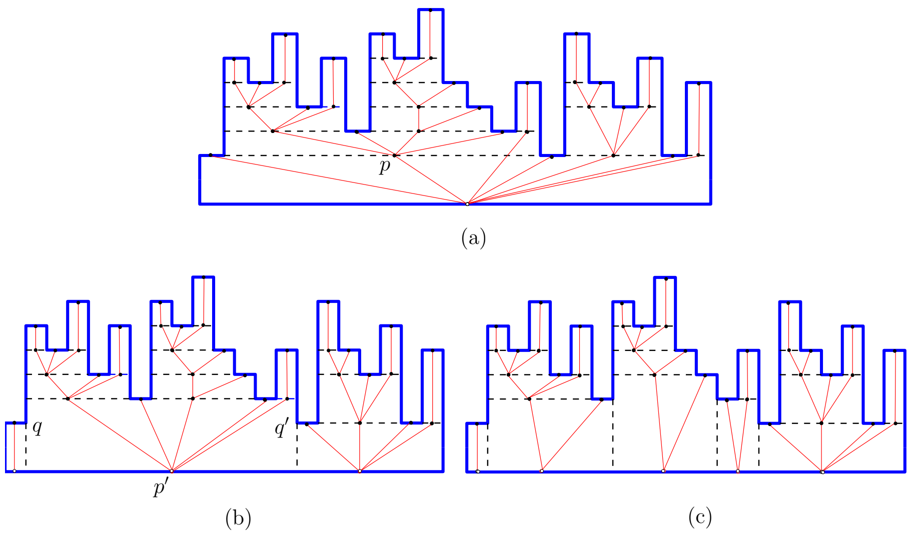

Figure 2: **(a)** A histogram H and the tree T_0 that corresponds to the initial partition of H. **(b)** The edge associated with node p is removed from the partition and is replaced by two vertical edges anchored at the reflex vertices q and q'. The white vertices denote the roots of the three new resulting trees. **(c)** The algorithm terminates after one more iteration, giving an optimal conforming partition of H (with stabbing number 5) along with the corresponding forest.

representing u'; if not, then, add a node representing u' to T_0 as a child of u. Recursively construct the subtree of u'. Apply an analogous procedure to the right neighbour of e_1 (indexed by r_1). Repeat for each edge $e_i \in \{e_2, \ldots, e_k\}$. Upon completion, the tree T_0 is constructed storing a representation of the initial horizontal partition (see Figure 2(a)). Finally, each tree node stores its height and links to its children in order of x-coordinates; the tree can be updated accordingly after construction. The running time for constructing T_0 is bounded by sorting $O(n)$ edges and a sequence of $O(n)$ searches and insertion on the search tree, resulting in $O(n \log n)$ time to construct T_0.

Algorithm. We now describe a greedy algorithm to construct an optimal conforming partition of H using T_0. Observe that the horizontal stabbing number of the initial partition is initially one, whereas its vertical stabbing number corresponds to the height of T_0. The algorithm stores the forest's trees in a priority queue indexed by height. While the vertical stabbing number of H remains greater than its horizontal stabbing number, split the tree of maximum height, say T. To do this, remove the horizontal partition edge stored in a tree node p, where p is a child of the root of T on a longest root-to-leaf path in T. The choice of T and p is not necessarily unique; it suffices to select any tallest tree T and any longest path in T. Observe that p has at least one and possibly two reflex vertices as endpoints, denoted a and b. Remove the horizontal partition edge associated with p and add a vertical partition edge (V_a or V_b) for each neighbour of p that lies above p on the boundary of H. The tree T is then divided into up to three new trees: a) the subtrees of the root of T to the left of p, b) the subtree rooted at p, and c) the subtrees of the root of T to the right of p. The root of each new tree corresponds to the base edge of H. See Figure 2(b). The following observation is straightforward:

Observation 3. The horizontal stabbing number of the partition associated with the forest corresponds to the number of trees in the forest, whereas its vertical stabbing number corresponds to the height of the tallest tree in the forest.

Once the height of the tallest tree becomes less than or equal to the number of trees in the forest, we return either the current partition or the previous partition, whichever has lower stabbing number. The number of steps is $O(n)$, where each step requires $O(\log n)$ time to determine the tree with maximum height using the priority queue.

The algorithm's correctness follows from Observations 2 and 3, and the fact that reducing the vertical stabbing number requires reducing the height of the tallest tree, which is exactly how the algorithm proceeds, decreasing the height of a tallest tree by one on each step. Therefore, we have the following theorem:

Theorem 1. *Given a histogram H, an optimal conforming partition of H can be found in $O(n \log n)$ time, where n is the number of vertices of H.*

3. 2-Approximation Algorithm

In this section, we give a 2-approximation algorithm for the optimal conforming partition problem. To this end, we formulate the problem as a k-sum integer linear program and show that a simple rounding of the relaxed program leads to a 2-approximation algorithm for this problem; we remark that our algorithm works even on orthogonal polygons with holes. We first review k-sum linear programs.

k-Sum Linear Program. Given an integer $k \geq 1$, a k-Sum Linear Program (KLP) [30] consists of an $m \times n$ matrix A, an m-vector b, an n-vector $X = (x_1, x_2, \ldots, x_n)$, and an n-vector $C = (c_1, c_2, \ldots, c_n)$ for which the objective is to

$$\begin{aligned} \text{minimize } & \max_{S \subseteq N : |S| = k} \sum_{j \in S} c_j x_j \qquad (1) \\ \text{subject to } & AX \geq b \\ & X \geq 0, \end{aligned}$$

where $N = \{1, 2, \ldots, n\}$. Observe that when $k = n$, the KLP is equivalent to a classical linear program.

Let P be an orthogonal polygon. We define two binary variables u_h and u_v for every reflex vertex $u \in \texttt{reflexV}(P)$ that correspond to H_u and V_u, respectively. Each variable's value (1 = present, 0 = absent) determines whether its associated partition edge is included in the partition. If two reflex vertices align, then they share a common variable. For each reflex vertex u in $\texttt{reflexV}(P)$, let ℓ_u^- and $\ell_u^|$ be respective maximal horizontal and vertical line segments that pass through $f_\epsilon(u)$ and are completely contained in P, where $f_\epsilon(u)$ denotes an ϵ translation of the

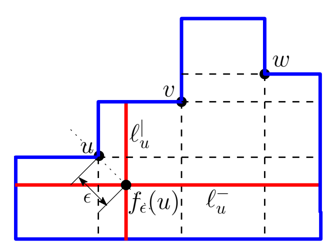

Figure 3: The maximal line segments ℓ_u^- and $\ell_u^|$ that pass through the point $f_\epsilon(u)$ are shown in red and blue, respectively. In this example, $u_{\Sigma^-} = 1 + u_v + v_v + w_v$ and $u_{\Sigma^|} = 1 + u_h$.

vertex u along the bisector of the interior angle determined by the boundary of P locally at u, for some ϵ less than the minimum distance between any two vertices of P. This perturbation ensures that ℓ_u^- and $\ell_u^|$ lie in the interior of P, as in the definition of stabbing number. See Figure 3. Let S_u^- (resp., $S_u^|$) be the set of reflex vertices in $\texttt{reflexV}(P)$, like v, such that V_v (resp., H_v) intersects ℓ_u^- (resp., $\ell_u^|$). For each reflex vertex $u \in \texttt{reflexV}(P)$, let

$$u_{\Sigma^-} = 1 + \sum_{p \in S_u^-} p_v, \quad \text{and} \quad u_{\Sigma^|} = 1 + \sum_{p \in S_u^|} p_h.$$

Thus, u_{Σ^-} and $u_{\Sigma^|}$ denote the number of rectangles stabbed by ℓ_u^- and $\ell_u^|$, respectively, and their maximum values among all reflex vertices u in P correspond to the respective horizontal and vertical stabbing numbers of P. Consequently, the stabbing number of the partition of P determined by the binary variables is

$$\max_{u \in \texttt{reflexV}(P)} \{\max\{u_{\Sigma^-}, u_{\Sigma^|}\}\}. \tag{2}$$

A partition divides the polygon into convex regions (more specifically, rectangles) if and only if at least one partition edge is rooted at every reflex vertex. Thus, a conforming partition of P corresponds to an assignment of truth values to the set of binary variables such that (i) no two edges of the partition cross, and (ii) for every reflex vertex u, at least one of V_u and H_u is present in the partition. Therefore, the optimal conforming partition problem can be formulated as a k-sum integer linear program as follows:

$$\begin{aligned} &\text{minimize (2)} && \qquad (3)\\ \text{subject to } & u_h + u_v \geq 1, && \forall u \in \texttt{reflexV}(P),\\ & v_h + u_v \leq 1, && \text{if } H_v \text{ intersects } V_u \text{ and u} \neq \text{v},\\ & u_h, u_v \in \{0, 1\}, && \forall u \in \texttt{reflexV}(P). \end{aligned}$$

To obtain an integer linear program, we introduce an additional variable y. The following integer linear program is equivalent to the above KLP:

$$
\begin{aligned}
& \text{minimize } y && && (4) \\
& \text{subject to } y - u_{\Sigma^-} \geq 0 && \forall u \in \texttt{reflexV}(P), && \\
& y - u_{\Sigma^|} \geq 0 && \forall u \in \texttt{reflexV}(P), && \\
& u_h + u_v \geq 1, && \forall u \in \texttt{reflexV}(P), && \\
& -v_h - u_v \geq -1, && \text{if } H_v \text{ intersects } V_u \text{ and u} \neq \text{ v}, && \\
& u_h, u_v \in \{0, 1\}, && \forall u \in \texttt{reflexV}(P). && (5)
\end{aligned}
$$

Since the number of sums in (2) is $O(n^2)$, the size of the above integer linear program is polynomial in n. By replacing (5) with $u_h, u_v \geq 0, \forall u \in$ `reflexV`(P), we obtain the final linear program; we call the resulting linear program `conformingLP`. [2]

Let s^* be a solution to `conformingLP`. We round s^* to a feasible solution for our problem as follows. For each vertex $u \in$ `reflexV`(P), let

$$
u_h = \begin{cases} 0, & \text{if } s^*(u_h) \leq 1/2, \\ 1, & \text{if } s^*(u_h) > 1/2, \end{cases} \quad \text{and} \quad u_v = \begin{cases} 0, & \text{if } s^*(u_v) < 1/2, \\ 1, & \text{if } s^*(u_v) \geq 1/2. \end{cases} \tag{6}
$$

We first show that, for every reflex vertex u, at least one of V_u and H_u is present in the partition.

Lemma 2. *For each vertex $u \in$ `reflexV`(P), at least one of u_h and u_v is equal to 1 after rounding a solution of `conformingLP`.*

Proof. We give a proof by contradiction. Suppose that after rounding a solution of `conformingLP`, $u_h = u_v = 0$ for some $u \in$ `reflexV`(P). Since $u_h = 0$ by (6) we have $s^*(u_h) \leq 1/2$ and, similarly, since $u_v = 0$ we have $s^*(u_v) < 1/2$. Therefore, $s^*(u_h) + s^*(u_v) < 1$, which contradicts the constraint $u_h + u_v \geq 1$ of `conformingLP`. □

[2]We observe that the constraints $u_h, u_v \leq 1$ are redundant since we can reduce any $u_h > 1$ (resp., $u_v > 1$) to u_h=1 (resp., u_v=1) without increasing the value of the objective function for any feasible solution.

The next lemma proves that no two edges of the partition obtained by `conformingLP` cross each other.

Lemma 3. *Let u, v be two vertices in* `reflexV`(P)*. Then, if H_v intersects V_u, then at most one of the variables v_h and u_v is 1 after rounding a solution of* `conformingLP`*.*

Proof. We give a proof by contradiction. Suppose that for two vertices $u, v \in$ `reflexV`(P): (i) H_v intersects V_u, and, (ii) both v_h and u_v are 1 after rounding. Since v_h=1, we have $s^*(v_h) > 1/2$ by (6). Similarly, $s^*(u_v) \geq 1/2$ by the rounding. Therefore, $s^*(v_h) + s^*(u_v) > 1$, which contradicts the constraint $v_h + u_v \leq 1$ (or equivalently $-v_h - u_v \geq -1$) of `conformingLP`. □

By combining Lemmas 2 and 3, we get the following result:

Lemma 4. *Let s^* be a feasible solution to* `conformingLP`*. Then, the partition determined by rounding s^* is a conforming partition.*

First, notice that the number of constraints of `conformingLP` is polynomial in $|$`reflexV`$(P)|$. Now, let u be a variable and consider $s^*(u)$, the real value of u after solving `conformingLP`. By (6), $u = 1$ if $s^*(u) > 1/2$ (in case of u corresponding to a horizontal partition edge) or if $s^*(u) \geq 1/2$ (in case of u corresponding to a vertical partition edge); otherwise, $u = 0$. Since $0 \leq s^*(u) \leq 1$, we conclude that the integer value of each variable is at most twice its real value. Therefore, we have the following theorem.

Theorem 5. *Let P be an orthogonal polygon possibly with holes. Then, there exists a polynomial-time 2-approximation algorithm for the optimal conforming partition problem on P.*

Remark. A preliminary attempt at obtaining a 2-approximation might be to assign to each reflex vertex u its vertical partition edge, V_u (or, equivalently, assigning the horizontal partition edge H_u to each u). This is not the case; Figure 4 shows an orthogonal polygon for which the optimal conforming partition has stabbing number 4. However, the partition obtained by assigning V_u

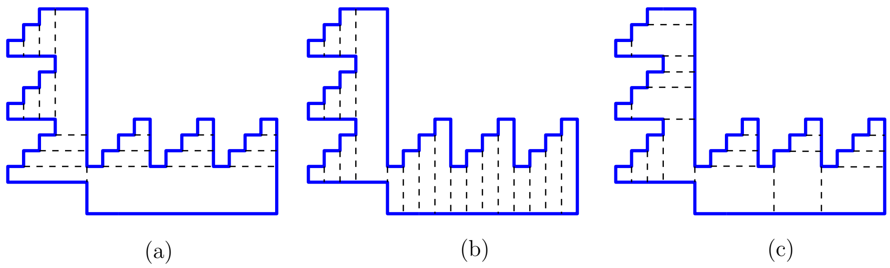

Figure 4: A simple orthogonal polygon P for which (a) the optimal partition has stabbing number 4 while (b) assigning V_u (or H_u) to every reflex vertex u of P results in a partition with stabbing number at least 10. (c) A possible partition produced by our 2-approximation algorithm with stabbing number 7.

(or H_u) consistently to every vertex $u \in \texttt{reflexV}(P)$ has stabbing number at least 10. In fact, the polygon in this example can be extended to show that this heuristic does not provide any constant-factor approximation.

4. Hardness

In this section, we show that the optimal conforming partition problem is NP-hard on orthogonal polygon with holes. We show the hardness by a reduction from Planar Variable Restricted 3SAT (Planar VR3SAT, for short).

An instance of the Planar 3SAT problem consists of a planar bipartite graph $G_I = (V, E)$, called a *variable-clause graph*, corresponding to a Boolean formula I in conjunctive normal form, where each clause contains three variables. The vertices in one partition of G_I correspond to the variables in I while the vertices in the other partition of G_I correspond to the clauses of I. Each clause vertex is connected by an edge to the variable vertices it contains. Knuth and Raghunathan [31] showed that such a graph can be drawn on a grid with all variable vertices on a horizontal line and the clause vertices connected in a comb-shape form above or below that line without any edge crossings. The Planar VR3SAT problem is a constrained version of Planar 3SAT in which each variable can appear in at most three clauses (and the corresponding *variable-*

clause graph is planar). Efrat et al. [32] showed that PLANAR VR3SAT is NP-hard.

Reduction Overview. Let $I = \{C_1, C_2, \ldots, C_k\}$ be an instance of PLANAR VR3SAT with k clauses and n variables, $X_1, X_2, \ldots, X_n$. We construct a polygon P with holes such that P has a conforming partition with stabbing number at most $5c$ if and only if I is satisfiable, where we determine the value of c later. Given I, we first construct the variable-clause graph of I in the non-crossing comb-shape form of Knuth and Raghunathan [31]. Without loss of generality, we assume that the variable vertices lie on a vertical line and the clause vertices are connected from left or right of that line; see Figure 5 for an illustration. Then, we replace each variable vertex X_i with a polygonal variable gadget to which three connecting corridors are attached from its left. The corridors are then connected to the clause gadgets whose associated clauses contain that variable. Each variable gadget has a special reflex vertex v such that choosing V_v or H_v in a conforming partition imposes constraints on how the rest of the variable gadget and its associated clause gadgets are partitioned. By having a sufficient number of reflex vertices in clause gadgets, we can force exactly one of the resulting partitions to have stabbing number at most $5c$. In the following, we first describe the details of the gadgets used in the reduction and then prove the correctness.

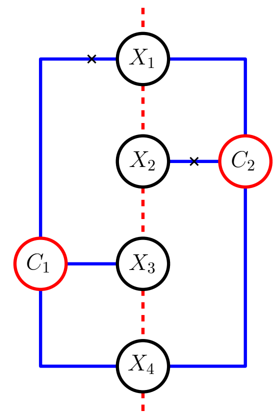

Figure 5: An instance of the PLANAR VR3SAT problem in the comb-shape form of Knuth and Raghunathan [31]. Crosses on the edges indicate negations; for example, $C_1 = (\overline{x_1} \vee x_3 \vee x_4)$.

Variable Gadgets. Figure 6 shows an example of a variable gadget. We denote the variable gadget corresponds to variable X_i by $\texttt{vGadget}(X_i)$. Moreover, we denote the two literals of a variable X_i by x_i and $\overline{x_i}$. Each variable gadget has three corridors, namely the *top*, *middle* and *bottom* corridors. Each corridor of

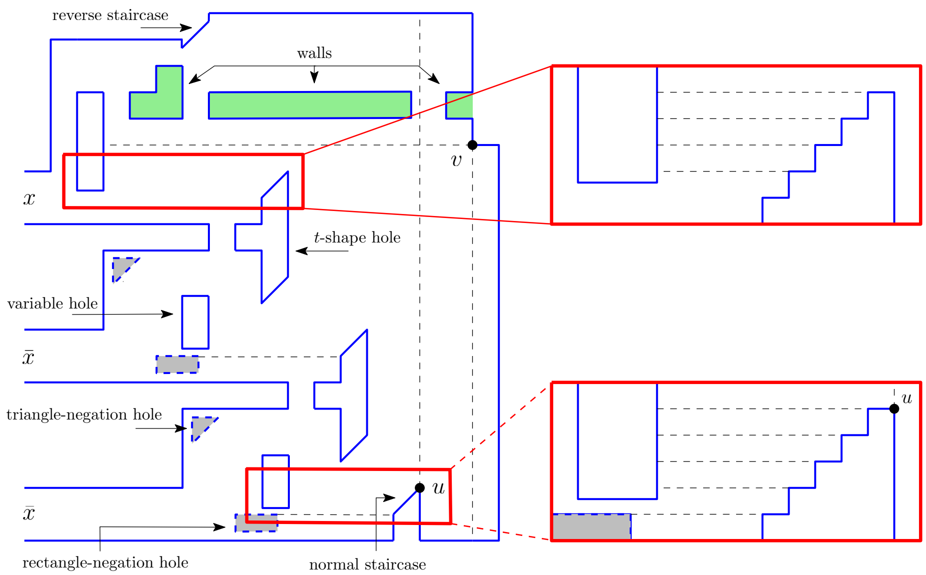

Figure 6: An example of a variable gadget X linked by three respective corridors to its occurrences (x, $\overline{x}$ and $\overline{x}$) in clauses. Each pair of dashed triangular and rectangular holes form a negation gadget that negates the truth value of X in the associated clause linked by the adjacent corridor. Each staircase consists of c reflex vertices.

$\texttt{vGadget}(X_i)$ is connected to one of the clauses that contains X_i. Let C_j be a clause that contains X_i. We denote the corridor connecting $\texttt{vGadget}(X_i)$ to C_j by $\texttt{corridor}(X_i, C_j)$. That is, $\texttt{corridor}(X_i, C_j)$ indicates the presence of a literal of X_i (i.e., x_i or $\bar{x_i}$) in the clause C_j. There are two holes in the beginning of every $\texttt{corridor}(X_i, C_j)$: a rectangular hole, called *variable hole*, and a *t-shaped hole* that has two staircases on its boundary, each consisting of c reflex vertices (each staircase is shown as a single diagonal edge in Figure 6), where the value of c is determined later. To avoid confusion, we call the upper staircase of each t-shaped hole a *normal staircase* and its lower staircase a *reverse staircase*. As Figure 6 shows, each variable gadget has also a normal staircase and a reverse staircase on its boundary. See Figure 7(Left) (resp., Figure 7(Right)) for an illustration of a normal staircase (resp., reverse staircase).

We separate the upper part of each variable gadget from the rest with two holes and a part of the boundary of $\texttt{vGadget}(X_i)$, called *walls*. See Figure 6.

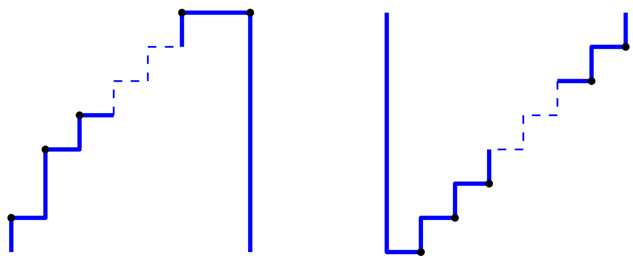

Figure 7: Details of a normal staircase (left), and a reverse staircase (right). Each of these staircases has n reflex vertices.

There is a gap between the two right walls such that V_u (u is the topmost reflex vertex of the lowest normal staircase of $\texttt{vGadget}(X_i)$) passes through this gap and enters into the upper part of P. Note that the vertical lines through all other vertices on this staircase intersect one of the walls.

Negation Gadget. If $\bar{x}_i \in C_j$; i.e., the negation literal of X_i appears in the clause C_j, then we locate a pair of holes inside $\texttt{corridor}(X_i, C_j)$ that together serve as a negation gadget. The dashed rectangle and triangle within the bottom corridor of the variable gadget shown in Figure 6 together form a negation gadget; we call these as *rectangle-negation hole* and *triangle-negation hole*, respectively. The rectangle-negation hole is located below the variable hole inside $\texttt{corridor}(X_i, C_j)$. The *triangle-negation hole* is located on the left and above the variable hole. By rescaling these two negation holes, we can ensure that no horizontal or vertical line segment inside $\texttt{vGadget}(X_i)$ can intersect both the triangle-negation and the variable holes or both the triangle-negation and the rectangle-negation holes at the same time. Note that the two upper vertices of the rectangle-negation hole have the same y-coordinate as the lowest reflex vertex of the normal staircase inside $\texttt{corridor}(X_i, C_j)$. Moreover, H_w is blocked by the variable hole for every reflex vertex w on this normal staircase except for the lowest one; see the magnified illustrations in Figure 6. Finally, the x-coordinate of the left side of the rectangle-negation hole is less than that of the left side of the variable hole inside $\texttt{corridor}(X_i, C_j)$.

Each triangle-negation gadget is a reverse staircase consisting of $4c$ reflex

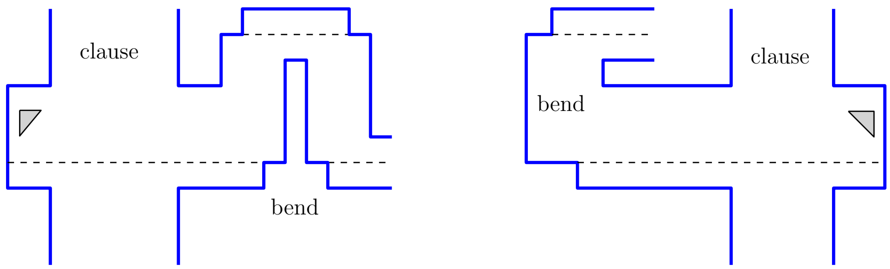

Figure 8: An illustration of a left-clause gadget (left), and a right-clause gadget (right).

vertices. Finally, recall the vertex v, the rightmost reflex vertex of $\texttt{vGadget}(X_i)$ (see Figure 6); we call this vertex the *decision vertex* of $\texttt{vGadget}(X_i)$.

Clause Gadgets. In the variable-clause graph, each variable vertex has degree at most three. Moreover, in the comb-shaped drawing of the variable-clause graph, edges might be incident to a variable vertex from both left and right of that vertex. Consider a clause vertex that lies on the left of the variable vertices; Figure 8(left) shows the clause gadget for such a clause. We call such clause gadget a *left-clause gadget*. Note that Figure 8 shows only a part of a left-clause gadget. To describe the complete gadget, we extend the top and bottom open parts of the gadget upwards and downwards until we connect the three corridors that come from the variables contained in this clause. Then, we close these open parts by a horizontal line segment in the top and bottom parts of the clause gadget. By the comb-shaped drawing of the variable-clause graph, the three corridors connecting variables to a clause must *all* be connected from left or right of the clause gadget.

In the opposite side of a corridor connected to a clause gadget, we locate a reverse staircase inside the clause gadget facing towards the corridor (see the triangle in Figure 8(left)). Each reverse staircase inside a clause gadget has $2c$ reflex vertices. Note that there is one such reverse staircase for each corridor connected to the clause gadget, and each such reverse staircase is located in a separate lacuna as shown in Figure 8. We create a bend in the middle of the corridor connecting a variable gadget to a left-clause gadget as shown in

Figure 8(left). There are four separate reflex vertices on the corners of the bend. These reflex vertices are created such that no vertical line segment inside the corridor can pass through two of them at the same time. A *right-clause gadget* is defined and constructed similar to that of a left-clause gadget. Figure 8(right) shows an example of a right-clause gadget. [3] Since we have to bend the corridor connecting a variable gadget to a right-clause gadget, we do not create any additional bend inside the corridor. There are two separate reflex vertices on the corners of the bend inside a right-clause gadget such that no vertical line segment can pass through both of them at the same time (see Figure 8(right)). Let P be the resulting polygon.

By re-scaling and making the gadgets and corridors small enough, we can ensure the construction of P and that the corridors will never intersect the gadgets or bends. See Figure 9 for polygon P corresponding to the instance of the Planar VR3SAT shown in Figure 5. Finally, c is greater than the number of reflex vertices of P that are neither on a staircase nor on a hole of P. More precisely, c is greater than the number of reflex vertices of P', a simple polygon obtained from P by removing the all holes and the staircases of P. We are now ready to prove the following lemma.

Lemma 6. *P has a conforming partition with stabbing number at most $5c$ if and only if I is satisfiable.*

Proof. ($\Leftarrow$) Suppose that I is satisfiable. We give a conforming partition of P that has stabbing number at most $5c$. For each variable X_i: if X_i is true, then we add V_v to the partition, where v is the decision vertex of $\texttt{vGadget}(X_i)$. Otherwise, if X_i is false, then we add H_v to the partition, where v is the decision vertex of $\texttt{vGadget}(X_i)$. In the following, we show that any orthogonal line segment inside $\texttt{vGadget}(X_i)$ or inside a clause gadget connecting to $\texttt{vGadget}(X_i)$ can intersect at most $5c$ rectangles induced by this partition.

[3]When it does not matter, we omit the left or right prefix when referring to a clause gadget.

Case 1. If X_i is true, then V_v forces all reverse staircases of $\texttt{vGadget}(X_i)$ to be partitioned vertically except for the reverse staircase on the boundary of $\texttt{vGadget}(X_i)$ (i.e., the topmost staircase of $\texttt{vGadget}(X_i)$). Thus, the normal staircases that face towards these reverse staircases are also forced to be partitioned vertically. Therefore, the vertical edge that passes through exactly one of the vertices of the normal staircase located on the boundary of $\texttt{vGadget}(X_i)$ (i.e., V_u in Figure 6) passes through the two right walls of $\texttt{vGadget}(X_i)$. This forces the topmost reverse staircase of $\texttt{vGadget}(X_i)$ to be partitioned vertically, which implies that all staircases of $\texttt{vGadget}(X_i)$ must be partitioned vertically. It is easy to see that no vertical or horizontal line segment inside $\texttt{vGadget}(X_i)$ can stab the rectangles induced by partitioning more than four staircases at the same time; hence, no more than $5c$ rectangles can be stabbed. Now, let K denote a corridor of $\texttt{vGadget}(X_i)$.

- If there is no negation gadget inside K, then we add additional vertical partition edges to partition K. The reflex vertices inside the bend of K force the bend and, consequently, the reverse staircase of the clause gadget facing towards K to be partitioned vertically. Therefore, any horizontal line segment through the corridor stabs at most $3c$ rectangles.

- If there is a negation gadget inside K, then we add H_u for every reflex vertex of the triangle-negation and the rectangle-negation holes. This forces the reflex vertices inside the bend of K and, therefore, the reverse staircase of the clause gadget facing towards K to be partitioned horizontally.

Note that since I is satisfiable and X_i is true, it is not possible for all three reverse staircases inside this clause gadget to be partitioned horizontally. Thus, any orthogonal line segment inside this clause gadget can stab at most $5c$ rectangles. Moreover, since there exists a bend in K, no horizontal or vertical line segment can stab all rectangles induced by partitioning a triangle-negation hole and a reverse staircase of this clause gadget simultaneously. Therefore, we conclude that the stabbing number of the conforming partition of $\texttt{vGadget}(X_i)$ is at most $5c$.

Case 2. If X_i is false, then we can use an analogous argument as in Case 1 to show that all staircases of `vGadget`(X_i) must be partitioned horizontally. Then, it is easy to see that no orthogonal line segment inside `vGadget`(X_i) can stab the rectangles induced by partitioning more than four staircases at the same time; hence, no more than $5c$ rectangles can be stabbed. Now, consider a corridor K of `vGadget`(X_i).

- If there is no negation gadget inside the corridor, then we add additional horizontal partition edges to partition the corridor. By an analogous argument as in the first part of Case 1, we can show that the entire corridor and the reverse staircase of the clause gadget facing towards the corridor must be partitioned horizontally. Since I is satisfiable, X_i is false and there is no negation-gadget inside K, it is not possible for all the three reverse staircases inside this clause gadget to be partitioned horizontally. Therefore, any vertical line segment through the clause gadget stabs at most $5c$ rectangles.

- If there is a negation gadget inside K, then we add V_u for every reflex vertex on the triangle-negation and the rectangle-negation holes. Note that this is possible as H_w is blocked by the left side of the variable hole, for all (except the lowest) reflex vertices w of the normal staircase inside `corridor`(X_i, K). By an analogous argument as in the second part of Case 1, we can show that the entire corridor and, consequently, the reverse staircase inside the clause gadget facing towards K must be partitioned vertically.

Therefore, the stabbing number of the partition of `vGadget`(X_i) and every clause gadget connecting to `vGadget`(X_i) is at most $5c$. This implies that the stabbing number of the resulting partition of P is at most $5c$.

($\Rightarrow$) Suppose that we are given a conforming partition of P that has stabbing number at most $5c$. We give a truth assignment for I as follows. For each variable X_i, we set X_i to `true` (resp., to `false`) if and only if the partition contains V_v (resp., contains H_v), where v is the decision vertex of `vGadget`(X_i).

Let $C(X_i) \in \{x_i, \bar{x_i}\}$ denote the literal of X_i that appears in a clause C. We denote the value of a literal x_i by $\texttt{val}(x_i)$. Suppose for a contradiction that this assignment does not result in a truth value for I. Thus, there exists a clause $C = (X_i, X_j, X_k)$ such that $\texttt{val}(C(X_i)) = \texttt{val}(C(X_j)) = \texttt{val}(C(X_k)) = \texttt{false}$. In the following, we show that the reverse staircase in C that faces towards $\texttt{corridor}(X_i, C)$ must be partitioned horizontally. The argument for the corresponding reverse staircases in C for $\texttt{corridor}(X_j, C)$ and $\texttt{corridor}(X_k, C)$ are analogous.

Case 1. If $C(X_i) = x_i$, then H_v is present in $\texttt{vGadget}(X_i)$ because $\texttt{val}(C(X_i)) = \texttt{false}$. Therefore, all (normal and reverse) staircases inside $\texttt{vGadget}(X_i)$ must have been partitioned horizontally. Since $C(X_i) = x_i$ there is no negation gadget in $\texttt{corridor}(X_i, C)$. Thus, the lowest reflex vertex of the normal staircase, which belongs to the t-shaped hole in the begin of $\texttt{corridor}(X_i, C)$, is forced to be an endpoint of a horizontal partition edge of the partition. This horizontal partition edge passes through $\texttt{corridor}(X_i, C)$ and forces the reflex vertices inside the bend of $\texttt{corridor}(X_i, C)$ to remain partitioned horizontally. Therefore, the given conforming partition contains the horizontal partition edge of $\texttt{corridor}(X_i, C)$ that goes through the interior of C and passes below the reverse staircase of C that faces towards $\texttt{corridor}(X_i, C)$; consequently, this reverse staircase is partitioned horizontally.

Case 2. If $C(X_i) = \bar{x_i}$, then V_v is present in $\texttt{vGadget}(X_i)$ because $\texttt{val}(C(X_i)) = \texttt{false}$. Since V_v is present in $\texttt{vGadget}(X_i)$, all (normal and reverse) staircases of $\texttt{vGadget}(X_i)$ are partitioned vertically. Since $C(X_i) = \bar{x_i}$, there exists a negation gadget (i.e., triangle-negation and rectangle-negation holes) inside $\texttt{corridor}(X_i, C)$. The triangle-negation hole inside $\texttt{corridor}(X_i, C)$ must be partitioned horizontally. Otherwise, there is a horizontal line segment inside the corridor that stabs all the rectangles induced by partitioning the triangle-negation hole and the reverse staircase on the t-shaped hole that is located just above $\texttt{corridor}(X_i, C)$; in particular, consider the horizontal line segment that passes through the space between the variable hole and the rectangle-negation hole of $\texttt{corridor}(X_i, C)$. See Figure 10. This implies that the stabbing number of the given conforming partition is greater than $5c$, which is a contradiction. The horizontal rectangles induced by partitioning the triangle-negation hole block the upper-left vertex of the rectangle-negation hole to be an endpoint of a vertical partition edge. Therefore, the partition edge through this vertex must be horizontal. Consequently, this horizontal partition edge forces the reflex vertices inside the bend of $\texttt{corridor}(X_i, C)$ to be partitioned horizontally. Therefore, the reverse staircase in the clause gadget of C must be partitioned horizontally.

We conclude that if $\texttt{val}(C(X_i)) = \texttt{false}$, then the reverse staircase inside the clause gadget of C that faces towards $\texttt{corridor}(X_i, C)$ is partitioned horizontally. Since $\texttt{val}(C(X_i)) = \texttt{val}(C(X_j)) = \texttt{val}(C(X_k)) = \texttt{false}$, all the reverse staircases inside the clause gadget of C are partitioned horizontally. Since each reverse staircase inside a clause gadget consists of $2c$ reflex vertices, there exists a vertical line segment inside the clause gadget of C that stabs more than $5c$ rectangles, which is a contradiction. This completes the second part of the proof. $\square$

It is straightforward to see that the reduction and construction of P can be done in polynomial time. Therefore, by Lemma 6, we have the following theorem.

Theorem 7. *The optimal conforming partition problem is* NP*-hard on orthogonal polygons with holes.*

5. Conclusion

In this paper, we studied the problem of computing a conforming partition of an orthogonal polygon P with minimum stabbing number over all such partitions of P; the stabbing number of a partition is defined as the maximum number of rectangles stabbed by any orthogonal line segment inside P. We first gave an $O(n \log n)$-time algorithm to solve the problem when P is a histogram with n vertices. We also gave a 2-approximation algorithm for the problem on any orthogonal polygon P, even if P has holes. Finally, we showed that the problem is NP-hard for orthogonal polygons with holes. We leave the following questions about conforming partitions open for future work:

1. What is the complexity of finding a conforming partition with minimum stabbing number on simple orthogonal polygons; i.e., polygons without holes?

2. Can a conforming partition with stabbing number at most c times the minimum be found in polynomial time, for some constant $c < 2$?

Another direction for future work is to study the problem of finding a general (not necessarily conforming) partition with minimum stabbing number in orthogonal polygons; i.e., the problem studied by Abam et al. [25]. The complexity of the general problem remains open even on orthogonal polygons with holes. Note that our reduction on polygons with holes does not work for general partitions. The best approximation algorithm for the general problem has approximation factor 3 [25]. Can our LP-based 2-approximation algorithm be generalized to get better approximation algorithms for the general problem?

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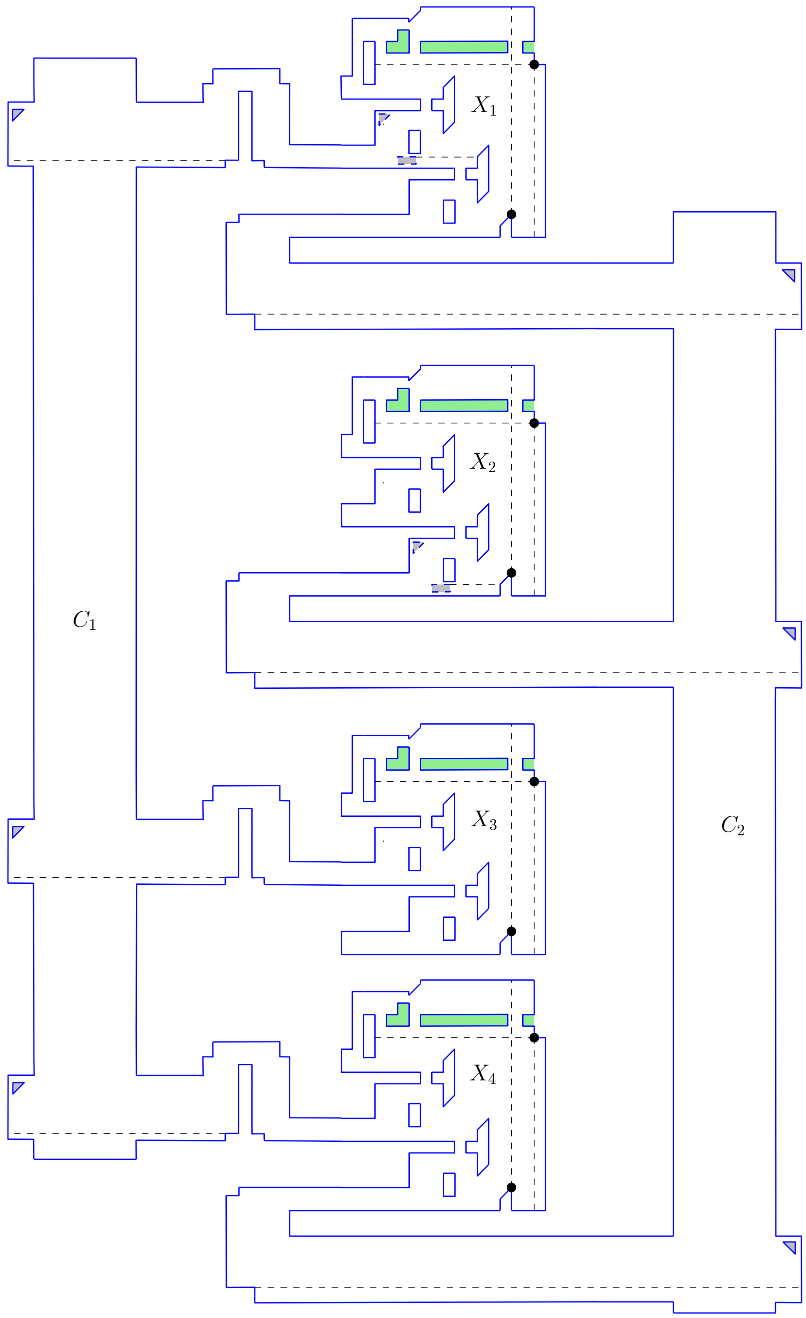

Figure 9: The complete polygon P corresponding to the instance of the Planar VR3SAT shown in Figure 5.

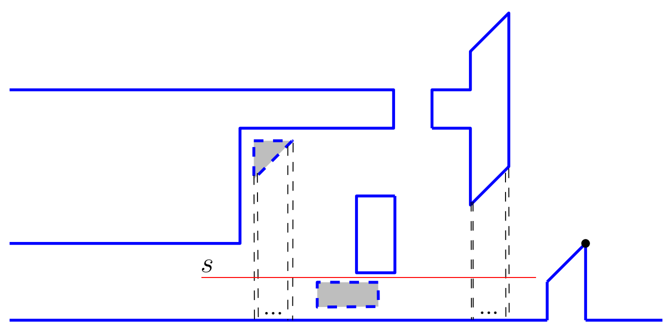

Figure 10: The line segment s stabs more than $5c$ rectangles.