Bounding Interference in Wireless Ad Hoc Networks with Nodes in Random Position

Majid Khabbazian, Stephane Durocher, Alireza Haghnegahdar, and Fabian Kuhn

Abstract—Given a set of positions for wireless nodes, the interference minimization problem is to assign a transmission radius (i.e., a power level) to each node such that the resulting communication graph is connected, while minimizing the maximum (respectively, average) interference. We consider the model introduced by von Rickenbach et al. (2005), in which each wireless node is represented by a point in Euclidean space on which is centred a transmission range represented by a ball, and edges in the corresponding graph are symmetric. The problem is NP-complete in two or more dimensions (Buchin 2008) and no polynomial-time approximation algorithm is known. We show how to solve the problem efficiently in settings typical for wireless ad hoc networks. If nodes are represented by a set P of n points selected uniformly and independently at random over a d-dimensional rectangular region, then the topology given by the closure of the Euclidean minimum spanning tree of P has $O(\log n)$ maximum interference with high probability and $O(1)$ expected interference. We extend the first bound to a general class of communication graphs over a broad set of probability distributions. We present a local algorithm that constructs a graph from this class; this is the first local algorithm to provide an upper bound on expected maximum interference. Finally, we disprove a conjecture of Devroye and Morin (2012) relating the maximum interference of the Euclidean minimum spanning tree to the optimal maximum interference attainable.

Index Terms—interference minimization, topology control, random points, wireless networks, communication graph.

1 Introduction

1.1 Motivation

Establishing connectivity in a wireless network can be a complex task for which various (sometimes conflicting) objectives must be optimized. To permit a packet to be routed between any two nodes in a network, the corresponding communication graph must be connected. In addition to requiring connectivity, various properties can be imposed on the network, including low power consumption [33], [45], bounded average traffic load [16], [24], small average hop distance between sender-receiver pairs [2], low dilation (t-spanner) [2], [7], [10], [11], [26], [34], [41], and minimal interference; this latter objective, minimizing interference, is the focus of much recent research [2], [5], [6], [9], [15], [22], [28], [30]–[32], [35], [37]–[39], [45]–[49] and of this paper.

The amplitude of a radio signal transmitted at a node p and received at a node q decreases as the distance between p and q increases. The signal from p must be sufficiently strong for q to receive it. That is, for a given transmission power level at node p, there exists some threshold, say $r(p)$, such that if q receives a message from p, then the distance from p to q can be at most $r(p)$. We model transmission in a wireless network by assigning to each wireless node p a radius of transmission $r(p)$, such that every node within distance $r(p)$ of p can receive a transmission from p, whereas no node at greater distance from p can. However, the distance between p and q alone is not sufficient to determine successful communication between p and q; even if q is within distance $r(p)$, signals sent from other nodes could interfere with the signal from p received at q. We adopt the interference model introduced by von Rickenbach et al. [48] which is related to the *geometric radio network* model of Dessmark and Pelc [13] and other early geometric models for wireless networks [18], [21].

We measure interference at node p by the number of nodes that have p within their respective radii of transmission. Given a set of wireless nodes whose positions are represented by a set of points P, we consider the problem of identifying a connected network on P that minimizes the maximum (respectively, average) interference. The problem of constructing the network is equivalent to that of assigning a transmission radius to each node (in general, $\Theta(n)$ distinct radii are assigned to a set of n nodes); once the transmission radius of each node is fixed, the corresponding communication graph and its associated maximum interference are also determined. Conversely, once a graph is fixed, each node's transmission radius is determined by the distance to its furthest neighbour. This model and, in particular,

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- *S. Durocher is with the Department of Computer Science, University of Manitoba, Winnipeg, Canada.*
  E-mail: durocher@cs.umanitoba.ca
- *A. Haghnegahdar is with the Department of Electrical and Computer Engineering, University of British Columbia, Vancouver, Canada.*
  E-mail: alirezah@ece.ubc.ca
- *M. Khabbazian is with the Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Canada.*
  E-mail: mkhabbazian@ualberta.ca
- *F. Kuhn is with the Department of Computer Science, University of Freiburg, Freiburg, Germany.*
  E-mail: kuhn@cs.uni-freiburg.de

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the interference minimization problem applied to this model are the focus of numerous publications and have generated significant recent research interest [2], [5], [9], [15], [22], [28], [30]–[32], [37], [38], [46]–[49]. Furthermore, a number of important algorithmic questions remain open with respect to interference minimization in this model. While models such as SINR arguably provide a more realistic physical representation of interference in a wireless network (e.g., [1], [19], [36], [40], [42]), algorithmic problems such as interference minimization are significantly more difficult to solve in the SINR model, motivating continued examination of interference minimization under both models. Finally, in some cases, a solution to a problem set in the model used in this work leads to an approximate solution to the corresponding problem under the SINR model (e.g., [31]). See Section 1.3 for a formal definition of the model and see Figure 1 for an example.

Given a set of points P in the plane, finding a connected graph on P that minimizes the maximum interference is NP-complete [9]. A polynomial-time algorithm exists that returns a solution with maximum interference $O(\sqrt{n})$, where $n = |P|$ [22]. Even in one dimension, for every n there exists a set of n points P such that any graph on P has maximum interference $\Omega(\sqrt{n})$ [48]. All such known examples involve specific constructions (i.e., exponential chains). We are interested in investigating a more realistic class of wireless networks: those whose node positions observe common random distributions that better model actual wireless ad hoc networks.

When nodes are positioned on a line (sometimes called the *highway model*), a simple heuristic is to assign to each node a radius of transmission that corresponds to the maximum of the distances to its respective nearest neighbours to the left and right. In the worst case, such a strategy can result in $\Theta(n)$ maximum interference when an optimal solution has only $\Theta(\sqrt{n})$ maximum interference [48]. Recently, Kranakis et al. [32] showed that if n nodes are positioned uniformly at random on an interval, then the maximum interference provided by this heuristic is $\Theta(\sqrt{\log n})$ with high probability.

1.2 Overview of Results

In this paper, we examine the corresponding interference minimization problems in two and higher dimensions. We generalize the nearest-neighbour path used in the highway model to the Euclidean minimum spanning tree (MST), and show that with high probability, the maximum interference of the MST of a set of n points selected uniformly at random over a d-dimensional region $[0,1]^d$ is $O(\log n)$, for any fixed $d \geq 1$. Our techniques differ significantly from those used by Kranakis et al. [32] to achieve their results in one dimension. As we show in Section 3, our results also apply to a broad class of random distributions, denoted $\mathcal{D}$, that includes both the uniform random distribution and realistic distributions for modelling random motion in mobile wireless networks, as well as to a large class of connected spanning graphs that includes the MST.

In Section 3.4 we present a local algorithm that constructs a topology whose maximum interference is $O(\log n)$ with high probability when node positions are selected according to a distribution in $\mathcal{D}$. Previous local algorithms for topology control (e.g., the cone-based local algorithm (CBTC) [33]) attempt to reduce transmission radii (i.e., power consumption), but not necessarily the maximum interference. Similarly, others attempt to minimize interference but do not guarantee connectivity (e.g., the k-neighbours algorithm [6]). Although reducing transmission radii at many nodes is often necessary to reduce the maximum interference, the two objectives differ; specifically, some nodes may require large transmission radii to minimize the maximum interference. Ours is the first local algorithm to provide a non-trivial upper bound on maximum interference. Our algorithm can be applied to any existing topology to refine it and further reduce its maximum interference. Consequently, our solution can be used either independently, or paired with another topology control strategy. Section 6 presents the analysis of an empirical evaluation of our algorithm with a suite of simulation results on static, mobile, and real GPS track data.

In Section 4 we consider the problem of minimizing the average interference and show that the expected interference of the MST of a set of n points selected uniformly at random over the unit d-cube $[0,1]^d$ is $O(1)$.

In Section 5 we briefly examine the worst-case maximum interference, i.e., points are not necessarily drawn from a random distribution and may be positioned adversarially. We disprove a conjecture of Devroye and Morin [15] relating the maximum interference of the Euclidean minimum spanning tree to the optimal maximum interference attainable. We do so by constructing a set P of n points on the line and show that every connected communication graph on P has maximum interference $\Omega(\sqrt{n})$ and, furthermore, that the MST of P has maximum interference $O(\sqrt{n})$.

1.3 Model and Definitions

We represent the position of a wireless node as a point in Euclidean space, $\mathbb{R}^d$, for some fixed[1] $d \geq 1$. For simplicity, we refer to each node by its corresponding point. Similarly, we represent a wireless network by its communication graph, a geometric graph whose vertices are a set of points $P \subseteq \mathbb{R}^d$. Given a (simple and undirected) graph G, we employ standard graph-theoretic notation, where $V(G)$ denotes the vertex set of G and $E(G)$ denotes[2] its edge set. We say vertices u and v are *k-hop neighbours* if there is a simple path of length k from

1. In the majority of instances, two or three dimensions suffice to model an actual wireless network. Our results are presented in terms of an arbitrary d since this permits expressing a more general result without increasing the complexity of the corresponding notation.

2. Note, $E(G)$ denotes the edge set of a graph G, whereas $\mathbb{E}[X]$ denotes the expected value of the random variable X.

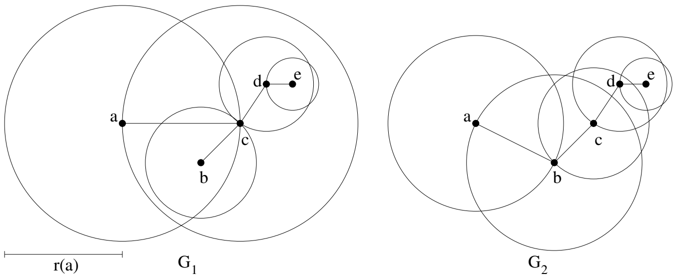


Fig. 1: A set of points $P = \{a, b, c, d, e\}$ in $\mathbb{R}^2$ and two communication graphs on P, denoted G_1 and G_2, illustrating radii of transmission by their corresponding discs. Nodes b and c can communicate in G_1 and G_2 because $\mathrm{dist}(b, c) \le \min\{r(b), r(c)\}$. Node b receives interference from node a in G_1 and G_2 because $\mathrm{dist}(a, b) \le r(a)$, but the two nodes cannot communicate in G_1 because $\mathrm{dist}(a, b) > r(b)$. The maximum interference in G_1 is 4, achieved at node c. The maximum interference in G_2 is 3, achieved at nodes b, c, and d. G_2 is an optimal solution for P, i.e., $\mathrm{OPT}(P) = 3$.

u to v in G. When $k = 1$ we say u and v are neighbours. The *k-hop neighbourhood* of a node u is the union of the sets of its k'-hop neighbours for all $k' \le k$.

We assume each node has a range of communication that is equal in every direction (i.e., a *radius of transmission*), that different nodes can have different transmission radii, and we consider bidirectional communication links, each of which is represented by an undirected graph edge connecting two nodes. Specifically, each node p has some *radius of transmission*, denoted by the function $r : P \to \mathbb{R}^+$, such that a node q receives a signal from p (possibly interference) if and only if $\mathrm{dist}(p, q) \le r(p)$, where $\mathrm{dist}(p, q) = \|p - q\|_2$ denotes the Euclidean distance between points p and q in $\mathbb{R}^d$. Similarly, a node q can communicate with p if and only if $\mathrm{dist}(p, q) \le \min\{r(p), r(q)\}$. Interference at a node q is defined by the number of nodes from which it can receive a signal, whereas connectivity in the communication graph is determined by the nodes with which q can communicate. For simplicity, suppose each node has an infinite radius of reception, regardless of its radius of transmission; that is, a node q can receive interference from any node p if $r(p)$ is sufficiently large, regardless of $r(q)$. See Figure 1 for an example.

Definition 1 (Communication Graph)**.** *A graph G is a* communication graph *with respect to a point set $P \subseteq \mathbb{R}^d$ and a function $r : P \to \mathbb{R}^+$ if*

1) *$V(G) = P$, and*
2) *for all vertices p and q in $V(G)$,*

$$\{p, q\} \in E(G) \Leftrightarrow \mathrm{dist}(p, q) \le \min\{r(p), r(q)\}. \tag{1}$$

Together, set P and function r uniquely determine the corresponding communication graph G. Alternatively, a communication graph can be defined as the closure of a given embedded graph. Specifically, if instead of being given P and r, we are given an arbitrary graph H embedded in $\mathbb{R}^d$, then the set P is trivially determined by $V(H)$ and a transmission radius for each node $p \in V(H)$ can be assigned to satisfy (1) by

$$r(p) = \max_{q \in \mathrm{Adj}(p)} \mathrm{dist}(p, q), \tag{2}$$

where $\mathrm{Adj}(p) = \{q \mid \{q, p\} \in E(H)\}$ denotes the set of vertices adjacent to p in H. The communication graph determined by H is the unique edge-minimal supergraph of H that satisfies Definition 1. We denote this graph by H' and refer to it as the *closure* of graph H. Therefore, a communication graph G can be defined either as a function of a set of points P and an associated mapping of transmission radii $r : P \to \mathbb{R}^+$, or as the closure of a given embedded graph H (where $G = H'$).

Definition 2 (Interference)**.** *Given a communication graph G, the* interference *at a node $p \in V(G)$ or at a point $p \in \mathbb{R}^d$ is*

$$\mathrm{inter}_G(p) = |\{q \mid q \in V(G),\ \mathrm{dist}(q, p) \le r(q)\}|,$$

the maximum interference *of G is*

$$\mathrm{inter}(G) = \max_{p \in V(G)} \mathrm{inter}_G(p),$$

and the average interference *of G is*

$$\mathrm{interAvg}(G) = \frac{1}{|V(G)|} \sum_{p \in V(G)} \mathrm{inter}_G(p).$$

In other words, the interference at p, denoted $\mathrm{inter}_G(p)$, is the number of nodes q such that p lies within q's radius of transmission[3]. This does not imply the existence of the edge $\{p, q\}$ in the corresponding communication graph G; such an edge exists if and only if the relationship is reciprocal, i.e., q also lies within p's radius of transmission.

Given a point set P, let $\mathcal{G}(P)$ denote the set of connected communication graphs on P. Let $\mathrm{OPT}(P)$ denote the optimal maximum interference attainable over graphs in $\mathcal{G}(P)$. That is,

$$\begin{aligned} \mathrm{OPT}(P) &= \min_{G \in \mathcal{G}(P)} \mathrm{inter}(G) \\ &= \min_{G \in \mathcal{G}(P)} \max_{p \in V(G)} \mathrm{inter}_G(p). \end{aligned}$$

Similarly, let $\mathrm{OPTAvg}(P)$ denote the optimal average interference attainable over graphs in $\mathcal{G}(P)$. That is,

$$\mathrm{OPTAvg}(P) = \min_{G \in \mathcal{G}(P)} \mathrm{interAvg}(G).$$

Thus, given a set of points P representing the positions of wireless nodes, the *maximum interference minimization problem* is to find a connected communication graph G on P that spans P such that the maximum interference

3. In some definitions of interference a node cannot cause interference with itself. When p is a node in $V(G)$, the respective values of interference for the two definitions differ by an additive factor of one. We include p in the tally to allow a more general measure of interference whose definition applies consistently at any point p in $\mathbb{R}^d$, regardless of whether p coincides with a node in $V(G)$.

is minimized (i.e., its maximum interference is $\mathrm{OPT}(P)$). Similarly, the *average interference minimization problem* is to find a connected communication graph G on P that spans P such that the average interference is minimized (i.e., its average interference is $\mathrm{OPTAvg}(P)$).

We examine the maximum and average interference of the communication graph determined by the closure of $\mathrm{MST}(P)$, where $\mathrm{MST}(P)$ denotes the Euclidean minimum spanning tree of the point set P. Our results apply with *high probability*, which refers to probability at least $1 - n^{-c}$, where $n = |P|$ denotes the number of network nodes and $c \geq 1$ is an arbitrary fixed constant.

2 Related Work

2.1 Minimizing Maximum Interference under the Bidirectional Model

We consider the bidirectional interference model (defined in Section 1.3). This model was introduced by von Rickenbach et al. [48], who gave a polynomial-time approximation algorithm that finds a solution with maximum interference $O(n^{1/4} \cdot \mathrm{OPT}(P))$ for any given set of points P on a line, and a one-dimensional construction showing that $\mathrm{OPT}(P) \in \Omega(\sqrt{n})$ in the worst case, where $n = |P|$. Halldórsson and Tokuyama [22] gave a polynomial-time algorithm that returns a solution with maximum interference $O(\sqrt{n})$ for any given set of n points in the plane. Buchin [9] showed that finding an optimal solution (one whose maximum interference is exactly $\mathrm{OPT}(P)$) is NP-complete in the plane. Tan et al. [47] gave an $O(n^3 n^{O(\mathrm{OPT}(P))})$-time algorithm for finding a solution with interference $\mathrm{OPT}(P)$ for any given set of points P on a line. Kranakis et al. [32] showed that for any set of n points P selected uniformly at random from the unit interval, the nearest-neighbour path $(\mathrm{MST}(P)')$ has maximum interference $\Theta(\sqrt{\log n})$ with high probability. Sharma et al. [46] consider heuristic solutions to the two-dimensional problem. Finally, recent results by Devroye and Morin [15] extend some of the results presented in this paper and answer a number of open questions definitively to show that with high probability, when P is a set of n points in $\mathbb{R}^d$ selected uniformly at random from $[0,1]^d$, $\mathrm{inter}(\mathrm{MST}(P)') \in \Theta((\log n)^{1/2})$, $\mathrm{OPT}(P) \in O((\log n)^{1/3})$, and $\mathrm{OPT}(P) \in \omega((\log n)^{1/4})$.

2.2 Minimizing Maximum Interference under the Unidirectional Model

If communication links are not bidirectional (i.e., edges are directed) and the communication graph is required to be strongly connected, then the worst-case maximum interference decreases. Under this model, von Rickenbach et al. [49] and Korman [30] give polynomial-time algorithms that return solutions with maximum interference $O(\log n)$ for any given set of points in the plane, and a one-dimensional construction showing that in the worst case $\mathrm{OPT}(P) \in \Omega(\log n)$. Korman also shows that for any P there exists a solution with interference $O(\mathrm{OPT}(P))$ for which no node requires a radius of transmission larger than the length of the longest edge in $\mathrm{MST}(P)$. Bilò and Proietti [5] show that no polynomial-time $o(\log n)$-approximation algorithm is possible unless $\mathrm{NP} \in \mathrm{DTIME}(n^{\log \log n})$ for any given set of n points in a general metric space. Bilò and Proietti extend this lower bound to the bidirectional interference model.

2.3 Minimizing Average Interference

In addition to results that examine the problem of minimizing the maximum interference, some work has addressed the problem of minimizing the average interference. Lou et al. [37] give respective algorithms for finding a set of radii that minimizes the average interference for any set P of n points in $O(n^3)$ time if $P \subseteq \mathbb{R}$ and $n^{O(n \log(d_{\max}/d_{\min}))}$ time if $P \subseteq \mathbb{R}^2$, where $d_{\max}(G)$ and $d_{\min}(G)$ are defined as in Theorem 3. Moscibroda and Wattenhofer [39] give a polynomial-time $O(\log n)$-approximation algorithm for any set of n points in a general metric space and show that no polynomial-time $o(\log n)$-approximation algorithm is possible unless $\mathrm{NP} \in \mathrm{DTIME}(n^{\log \log n})$.

3 Minimizing Maximum Interference in Random Networks

3.1 Generalizing One-Dimensional Solutions

Before presenting our results on random sets of points, we begin with a brief discussion regarding the possibility of generalizing existing algorithms that provide approximate solutions for one-dimensional instances of the maximum interference minimization problem (in an adversarial deterministic input setting).

Since the problem of identifying a graph that achieves the optimal (minimum) interference is NP-hard in two or more dimensions [9], it is natural to ask whether one can design a polynomial-time algorithm to return a good approximate solution. Although von Rickenbach et al. [48] give a $\Theta(n^{1/4})$-approximate algorithm in one dimension [48], the current best polynomial-time algorithm in two (or more) dimensions by Halldórsson and Tokuyama [22] returns a solution with maximum interference $O(\sqrt{n})$; as noted by Halldórsson and Tokuyama, this algorithm is not known to guarantee any approximation factor better than the immediate bound of $O(\sqrt{n})$. The algorithm of von Rickenbach et al. uses two strategies for constructing respective communication graphs, and returns the graph with the lower maximum interference; an elegant argument that depends on Lemma 1 bounds the resulting worst-case maximum interference by $\Theta(n^{1/4} \cdot \mathrm{OPT}(P))$. The two strategies correspond roughly to a) $\mathrm{MST}(P)'$ and b) classifying every $\sqrt{n}$th node as a hub, joining each hub to its left and right neighbouring hubs to form a network backbone, and connecting each remaining node to its closest hub. The algorithm of Halldórsson and Tokuyama applies ϵ-nets, resulting in a strategy that is loosely analogous to a generalization of the hub strategy of von Rickenbach et al. to higher dimensions. One might wonder whether the hybrid approach of von Rickenbach et al. might be applicable in higher dimensions by returning $\mathrm{MST}(P)'$ or the communication graph constructed by the algorithm of Halldórsson and Tokuyama, whichever has lower maximum interference. To apply this idea directly would require generalizing the following property established by von Rickenbach et al. to higher dimensions:

Lemma 1 (von Rickenbach et al. [48] (2005))**.** *For any set of points $P \subseteq \mathbb{R}$,*

$$\mathrm{OPT}(P) \in \Omega\left(\sqrt{\mathrm{inter}(\mathrm{MST}(P)')}\right).$$

However, von Rickenbach et al. also show that for any n, there exists a set of n points $P \subseteq \mathbb{R}^2$ such that $\mathrm{OPT}(P) \in O(1)$ and $\mathrm{inter}(\mathrm{MST}(P)') \in \Theta(n)$, which implies that Lemma 1 does not hold in higher dimensions. Consequently, techniques such as those used by von Rickenbach et al. do not immediately generalize to higher dimensions.

3.2 Randomized Point Sets

Although using the hybrid approach of von Rickenbach et al. [48] directly may not be possible, Kranakis et al. [32] recently showed that if a set P of n points is selected uniformly at random from an interval, then the maximum interference of the communication graph determined by $\mathrm{MST}(P)'$ is $\Theta(\sqrt{\log n})$ with high probability. Here we show that if points are selected randomly from d-dimensional Euclidean space, where $d \in O(1)$, the maximum interference of $\mathrm{MST}(P)'$ is $O(\log n)$ with high probability. We start with some basic definitions.

Definition 3 (Primitive Edge)**.** *Assume that a communication graph G is the closure of some embedded graph H. An edge $\{p, q\} \in E(G)$ is called* primitive *with respect to H if $\{p, q\} \in E(H)$ and $\min\{r(p), r(q)\} = \mathrm{dist}(p, q)$.*

Observe that because G is the closure of H, the radius of any node u is equal to the distance to its farthest neighbour in H and therefore, every node is incident to at least one primitive edge.

Definition 4 (Bridge)**.** *An edge $\{p, q\} \in E(G)$ in a communication graph G is* bridged *if there is a path joining p and q in G consisting of at most three edges distinct from $\{p, q\}$ such that for each of the three edges $\{x, y\}$, $\mathrm{dist}(x, y) < \mathrm{dist}(p, q)$, or $\mathrm{dist}(x, y) = \mathrm{dist}(p, q)$ and $\{x, y\}$ is primitive.*

Given a set of points P in $\mathbb{R}^d$, let $\mathcal{T}(P)$ denote the set of all communication graphs G with $V(G) = P$ such that G is the closure of some embedded graph H and such that no primitive edge in $E(G)$ is bridged.

Further, let $\mathcal{C}(R, r, d)$ be the minimum number of d-dimensional balls of radius r required to cover a d-dimensional ball of radius R. The following property holds since $\mathbb{R}^d$ is a doubling metric space for any constant d [23] (equivalently, $\mathbb{R}^d$ has constant doubling dimension [17], [20]):

Proposition 2. *If $d \in \Theta(1)$ and $R/r \in \Theta(1)$, then $\mathcal{C}(R, r, d) \in \Theta(1)$.*

For a given communication graph G, we define $d_{\max}(G)$ and $d_{\min}(G)$ as the lengths of the longest and shortest edges of G, respectively. That is, $d_{\max}(G) := \max_{\{s,t\}\in E(G)} \mathrm{dist}(s, t)$ and $d_{\min}(G) = \min_{\{s,t\}\in E(G)} \mathrm{dist}(s, t)$. Halldórsson and Tokuyama [22], Maheshwari et al. [38], and Lou et al. [37] give centralized algorithms for constructing graphs G, each with maximum interference $O(\log(d_{\max}(G)/d_{\min}(G)))$. As we show in Theorem 3, this bound holds for any graph G in the class $\mathcal{T}(P)$. In Section 3.4 we describe a local algorithm for constructing a connected graph in $\mathcal{T}(P)$ on any given point set P.

Theorem 3. *Let P be a set of points in $\mathbb{R}^d$. For any graph $G \in \mathcal{T}(P)$,*

$$\mathrm{inter}(G) \in O\left(\log\left(\frac{d_{\max}(G)}{d_{\min}(G)}\right)\right).$$

Proof: We first normalize the scale of P to simplify the proof. Let $Q = \{p\cdot\alpha \mid p \in P\}$ denote a uniform scaling of P by a factor of $\alpha = 1/d_{\min}(G)$ and let H denote the corresponding communication graph. That is, $\{u, v\} \in E(G) \Leftrightarrow \{u\cdot\alpha, v\cdot\alpha\} \in E(H)$. Similarly, scale transmission radii such that each node's transmission radius in Q is α times its corresponding node's transmission radius in P. Thus,

$$d_{\min}(H) = 1 \quad \text{and} \quad d_{\max}(H) = \frac{d_{\max}(G)}{d_{\min}(G)}. \tag{3}$$

Consider some node p and let U be the set of nodes that cause interference at p, i.e., $U = \{u \in V : \mathrm{dist}(u, p) \le r(u)\}$. Let $g = \lceil \log d_{\max}(H) \rceil$. We partition the set U into $g+1$ subsets $U_0, U_1, \ldots, U_g$, such that for each $0 \le i \le g$, $U_i = \{u \in U : r(u) \in [2^i, 2^{i+1})\}$. We will show that for all $i \in \{0, \ldots, g\}$,

$$|U_i| \le \mathcal{C}(2^{i+1}, 2^{i-2}, d) \cdot \mathcal{C}(2^{i+2}, 2^{i-2}, d). \tag{4}$$

Applying Proposition 2, this implies $|U_i| \in O(1)$ and thus $|U| \in O(\log(d_{\max}(H)))$, from which the claim of the theorem follows.

Let us therefore fix some $i \in \{0, \ldots, g\}$ and assume for the sake of contradiction that (4) does not hold. First recall that every node $v \in V$ is adjacent to some primitive edge of length $r(v)$. Hence, every node $u \in U_i$ is adjacent to some primitive edge of length $r(u) \in [2^i, 2^{i+1})$. We can thus define a mapping $\omega : U_i \to V$ such that for every $u \in U_i$, $\{u, \omega(u)\}$ is a primitive edge of length $r(u) \in [2^i, 2^{i+1})$. Also note that all nodes in U_i are contained in a ball with centre p and radius 2^{i+1}. By Proposition 2, this ball can be covered with $\mathcal{C}(2^{i+1}, 2^{i-2}, d)$ balls of radius 2^{i-2}. Thus, because we assume (for contradiction) that (4) does not hold, by the pigeonhole principle, there must be a ball B_i of radius 2^{i-2} that contains a set U_i' of at least $\mathcal{C}(2^{i+2}, 2^{i-2}, d) + 1$ nodes from U_i. We define $W_i := \{\omega(u) : u \in U_i'\}$. Because nodes in U_i' are in ball B_i

(of radius 2^{i-2}) and for all $u \in U_i' \subseteq U_i$, $\mathrm{dist}(u, \omega(u)) \geq 2^i$, we have

$$W_i \cap U_i' = \varnothing. \tag{5}$$

We consider two cases: i) there are two nodes $u_1, u_2 \in U_i'$ such that $\omega(u_1) = \omega(u_2)$, and ii) for any two nodes $u_1, u_2 \in U_i'$, $\omega(u_1) \neq \omega(u_2)$.

CASE 1. We define $w := \omega(u_1) = \omega(u_2)$. Without loss of generality, assume that $\mathrm{dist}(u_1, w) \leq \mathrm{dist}(u_2, w)$. Because u_1 and u_2 are both in ball B_i, $\mathrm{dist}(u_1, u_2) \leq 2^{i-1}$ and therefore $\mathrm{dist}(u_1, u_2) < \mathrm{dist}(u_1, w) \leq \mathrm{dist}(u_2, w)$. Since $\{u_1, w\}$ is a primitive edge, the edge $\{u_2, w\}$ is bridged. Because also $\{u_2, w\}$ is a primitive edge, this is a contradiction to the assumption that $G \in \mathcal{T}(P)$.

CASE 2. We have $|W_i| = |U_i'| \geq \mathcal{C}(2^{i+2}, 2^{i-2}, d) + 1$. Since every node in W_i lies in a ball of radius $2^{i-2} + 2^{i+1} < 2^{i+2}$, and because a ball of radius 2^{i+2} can be covered with $\mathcal{C}(2^{i+2}, 2^{i-2}, d) + 1$ balls of radius 2^{i-2}, there must be a ball of radius 2^{i-2} that contains at least two nodes w_1 and w_2 from W_i. Assume that $u_1, u_2 \in U_i'$ such that $\omega(u_1) = w_1$ and $\omega(u_2) = w_2$. Without loss of generality, assume that either $\mathrm{dist}(u_2, w_2) \geq \mathrm{dist}(u_1, w_1) \geq 2^i$. Because $\mathrm{dist}(u_1, u_2) \leq 2^{i-1}$, $\mathrm{dist}(w_1, w_2) \leq 2^{i-1}$, and because $\{u_1, w_1\}$ is primitive, this implies that $\{u_2, w_2\}$ is bridged. Because $\{u_2, w_2\}$ is a primitive edge too, this is a contradiction to the assumption that $G \in \mathcal{T}(P)$.

A contradiction is derived in both cases. Therefore, (4) holds and thus the claim of the theorem follows. □

In the next lemma, we show that $\mathrm{MST}(P)'$ is in $\mathcal{T}(P)$. Consequently, $\mathcal{T}(P)$ always includes a connected communication graph. To ensure that $\mathrm{MST}(P)$ is unique we assume a global order $\prec$ on the set of edges such that for any four nodes a, b, c, and d, $\mathrm{dist}(a, b) < \mathrm{dist}(c, d) \Rightarrow \{a, b\} \prec \{c, d\}$. E.g., it suffices to compare the coordinates of each edge's endpoints lexicographically when $\mathrm{dist}(a, b) = \mathrm{dist}(c, d)$. We assume that $\mathrm{MST}(P)$ is the unique minimum spanning tree that is also minimal with respect to the global order $\prec$.

Lemma 4. *For any set of points $P \subseteq \mathbb{R}^d$, $\mathrm{MST}(P)' \in \mathcal{T}(P)$.*

Proof: By definition, all primitive edges are edges of $\mathrm{MST}(P)$. Suppose there is a primitive edge $\{p, q\} \in E(\mathrm{MST}(P))$ that is bridged. Therefore, there is a path T from p to q in $\mathrm{MST}(P)'$ that contains at most three edges, such that for each edge $\{x, y\} \neq \{p, q\}$ of T, $\mathrm{dist}(x, y) < \mathrm{dist}(p, q)$ or $\mathrm{dist}(x, y) = \mathrm{dist}(p, q)$ and $\{x, y\}$ is primitive and thus also $\{x, y\} \in E(\mathrm{MST}(P))$. Assume that $\{p', q'\}$ is the largest edge in T with respect to the global order $\prec$. We must have $\mathrm{dist}(p', q') = \mathrm{dist}(p, q)$, as otherwise all edges in T will be smaller than $\{p, q\}$ with respect to the order $\prec$, hence a smaller MST can be constructed by replacing $\{p, q\}$ with one of edges of T. Since $\{p, q\}$ is bridged by T, the edge $\{p', q'\}$ has to be a primitive edge and therefore $\{p', q'\} \in E(\mathrm{MST}(P))$. However, this is a contradiction to the assumption that $\mathrm{MST}(P)$ is the minimal MST with respect to the order $\prec$ as it is possible to get a smaller MST by replacing $\{p', q'\}$ with another edge of T. □

Theorem 3 implies that the interference of any graph G in $\mathcal{T}(P)$ is bounded asymptotically by the logarithm of the ratio of the longest and shortest edges in G. While this ratio can be arbitrarily large in the worst case, we show that the ratio is bounded for many typical distributions of points. Specifically, if the ratio is $O(n^c)$ for some constant c, then the maximum interference is $O(\log n)$.

Definition 5 ($\mathcal{D}$)**.** *Let $\mathcal{D}$ denote the class of distributions over $[0,1]^d$ such that for any $D \in \mathcal{D}$ and any set P of $n \geq 2$ points selected independently at random according to D, the minimum distance between any two points in P is greater than n^{-c} with high probability, for some constant c (independent of n).*

Corollary 5. *For any integers $d \geq 1$ and $n \geq 2$, any distribution $D \in \mathcal{D}$, and any set P of n points, each of which is selected independently at random over $[0,1]^d$ according to distribution D, with high probability, for all graphs $G \in \mathcal{T}(P)$, $\mathrm{inter}(G) \in O(\log n)$.*

Proof: Let $d_{\min}(G) = \min_{\{s,t\} \in E(G)} \mathrm{dist}(s, t)$ and $d_{\max}(G) = \max_{\{s,t\} \in E(G)} \mathrm{dist}(s, t)$. Since points are contained in $[0,1]^d$, $d_{\max}(G) \leq \sqrt{d}$. Points in P are distributed according to a distribution $D \in \mathcal{D}$. By Definition 5, with high probability, $d_{\min}(G) \geq n^{-c}$ for some constant c. Thus, with high probability, we have

$$\log\left(\frac{d_{\max}(G)}{d_{\min}(G)}\right) \leq \log\left(\frac{\sqrt{d}}{n^{-c}}\right). \tag{6}$$

The result follows from (6), Theorem 3, and the fact that $\log(n^c\sqrt{d}) \in O(\log n)$ when d and c are constant. □

Lemma 6. *Let D be a distribution with domain $[0,1]^d$, for which there is a constant c' such that for any point $x \in [0,1]^d$, we have $D(x) \leq c'$, where $D(x)$ denotes the probability density function of D at $x \in [0,1]^d$. Then $D \in \mathcal{D}$.*

Proof: Let $p_1, p_2, \ldots, p_n$, be $n \geq 2$ independent random points in $[0,1]^d$ with distribution D. Let $c'' = 1 + \frac{\log c' + 2}{d}$ and let $\mathcal{E}_i$, $1 \leq i \leq n$, denote the event that there is a point p_j, $j \neq i$, such that $\mathrm{dist}(p_i, p_j) \leq n^{-c''}$. Let the random variable $d_{\min}$ be equal to $\min_{i \neq j} \mathrm{dist}(p_i, p_j)$. We have

$$\Pr(d_{\min} \leq n^{-c''}) = \Pr\left(\bigvee_{1 \leq i \leq n} \mathcal{E}_i\right) \leq \sum_{1 \leq i \leq n} \Pr(\mathcal{E}_i), \tag{7}$$

where the inequality holds by the union bound. To establish an upper bound on $\Pr(\mathcal{E}_i)$, consider a d-dimensional ball B_i with centre p_i and radius $n^{-c''}$. The probability that there is point p_j, $j \neq i$, in that ball is at most c' times the volume of $B_i \cap [0,1]^d$. The volume of $B_i \cap [0,1]^d$ is at most $(2n^{-c''})^d$. Therefore, $\Pr(\mathcal{E}_i) \leq c'(2n^{-c''})^d$ for every

$1 \leq i \leq n$. Thus, by (7), we get

$$\begin{aligned}
\Pr(d_{\min} > n^{-c''}) &\geq 1 - \sum_{1 \leq i \leq n} \Pr(\mathcal{E}_i) \\
&\geq 1 - n \cdot c' \left(2n^{-c''}\right)^d \\
&= 1 - \frac{c' 2^d}{n^{d+\log c' + 1}} \\
&\geq 1 - \frac{c' 2^d}{n \cdot 2^{d+\log c'}} \\
&= 1 - \frac{1}{n} \ .
\end{aligned}$$

Therefore, $D \in \mathcal{D}$. Note, here $c = c''$ in Definition 5. □

Corollary 7. *The uniform distribution with domain $[0,1]^d$ is in $\mathcal{D}$.*

By Corollaries 5 and 7 we can conclude that if a set P of $n \geq 2$ points is distributed uniformly in $[0,1]^d$, then with high probability, any communication graph in $G \in \mathcal{T}(P)$ will have maximum interference $O(\log n)$. This is expressed formally in the following corollary:

Corollary 8. *Choose any integers $d \geq 1$ and $n \geq 2$. Let P be a set of n points, each of which is selected independently and uniformly at random over $[0,1]^d$. With high probability, for all graphs $G \in \mathcal{T}(P)$, $\mathrm{inter}(G) \in O(\log n)$.*

3.3 Mobility

Our results apply to the setting of mobility (e.g., mobile ad hoc wireless networks). Each node in a mobile network must periodically exchange information with its neighbours to update its local data storing positions and transmission radii of nodes within its local neighbourhood. The distribution of mobile nodes depends on the mobility model, which is not necessarily uniform. For example, when the network is distributed over a disc or a box-shaped region, the probability distribution associated with the random waypoint model (RWP) [25] achieves its maximum at the centre of the region, whereas the probability of finding a node close to the region's boundary approaches zero [24]. Since the maximum value of the probability distribution associated with a RWP model is constant [24], by Corollary 5 and Lemma 6 we can conclude that at any point in time, the maximum interference of the network is $O(\log n)$ with high probability. In general, this holds for any random mobility model whose corresponding probability distribution has a constant maximum value.

3.4 Local Algorithm

As discussed in Section 1.1, existing local algorithms for topology control attempt to reduce transmission radii, but not necessarily the maximum interference. By Lemma 4 and Corollary 5, if P is a set of n points selected according to a distribution in $\mathcal{D}$, then with high probability $\mathrm{inter}(\mathrm{MST}(P)') \in O(\log n)$. Unfortunately, a minimum spanning tree cannot be generated using only local information [29]. Thus, an interesting question is whether each node can assign itself a transmission radius using only local information such that the resulting communication graph belongs to $\mathcal{T}(P)$ while remaining connected. We answer this question affirmatively by presenting a local algorithm (LOCALRADIUSREDUCTION), that assigns a transmission radius to each node such that if an initial communication graph $G_{\max}$ is connected, then the resulting communication graph is a connected spanning subgraph of $G_{\max}$ that belongs to $\mathcal{T}(P)$. Consequently, the resulting topology has maximum interference $O(\log n)$ with high probability when nodes are selected according to any distribution in $\mathcal{D}$. Our algorithm can be applied to any existing topology to refine it and reduce its maximum interference. Thus, our solution can be used either independently, or paired with another topology control strategy.

For the distributed algorithm, we assume that each edge e has a unique identifier $\mathrm{ID}(e)$. For example, these can be obtained locally by using unique node identifiers. The edge identifiers allow to define a global order $\prec$ on all the possible edges of the communication graph as follows. For any two edges $\{u_1, v_1\}, \{u_2, v_2\} \in E(G_{\max})$, we have $\{u_1, v_1\} \prec \{u_2, v_2\}$ if and only if $\mathrm{dist}(u_1, v_1) < \mathrm{dist}(u_2, v_2)$ or $\mathrm{dist}(u_1, v_1) = \mathrm{dist}(u_2, v_2)$ and $\mathrm{ID}(\{u_1, v_1\}) < \mathrm{ID}(\{u_2, v_2\})$.

Let P be a set of $n \geq 2$ points in $\mathbb{R}^d$ and let $r_{\max} : P \to \mathbb{R}^+$ be a function that returns the maximum transmission radius allowable at each node. Let $G_{\max}$ denote the communication graph determined by P and $r_{\max}$. We suppose that $G_{\max}$ is connected. Further, assume that $\mathrm{Adj}_{\max}(u)$ is the set of neighbours of a node u in $G_{\max}$. Algorithm LOCALRADIUSREDUCTION assumes that each node is initially aware of its maximum transmission radius, its spatial coordinates, and its unique identifier.

The algorithm begins with a local data acquisition phase, during which every node broadcasts its identity, maximum transmission radius, and coordinates in a node data message. Each message also specifies whether the data is associated with the sender or whether it is forwarded from a neighbour. Every node records the node data it receives and retransmits those messages that were not previously forwarded. Upon completing this phase, each node is aware of the corresponding data for all nodes within its 2-hop neighbourhood. The algorithm then proceeds to a local transmission radius reduction phase, which does not require any additional communication. Consequently, each node only requires knowledge of its 2-hop neighbourhood and the algorithm is local.

We say that an edge $\{u, v\}$ of $G_{\max}$ is *redundant* iff there is a path at most 3 connecting u and v such that for every edge $\{x, y\}$ of the path, we have $\{x, y\} \prec \{u, v\}$. Let H be the graph consisting of all edges of $G_{\max}$ that are not redundant. The communication graph G is defined as the closure of graph H, i.e., $G = H'$. Consequently, each node u chooses $r(u)$ to be the largest distance to a neighbour $v \in \mathrm{Adj}_{\max}(u)$ such that $\{u, v\}$ is not

LocalRadiusReduction(u)
$r'(u) \leftarrow 0$
for each $v \in \mathrm{Adj}_{\max}(u)$ **do**
  if $\mathrm{dist}(u, v) > r'(u)$ **and** $\neg$Redundant(u, v) **then**
    $r'(u) \leftarrow \mathrm{dist}(u, v)$
return $r'(u)$

Alg. 1: LocalRadiusReduction(u)

Redundant(a, b)
$result \leftarrow$ **false**
for each $v \in \mathrm{Adj}_{\max}(a)$ **do**
  if $v \in \mathrm{Adj}_{\max}(b)$ **and**
    $\{a, v\} \prec \{a, b\}$ **and** $\{v, b\} \prec \{a, b\}$ **then**
    $result \leftarrow$ **true**
  for each $w \in \mathrm{Adj}_{\max}(v)$ **do**
    if $w \in \mathrm{Adj}_{\max}(b)$ **and** $\{a, v\} \prec \{a, b\}$ **and**
      $\{v, w\} \prec \{a, b\}$ **and** $\{w, b\} \prec \{a, b\}$ **then**
      $result \leftarrow$ **true**
return $result$

Alg. 2: Redundant(a, b)

redundant. The details are given in Algorithm 1.

Algorithm LocalRadiusReduction is 2-local, that is, each node only needs to learn about the initial state of nodes at distance at most 2 in $G_{\max}$. Further, the local computation time at each node is bounded by $O(\Delta^3)$, where Δ denotes the maximum vertex degree in $G_{\max}$. Each call to the subroutine Bridge costs at most $O(\Delta^2)$ time and there are at most Δ calls to Bridge from each node.

Theorem 9. *The communication graph constructed by Algorithm* LocalRadiusReduction *is in $\mathcal{T}(P)$ and it is connected if the initial communication graph $G_{\max}$ is connected.*

Proof: Let G denote the communication graph constructed by Algorithm LocalRadiusReduction. We first prove that G is in $\mathcal{T}(P)$. By construction, G is the closure of the graph H consisting of all edges of $G_{\max}$ that are not redundant. An edge $\{u, v\}$ is redundant if it is the largest with respect to the global order $\prec$ in some cycle of length at most 4 in $G_{\max}$. Consequently, H has no cycles of length less than 5. For contradiction, assume that there is an edge $\{u, v\} \in E(G)$ which is bridged and which is primitive (with respect to H). Then, there is a path T of length at most 3 that connects u and v such that for each edge $\{x, y\}$ of T, either $\mathrm{dist}(x, y) < \mathrm{dist}(u, v)$ or $\mathrm{dist}(x, y) = \mathrm{dist}(u, v)$ and $\{x, y\}$ is primitive (with respect to H). Hence, G contains a cycle of length at most 4 such that the longest edges of the cycle are all primitive (and thus also edges of H). This cannot be because from each such cycle of $G_{\max}$ the largest edge with respect to $\prec$ is not included in H.

It remains to prove that G is connected if $G_{\max}$ is connected. For contradiction, assume that G and therefore also H is not connected and consider a set $S \subset V, S \neq \varnothing$ such that H does not contain an edge between S and $V \setminus S$. Let $e = \{u, v\}$ be the smallest edge of $G_{\max}$ (with respect to $\prec$) over the cut $(S, V \setminus S)$. Edge $\{u, v\}$ cannot be redundant because every cycle of $G_{\max}$ that contains $\{u, v\}$ has to contain at least one other edge $\{u', v'\}$ across the cut $(S, V \setminus S)$ and by assumption $\{u, v\} \prec \{u', v'\}$. □

More generally, since transmission radii are only decreased, it can be shown that $G_{\min}$ and $G_{\max}$ have the same number of connected components by applying Theorem 9 on every connected component of $G_{\max}$.

4 Minimizing Average Interference in Random Networks

We now examine the problem of minimizing the average interference in a set of points whose positions are selected uniformly and independently at random over the unit square in the plane. To the authors' best knowledge, this work is the first to examine average interference in a random setting.

We refer to the following lemma by Wan et al., where the *radius* of a set $P \subseteq \mathbb{R}^d$ is

$$\mathrm{rad}_P = \min_{p \in P} \max_{q \in P} \mathrm{dist}(p, q).$$

Consequently, there exists a point $p \in P$ such that a disc of radius rad_P centred at p covers P [50].

Lemma 10 (Wan et al. [50] (2002))**.** *For any set $P \subseteq \mathbb{R}^2$ of points with radius one,*

$$\sum_{e \in E(\mathrm{MST}(P))} \|e\|^2 \leq 12.$$

In this section, we show that for any set P of points selected uniformly and independently at random in a unit square, $\mathbb{E}[\mathrm{interAvg}(\mathrm{MST}'(P))] \in O(1)$. Interestingly, this result does not hold for every distribution in $\mathcal{D}$ (see Definition 5). For clarity of the proofs, throughout this section, we assume that the distance between each pair of nodes is unique, i.e., that points in P are in general position.

Lemma 11 is similar to an earlier result of Penrose [43]. Penrose's result does not immediately imply Lemma 11 and, as such, we include our proof for completeness.

Lemma 11. *For any set $P \subseteq [0, 1]^2$ of n points selected uniformly and independently at random, with high probability, the longest edge in $\mathrm{MST}(P)$ has length $l_{\max} \in O\left(\sqrt{(\log n)/n}\right)$.*

Proof: Choose any real number $c \geq 2$. We divide $[0, 1]^2$ into a square grid with $\left\lfloor \sqrt{n/(c \log n)} \right\rfloor^2$ square cells, each with side length $\left\lfloor \sqrt{c \log n/n} \right\rfloor \geq \sqrt{c \log n/n}$. We say two distinct cells are adjacent if they share a side. The distance between any two points in adjacent cells is at most

$$l = \sqrt{5} \left\lfloor \sqrt{c \log n/n} \right\rfloor \in \Theta\left(\sqrt{\log n/n}\right).$$

Assume each cell contains at least one node. If we select one representative node in each cell, connect every node in each cell to its representative node, and connect representative nodes in adjacent cells, the resulting graph will be connected with a maximum edge length of l. If every cell contains a node, then the longest edge in $\mathrm{MST}(P)$ has length at most l. Therefore, to prove the lemma it suffices to show that every cell contains at least one node with high probability.

The probability that a given cell does not contain any node is at most

$$\begin{aligned}\left(1-\left\lfloor\sqrt{\frac{n}{c\log n}}\right\rfloor^{-2}\right)^{n} &\leq \left(1-\frac{c\log n}{n}\right)^{n}\\ &\leq \exp\left(-\frac{c\log n}{n}\cdot n\right)\\ &= n^{-c}.\end{aligned}$$

By a union bound, the probability that every cell contains at least one node is at least $1-n^{1-c}$, which completes the proof. □

Definition 6 (Communication Coverage)**.** *Given a communication graph G and any node $p \in V(G)$, the* communication coverage *of p is the region within the transmission range of node p, which we denote $\mathrm{cov}_G(p)$. That is $\mathrm{cov}_G(p)$ is the disc of radius $r(p)$ centred at p. We define the communication coverage of G as*

$$\mathrm{cov}(G) = \bigcup_{p\in V(G)} \mathrm{cov}_G(p).$$

The following lemma shows that for any set of points P, the average interference within $\mathrm{cov}(\mathrm{MST}'(P))$ is constant. Note that this result applies to a continuum of points in the plane, and not only to interference at discrete points in P.

Lemma 12. *For any set of points $P \subseteq \mathbb{R}^2$, the average interference in $\mathrm{cov}(\mathrm{MST}'(P))$ is $O(1)$, i.e.,*

$$\frac{1}{|\mathrm{cov}(\mathrm{MST}'(P))|}\iint_{\mathrm{cov}(\mathrm{MST}'(P))}\mathrm{inter}_{\mathrm{MST}'(P)}(x,y)\ dx\ dy \in O(1).$$

Proof: Choose any set of points $P=\{p_1,\ldots,p_n\}$ in $\mathbb{R}^2$. Let q be a point selected uniformly at random in $\mathrm{cov}(\mathrm{MST}'(P))$. It suffices to show that $\mathbb{E}[\mathrm{inter}_{\mathrm{MST}'(P)}(q)] \in O(1)$. Without loss of generality, suppose that P has radius one. We partition $\mathrm{cov}(\mathrm{MST}'(P))$ into n disjoint regions $\mathcal{R}_1,\ldots,\mathcal{R}_n$ such that for each i, $\mathcal{R}_i$ includes all the points in $\mathbb{R}^2$ that are within the transmission ranges of exactly i nodes in P. For each i, let e_i denote the longest edge in $E(\mathrm{MST}(P))$ incident to the point $p_i \in P$. Note that e_i and e_j are not necessarily distinct for $i \neq j$. However, for every i, there is at most one $j \neq i$ such that $e_i = e_j$.

$$\begin{aligned}\mathbb{E}[\mathrm{inter}_{\mathrm{MST}'(P)}(q)] &= \frac{\sum_{i=1}^{n} i|\mathcal{R}_i|}{|\mathrm{cov}(\mathrm{MST}'(P))|}\\ &= \frac{\sum_{i=1}^{n} i|\mathcal{R}_i|}{\sum_{i=1}^{n}|\mathcal{R}_i|}\\ &= \frac{\sum_{i=1}^{n}|\mathrm{cov}_{\mathrm{MST}'(P)}(p_i)|}{\sum_{i=1}^{n}|\mathcal{R}_i|}\\ &= \frac{\sum_{i=1}^{n}\pi\|e_i\|^2}{\sum_{i=1}^{n}|\mathcal{R}_i|}\\ &\leq \frac{2\sum_{e\in E(\mathrm{MST}(P))}\pi\|e\|^2}{\sum_{i=1}^{n}|\mathcal{R}_i|} \qquad (8)\\ &\leq \frac{2\sum_{e\in E(\mathrm{MST}(P))}\pi\|e\|^2}{\sum_{e\in E(\mathrm{MST}(P))}\frac{\|e\|^2}{4\sqrt{3}}} \qquad (9)\\ &= 8\pi\sqrt{3}.\end{aligned}$$

(9) follows from (8) by the fact that the lozenges with e_i as their largest diameters (with angles $\pi/3-\epsilon$) do not overlap [50]. □

Lemma 13. *Let l be a positive real number, let $\mathcal{S}$ be a square of size $l\times l$, and let P a set of n points in $\mathcal{S}$. If the transmission range of nodes is determined by $\mathrm{MST}(P)$, then the average interference in $\mathcal{S}$ is $O(1)$, that is*

$$\frac{1}{l^2}\iint_{\mathcal{S}}\mathrm{inter}_{\mathrm{MST}'(P)}(x,y)\ dx\ dy \in O(1).$$

Proof: Choose any set of points $P=\{p_1,\ldots,p_n\}$ in $\mathcal{S}$. Let q be a point selected uniformly at random in $\mathcal{S}$. It suffices to show that $\mathbb{E}[\mathrm{inter}_{\mathrm{MST}'(P)}(q)] \in O(1)$. Without loss of generality, suppose $l = 1/\sqrt{2}$. Note that the radius of P is at most one as the diameter of $\mathcal{S}$ is one. We partition $\mathcal{S}$ into n disjoint regions $\mathcal{R}_1,\ldots,\mathcal{R}_n$ such that for each i, $\mathcal{R}_i$ includes all the points in $\mathbb{R}^2$ that are within the transmission ranges of exactly i nodes in P. For each i, let e_i denote the longest edge in $E(\mathrm{MST}(P))$ incident to the point $p_i \in P$. We have

$$\begin{aligned}\mathbb{E}[\mathrm{inter}_{\mathrm{MST}'(P)}(q)] &= \frac{\sum_{i=1}^{n} i|\mathcal{R}_i|}{|\mathcal{S}|}\\ &= 2\sum_{i=1}^{n} i|\mathcal{R}_i|\\ &= 2\sum_{i=1}^{n}|\mathrm{cov}_{\mathrm{MST}'(P)}(p_i)\cap\mathcal{S}|\\ &\leq 2\sum_{i=1}^{n}|\mathrm{cov}_{\mathrm{MST}'(P)}(p_i)|\\ &= 2\sum_{i=1}^{n}\pi\|e_i\|^2\\ &\leq 4\sum_{e\in E(\mathrm{MST}(P))}\pi\|e\|^2\\ &\leq 4\cdot 12 \qquad (10)\\ &= 48,\end{aligned}$$

where (10) is by Lemma 10. □

Lemma 14. *For any set of points $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^2$ and any $p_x \in P$,*

$$\mathrm{inter}_{\mathrm{MST}'(P)}(p_x) \le \mathrm{inter}_{\mathrm{MST}'(P\setminus\{p_x\})}(p_x) + 6. \tag{11}$$

Proof: Without loss of generality, we show (11) holds when $p_x = p_1$. For any pair $\{p_i, p_j\} \subseteq \{p_2, \ldots, p_n\}$, we show that if the edge $\{p_i, p_j\} \in E(\mathrm{MST}(P))$, then $\{p_i, p_j\} \in E(\mathrm{MST}(P \setminus \{p_1\}))$. By contradiction, suppose there exists a pair $\{p_i, p_j\} \subseteq \{p_2, \ldots, p_n\}$ such that $\{p_i, p_j\} \in E(\mathrm{MST}(P))$ and $\{p_i, p_j\} \notin E(\mathrm{MST}(P \setminus \{p_1\}))$. By removing $\{p_i, p_j\}$, $\mathrm{MST}(P)$ is divided into two connected components, C_1 and C_2. Since $\mathrm{MST}(P \setminus \{p_1\})$ is connected, there must an edge $\{p_k, p_l\}$ such that $p_k \in C_1$ and $p_l \in C_2$. Note that $\{p_k, p_l\} \neq \{p_i, p_j\}$ by our assumption that $\{p_i, p_j\} \notin E(\mathrm{MST}(P \setminus \{p_1\}))$. Also, $k \neq 1$ and $l \neq 1$ since p_1 is not in $V(\mathrm{MST}(P \setminus \{p_1\}))$. Furthermore, $\mathrm{dist}(p_k, p_l) < \mathrm{dist}(p_i, p_j)$ because otherwise by replacing the edge $\{p_k, p_l\}$ with $\{p_i, p_j\}$ we can reduce the sum of the lengths of edges in $\mathrm{MST}(P \setminus \{p_1\})$. This derives a contradiction, however, as we can reduce the sum of length of edges in $\mathrm{MST}(P)$ by replacing $\{p_i, p_j\}$ with $\{p_k, p_l\}$. The above argument shows that the transmission range of any non-neighbour of p_1 determined by $\mathrm{MST}(P \setminus \{p_1\})$ is not more than its transmission range determined by $\mathrm{MST}(P)$. We conclude the proof by noting that for any set of points Q, the maximum degree of $\mathrm{MST}(Q)$ is at most six, hence p_1 has at most six neighbours in $\mathrm{MST}(P)$. □

Theorem 15. *Let n be a positive integer. Let $\mathrm{interAvg}(\mathrm{MST}'(P))$ be a random variable equal to the average interference of a set P of n points distributed uniformly and independently at random in $[0,1]^2$. Then*

$$\mathbb{E}[\mathrm{interAvg}(\mathrm{MST}'(P))] \in O(1).$$

Proof: Let $P = \{p_1, \ldots, p_n\}$ be a set of n points selected uniformly at random in $[0,1]^2$.

$$\begin{aligned}\mathbb{E}[\mathrm{interAvg}(\mathrm{MST}'(P))] &= E\left[\frac{1}{n}\sum_{i=1}^{n} \mathrm{inter}_{\mathrm{MST}'(P)}(p_i)\right] \\ &= \frac{1}{n}\sum_{i=1}^{n} E\left[\mathrm{inter}_{\mathrm{MST}'(P)}(p_i)\right] \qquad (12) \\ &= E\left[\mathrm{inter}_{\mathrm{MST}'(P)}(p_1)\right], \qquad (13)\end{aligned}$$

where (12) holds because the expected value of a sum of random variables (independent or not) is equal to the sum of the individual expectations, and (13) holds by the fact that, due to symmetry, for every $\{p_i, p_j\} \subseteq \{p_1, \ldots, p_n\}$,

$$E\left[\mathrm{inter}_{\mathrm{MST}'(P)}(p_i)\right] = E\left[\mathrm{inter}_{\mathrm{MST}'(P)}(p_j)\right].$$

Since p_1 is selected uniformly and independently at random, by Lemma 13, we get

$$\mathbb{E}[\mathrm{inter}_{\mathrm{MST}'(P\setminus\{p_1\})}(p_1)] \in O(1).$$

Furthermore, by Lemma 14, we have

$$\mathrm{inter}_{\mathrm{MST}'(P)}(p_1) \le \mathrm{inter}_{\mathrm{MST}'(P\setminus\{p_1\})}(p_1) + 6.$$

Thus,

$$\begin{aligned}\mathbb{E}[\mathrm{inter}_{\mathrm{MST}'(P)}(p_1)] &\le \mathbb{E}[\mathrm{inter}_{\mathrm{MST}'(P\setminus\{p_1\})}(p_1)] + 6 \\ &\in O(1),\end{aligned}$$

which completes the proof. □

5 Worst-Case Maximum Interference

We disprove a conjecture of Devroye and Morin [15] relating the maximum interference of the Euclidean minimum spanning tree to the optimal maximum interference attainable. We do so by constructing a set P of n points on the line and show that every connected communication graph on P has maximum interference $\Omega(\sqrt{n})$ and, furthermore, that the MST of P has maximum interference $O(\sqrt{n})$.

As shown by von Rickenback et al. [48], for any set of n points $P \subseteq \mathbb{R}$,

$$\mathrm{OPT}(P) \in \Omega\left(\sqrt{\mathrm{inter}(\mathrm{MST}'(P))}\right), \tag{14}$$

$$\text{and} \qquad \mathrm{OPT}(P) \in O\left(\sqrt{n}\right). \tag{15}$$

Halldórsson and Tokuyama [22] showed (15) also holds for any set of n points $P \subseteq \mathbb{R}^2$. Devroye and Morin conjectured the following:

Conjecture 16 (Devroye and Morin [15] (2012))**.** *For any fixed d and any set of n points $P \subseteq \mathbb{R}^d$,*

$$\mathrm{OPT}(P) \in O\left(\sqrt{\mathrm{inter}(\mathrm{MST}'(P))}\right). \tag{16}$$

If true, Conjecture 16 and (14) would imply that for any set of n points $P \subseteq \mathbb{R}$,

$$\mathrm{OPT}(P) \in \Theta\left(\sqrt{\mathrm{inter}(\mathrm{MST}'(P))}\right). \tag{17}$$

We disprove Conjecture 16 by the following proposition:

Proposition 17. *For any $n \ge 1$ there exists a set of n points $P \subseteq \mathbb{R}$ such that $\mathrm{OPT}(P) \in \Theta\left(\sqrt{n}\right)$ and $\mathrm{inter}(\mathrm{MST}'(P)) \in \Theta\left(\sqrt{n}\right)$.*

Proof: Choose any $n' \ge 1$ and let $n = \lfloor \sqrt{n'} \rfloor^2$. We define a set P of n points. If $n < n'$, define a set P' of n' points by adding any $n' - n$ points to P. Observe that $\mathrm{OPT}(P') \in \Theta(\mathrm{OPT}(P))$ and $\mathrm{inter}(\mathrm{MST}'(P')) \in \Theta(\mathrm{inter}(\mathrm{MST}'(P)))$.

von Rickenback et al. [48] define an *exponential chain* as a sequence of points on the line such that each gap between adjacent points is twice the length of the preceding gap. Let P_0 denote an exponential chain of $\lceil n/2 \rceil$ points in which the first two points are separated by a gap of length 1. Since for each $k > 1$, $2^k > \sum_{i=0}^{k-1} 2^i$, it follows that every node in $\mathrm{MST}'(P_0)$ causes interference at the first node, resulting in $\mathrm{inter}(\mathrm{MST}'(P_0)) \in \Theta(n)$.

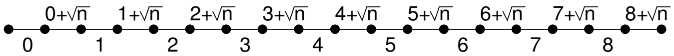


Fig. 2: logarithmic-scale illustration of spacing between adjacent points in P (not to scale)

We modify the exponential chain in set P_0 by adding a gap of length $2^{\sqrt{n}+k}$ following the gap of length 2^k in P_0, for each $k \in \{0, 1, \ldots, \lfloor n/2 \rfloor - 1\}$. Let P denote the new set of n points (see Figure 2).

Observe that any node can cause interference to at most two nodes to its right; interference at nodes to its left, however, can be significantly greater. Therefore, each of the first $\sqrt{n}$ nodes causes interference in at most $\sqrt{n}+1$ other nodes.

We now show that $\text{inter}(\text{MST}'(P)) \leq 2\sqrt{n}+3$. Choose any $i \in \{\sqrt{n}+1, \ldots, n/2-1\}$.

$$\begin{aligned} 2^{\sqrt{n}+i} &< 2^{\sqrt{n}+i} + 2^{i-1} - 2^{i-\sqrt{n}-2} \\ &= 2^{i-1} + \sum_{j=i-\sqrt{n}-1}^{i-1+\sqrt{n}} 2^j \\ &= \sum_{j=i-1}^{i-1+\sqrt{n}} 2^j + \sum_{j=i-\sqrt{n}-1}^{i-1} 2^j \\ &= \sum_{j=i-1-\sqrt{n}}^{i-1} \left[2^{\sqrt{n}+j} + 2^j \right]. \end{aligned} \tag{18}$$

By (18), no edge of length $2^{\sqrt{n}+i}$ in $\text{MST}'(P)$ can cause interference at any edge beyond the first $2\sqrt{n}$ edges immediately to its left. Since interference caused to the right of any edge is limited to at most its next two right neighbours and each node interferes with itself, therefore,

$$\text{inter}(\text{MST}'(P)) \leq 2\sqrt{n}+3 \in O(\sqrt{n}). \tag{19}$$

Next, we show that $\text{inter}(\text{MST}'(P)) \in \Omega(\sqrt{n})$. Choose any $i \in \{\sqrt{n}+1, \ldots, n/2-1\}$.

$$\begin{aligned} 2^{\sqrt{n}+i} &= \sum_{j=0}^{\sqrt{n}+i-1} 2^j + 1 \\ &> \sum_{j=i-\sqrt{n}}^{\sqrt{n}+i-1} 2^j \\ &= \sum_{j=i}^{\sqrt{n}+i-1} 2^j + \sum_{j=i-\sqrt{n}}^{i-1} 2^j \\ &= \sum_{j=i-\sqrt{n}}^{i-1} \left[2^{\sqrt{n}+j} + 2^j \right]. \end{aligned} \tag{20}$$

By (20), any edge of length $2^{\sqrt{n}+i}$ in $\text{MST}'(P)$ causes interference at the $2(\sqrt{n}-1)$ edges immediately to its left. Therefore,

$$\text{inter}(\text{MST}'(P)) \geq 2(\sqrt{n}-1) \in \Omega(\sqrt{n}). \tag{21}$$

By (19) and (21), $\text{inter}(\text{MST}'(P)) \in \Theta(\sqrt{n})$.

Next we show that any longer edge (any edge not in $\text{MST}(P)$) causes interference at the first node. Any such edge must span two adjacent edges of $\text{MST}(P)$, whose respective lengths are 2^i and $2^{i+\sqrt{n}}$ for some i. Choose any $i \in \{\sqrt{n}+1, \ldots, n/2-1\}$. Since

$$2^{i+\sqrt{n}} + 2^i > \sum_{j=0}^{i+\sqrt{n}-1} 2^j + \sum_{j=0}^{i-1} 2^j,$$

any edge not in $\text{MST}(P)$ causes interference at the leftmost node.

Choose any communication graph G_P on P. Consider the partition of P into a sequence of $\Theta(\sqrt{n})$ blocks, each containing $\Theta(\sqrt{n})$ points.

CASE 1. Suppose every node in some block is adjacent only to its immediate neighbours. Locally, this block is analogous to $\text{MST}'(P)$, resulting in interference $\Omega(\sqrt{n})$ within the block and $\text{inter}(G_P) \in \Omega(\sqrt{n})$.

CASE 2. Suppose every block contains some node that is not in $\text{MST}(P)$. Consequently, each block contains a node that interferes with the first node, resulting in $\Omega(\sqrt{n})$ at the first node, since there are $\Theta(\sqrt{n})$ blocks.

Therefore,

$$\text{OPT}(P) \in \Omega\left(\sqrt{n}\right). \tag{22}$$

The result follows by (19) and (22) since $\text{OPT}(P) \leq \text{inter}(\text{MST}'(P))$. □

6 SIMULATION

We evaluated the performance of Algorithm LOCALRADIUSREDUCTION in three settings (simulated static wireless networks, simulated mobile wireless networks, and real GPS track data) and compared it against four topology control algorithms: i) the cone-based local topology control (CBTC) algorithm [33], ii) the k-neighbour algorithm [6], iii) local computation of the intersection of the Gabriel graph and the unit disc graph (with unit radius $r_{\max}$) [8], and iv) fixed-radius topologies (unit disk graphs of radius $r_{\max} \in \{100, 200, 300\}$). Performance was evaluated by comparing average maximum interference, expected average interference, average physical degree, and average energy cost (the sum of the squares of the transmission radii [6]). These results are displayed in Figures 3 (static), 4 (mobile), and 5 (GPS).

An edge $\{u, v\}$ exists in the communication graph generated by the k-neighbour algorithm if and only if u is one of the k nodes nearest to v (by Euclidean distance) and v is one of the k nodes nearest to u. Given the value k, nodes can generate such a communication graph locally. Given $p \in (0, 1)$, the value of k is assigned such that the resulting communication graph is connected with probability at least p.

When simulating Algorithm LOCALRADIUSREDUCTION, each node collects the list of nodes in its 2-hop neighbourhood in two rounds, applies the algorithm to reduce its transmission radius and then broadcasts its computed transmission radius, allowing neighbouring nodes a final opportunity to eliminate asymmetric edges and further reduce their transmission radii while maintaining connectivity in the network.

We used the random waypoint model (RWP) [25] and real mobility trace data to simulate mobile networks. For the RWP model we applied the approximated probability distribution described by Bettstetter and Wagner [3] to position nodes independently at random across the network and generate independent snapshots in each simulation iteration. Using the mobility trace data, we estimated a probability density distribution which was used to generate independent snapshots.

6.1 Simulation Parameters

We set the simulation region's dimensions to 1000 metres $\times$ 1000 metres. For both static and dynamic networks, we varied the number of nodes n from 50 to 1000 in increments of 50. We fixed the maximum transmission radius $r_{\max}$ for each network to 100, 200, or 300 metres. To compute the average maximum interference and the expected average interference for static networks, for each n and $r_{\max}$ we generated 100,000 static networks, each with n nodes and maximum transmission radius $r_{\max}$, distributed uniformly at random in the simulation region. In the RWP model, for each n and $r_{\max}$, we randomly generated 100,000 independent networks using the following approximation for nodes' spatial distribution [3], [4]:

$$f(x,y) \approx \frac{9}{16x_m^3 y_m^3}(x^2 - x_m^2)(y^2 - y_m^2),$$

where $x_m = 500$, $y_m = 500$, $x \in [-x_m, x_m]$, and $y \in [-y_m, y_m]$. For a better approximation, we refer readers to [24]. To use the real mobility trace data of Piorkowski et al. [44], which includes GPS coordinates for trajectories of 537 taxi vehicles, we selected 500 vehicles with the largest trace samples, each has over 8000 sample points. We varied the number of nodes from 50 to 500 in increments of 50.

6.2 Simulation Results

As demonstrated by our simulation results, the average maximum interference of unit disc graph topologies increases linearly with n (see Figure 3a). Since these plots are significantly larger (i.e., they correspond to worse performance) than the other plots in all four evaluation criteria, unit disc graph plots are excluded from subsequent figures to permit more detailed comparison. Although both the local Gabriel and CBTC algorithms performed significantly better than the unit disc graphs, the lowest average maximum interference was achieved by the LocalRadiusReduction and k-neighbour algorithms, for which the corresponding plots grow logarithmically with n, as seen in Figures 3b, 4a, and 5a. Note that the LocalRadiusReduction algorithm reduces the maximum interference to $O(\log n)$ with high probability, irrespective of the initial maximum transmission radius $r_{\max}$. LocalRadiusReduction has average maximum interference slightly greater than k-neighbour (Figures 3b, 4a, and 5a), but lower expected average interference, average physical degree, and average energy cost (Figures 3c–3e, 4b–4d, and 5b–5d).

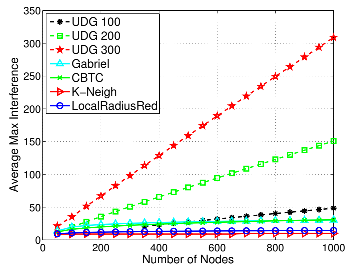
Simulation results obtained using a RWP model closely match those obtained on a static network because the distribution of nodes at any time during a random walk is nearly uniform [12]. The spatial distribution of nodes moving according to a RWP model is not uniform, and is maximized at the centre of the simulation region [24]. Consequently, the density of nodes is high near the centre, resulting in greater interference at these nodes.

Finally, we evaluated the algorithm LocalRadiusReduction using real mobility trace data of Piorkowski et al. [44], consisting of GPS coordinates for trajectories of 537 taxi vehicles recorded over one month in 2008, driving throughout the San Fransisco Bay area. We selected the 500 largest traces, each of which has over 8000 sample points. To implement our algorithm, we selected n taxis among the 500 uniformly at random, ranging from $n = 50$ to $n = 500$ in increments of 50. As seen in Figure 5, the results are similar to those measured in our simulation. The k-neighbours algorithm produced disconnected communication graphs (containing multiple connected components) in 2.5% of instances, even when the value of k was increased significantly (e.g., up to $k = 30$). This is likely explained by the highly non-uniform distribution of nodes in the track data. This difference is significant, however, because the k-neighbours algorithm does not guarantee that the returned topology is connected, failing to satisfy the primary objective of the interference minimization problem for some input instances.

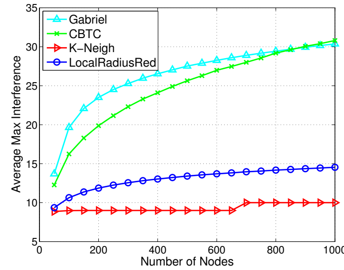
7 Conclusion

Using Algorithm LocalRadiusReduction, each node determines its transmission radius as a function of its 2-hop neighbourhood. Alternatively, suppose each node could select its transmission radius at random using a suitable distribution over $[d_{\min}(G), d_{\max}(G)]$. Can such a strategy for assigning transmission radii ensure connectivity and low maximum interference with high probability? Similarly, additional topologies and local algorithms for constructing them might achieve $O(\log n)$ expected maximum interference. It can be shown that every graph whose longest edge has length $O(\sqrt{\log n/n})$ has expected maximum interference $O(\log n)$. Devroye et al. [14, Section 2.3] show that the longest edge in a Gabriel graph has length $O(\sqrt{\log n/n})$ with high probability. Our experimental results suggest that the CBTC local topology control algorithms may also provide $O(\log n)$ expected maximum interference. Since the CBTC topology of a set of points P is not in $\mathcal{T}(P)$ in general, whether this bound holds remains to be proved.

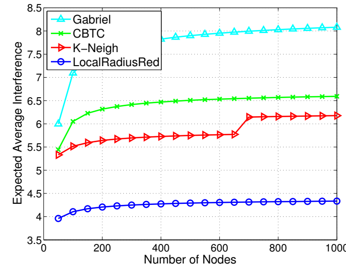
As mentioned in Section 2, multiple open questions related to interference on random sets of points were resolved recently by Devroye and Morin [15]. Several



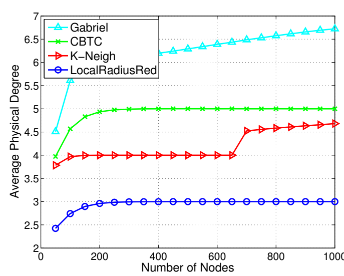
(a) Average Maximum Interference



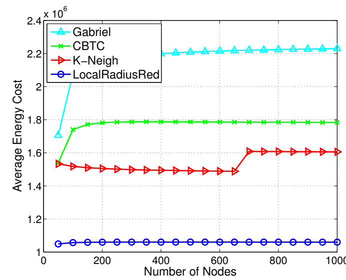
(b) Average Maximum Interference



(c) Expected Average Interference

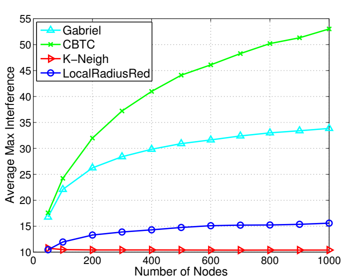


(d) Average Physical Degree

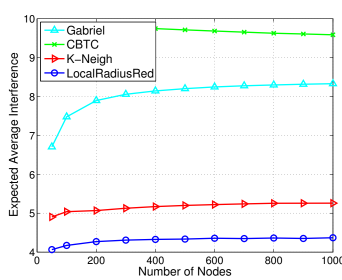


(e) Average Energy Cost

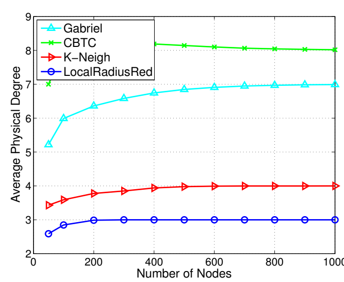
Fig. 3: Comparing the LOCALRADIUSREDUCTION algorithm against other local topology control algorithms on a simulated static wireless network. Plots for the fixed-radius algorithm are omitted from Figures 3b–3c to allow the remaining plots to be more easily differentiated.



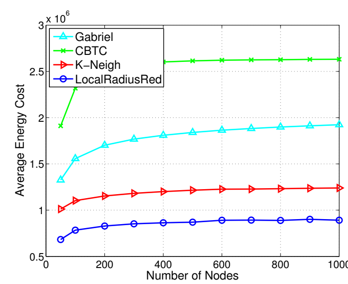
(a) Average Maximum Interference



(b) Expected Average Interference

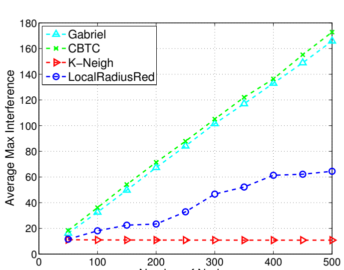


(c) Average Physical Degree

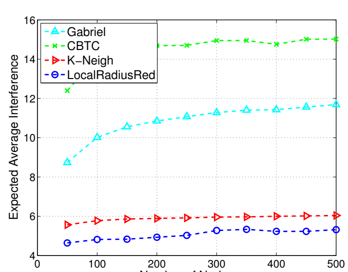


(d) Average Energy Cost

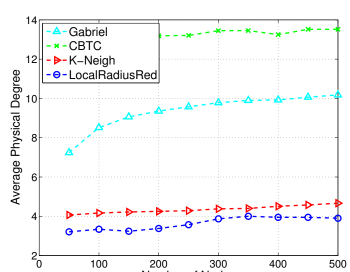
Fig. 4: Comparing the LOCALRADIUSREDUCTION algorithm against other local topology control algorithms on a simulated mobile network.



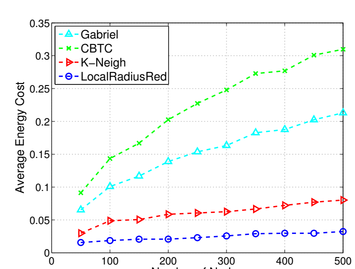
(a) Average Maximum Interference



(b) Expected Average Interference



(c) Average Physical Degree



(d) Average Energy Cost

Fig. 5: Comparing the LOCALRADIUSREDUCTION algorithm against other local topology control algorithms on recorded mobile vehicular GPS tracks. The communication graphs returned by the k-neighbours algorithm were disconnected in 2.5% of cases. Performance results for the connected and disconnected graphs are not differentiated in the plots.

questions remain open related to the algorithmic problem of finding an *optimal* solution (one whose maximum interference is exactly $\mathrm{OPT}(P)$) when node positions may be selected adversarially. The complexity of the interference minimization problem in one dimension remains open; at present, it is unknown whether the problem is polynomial-time solvable or NP-hard [47]. While the problem is known to be NP-complete in two dimensions [9], no polynomial-time approximation algorithm nor any inapproximability hardness results are known. Several closely related problems also remain open, including the problem of finding a c-connected graph whose associated maximum interference is minimized for a given input point set and a given fixed integer c, as well as to examine the interference minimization problem using models for wireless networks that consider physically-based representations for interference (see Section 1.1).

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