Thickness and Colorability of Geometric Graphs

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Abstract. The geometric thickness $\overline{\theta}(G)$ of a graph G is the smallest integer t such that there exist a straight-line drawing Γ of G and a partition of its straight-line edges into t subsets, where each subset induces a planar drawing in Γ. Over a decade ago, Hutchinson, Shermer, and Vince proved that any n-vertex graph with geometric thickness two can have at most $6n - 18$ edges, and for every $n \geq 8$ they constructed a geometric thickness two graph with $6n - 20$ edges. In this paper, we construct geometric thickness two graphs with $6n - 19$ edges for every $n \geq 9$, which improves the previously known $6n - 20$ lower bound. We then construct a thickness two graph with 10 vertices that has geometric thickness three, and prove that the problem of recognizing geometric thickness two graphs is NP-hard, answering two questions posed by Dillencourt, Eppstein and Hirschberg. Finally, we prove the NP-hardness of coloring graphs of geometric thickness t with $4t - 1$ colors, which strengthens a result of McGrae and Zito, when $t = 2$.

1 Introduction

The *thickness* $\theta(G)$ of a graph G is the smallest integer t such that the edges of G can be partitioned into t subsets, where each subset induces a planar graph. Since 1963, when Tutte [21] first formally introduced the notion of graph thickness, this property of graphs has been extensively studied for its interest from both the theoretical [2, 5, 7] and practical point of view [17, 19]. A wide range of applications, e.g., circuit layout design, simultaneous geometric embedding, and network visualization, have motivated the examination of thickness in the geometric setting [7, 11, 12, 14]. The *geometric thickness* $\overline{\theta}(G)$ of a graph G is the smallest integer t such that there exist a *straight-line drawing* (i.e., a drawing on the Euclidean plane, where every vertex is drawn as a point and every edge is drawn as a straight line segment) Γ of G and a partition of its straight-line edges into t subsets, where each subset induces a planar drawing in Γ. If $t = 2$, then G is called a *geometric thickness two graph* (or, a doubly-linear graph [14]), and Γ is called a *geometric thickness two representation* of G. While graph theoretical

⋆ Work of the author is supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC).

⋆⋆ Work of the author is supported in part by a University of Manitoba Graduate Fellowship.

thickness does not impose any restriction on the placement of the vertices in each planar layer, the geometric thickness forces the same vertices in different planar layers to share a fixed point in the plane. Eppstein [11] clearly established this difference by constructing thickness three graphs that have arbitrarily large geometric thickness.

Structural Properties. Geometric thickness has been broadly examined on several classes of graphs, e.g., complete graphs [7], bounded-degree graphs [4, 10, 11], and graphs with bounded treewidth [8, 9]. Hutchinson, Shermer, and Vince [14] examined properties of graphs with geometric thickness two. They proved that these graphs at most $6n - 18$ edges, and for every $n \geq 8$ they constructed a geometric thickness two graph with $6n - 20$ edges. Even after several attempts [7, 10] to understand the structural properties of geometric thickness two graphs, the question whether there exists a geometric thickness two graph with $6n - 18$ edges remained open for over a decade. Answering this question is quite challenging since although one can generate many thickness two graphs with $6n - 18$ or $6n - 19$ edges, no efficient algorithm is known that can determine the geometric thickness of such a graph. However, by examining the point configurations that are likely to support geometric thickness two graphs with large numbers of edges, we have been able to construct geometric thickness two graphs with $6n - 19$ edges (see Section 2).

Recognition. Although graph theoretical thickness is known for all complete graphs [2] and complete bipartite graphs [5], geometric thickness for these graph classes is not completely characterized. Dillencourt, Eppstein and Hirschberg [7] proved an $\lceil n/4 \rceil$ upper bound on the geometric thickness of K_n, giving a nice construction for representations with geometric thickness $t = \lceil n/4 \rceil$. They also gave a lower bound on the geometric thickness of complete graphs that matches the upper bound for several smaller values of n. Their bounds show that the geometric thickness of K_{15} is greater than its graph theoretical thickness, i.e., $\overline{\theta}(K_{15}) = 4 > \theta(K_{15}) = 3$, which settles the conjecture of [16] on the relation between geometric and graph theoretical thickness. Since the exact values of $\overline{\theta}(K_{13})$ and $\overline{\theta}(K_{14})$ are still unknown, Dillencourt et al. [7] hoped that the relation $\overline{\theta}(G) > \theta(G)$ could be established with a graph of smaller cardinality. In Section 3 we prove that the smallest such graph contains 10 vertices.

Since determining the thickness of an arbitrary graph is NP-hard [17], Dillencourt et al. [7] suspected that determining geometric thickness might be also NP-hard, and mentioned it as an open problem. The hardness proof of Mansfield [17] relies heavily on the fact that $\theta(K_{6,8}) = 2$. Dillencourt et al. [7] mentioned that this proof cannot be immediately adapted to prove the hardness of the problem of recognizing geometric thickness two graphs by showing that $\overline{\theta}(K_{6,8}) = 3$. This complexity question has been repeated several times in the literature [8, 11] since 2000, and also appeared as one of the selected open questions in the 11th International Symposium on Graph Drawing (GD 2003) [6]. In Section 4 we settle the question by proving the problem to be NP-hard.

Colorability. As a natural generalization of the well-known Four Color Theorem for planar graphs [3], a long-standing open problem in graph theory is to determine the relation between thickness and colorability [15, 20]. For every $t \geq 3$, the best known lower bound on the chromatic number of the graphs with thickness t is $6t-2$, which can be achieved by the largest complete graph of thickness t. On the other hand, every graph with thickness t is $6t$ colorable [15]. Recently, McGrae and Zito [18] examined a variation of this problem that given a planar graph and a partition of its vertices into subsets of at most r vertices, asks to assign a color (from a set of s colors) to each subset such that two adjacent vertices in different subsets receive different colors. They proved that the problem is NP-complete when $r = 2$ (respectively, $r > 2$) and $s \leq 6$ (respectively, $s \leq 6r-4$) colors. In Section 5 we prove the NP-hardness of coloring geometric thickness t graphs with $4t-1$ colors. As a corollary, we strengthen the result of McGrae and Zito [18] that coloring thickness $t = r = 2$ graphs with 6 colors is NP-hard. Our hardness result is particularly interesting since no geometric thickness t graph with chromatic number more than $4t$ is known.

2 Geometric Thickness Two Graphs with $6n-19$ Edges

Let K'_9 be the graph obtained by deleting an edge from K_9. In this section we first construct a geometric thickness two representation Γ of K'_9 that has $6n-19$ edges. We then show how to add vertices in Γ such that for any $n \geq 9$ one can construct a geometric thickness two graph with $6n-19$ edges.

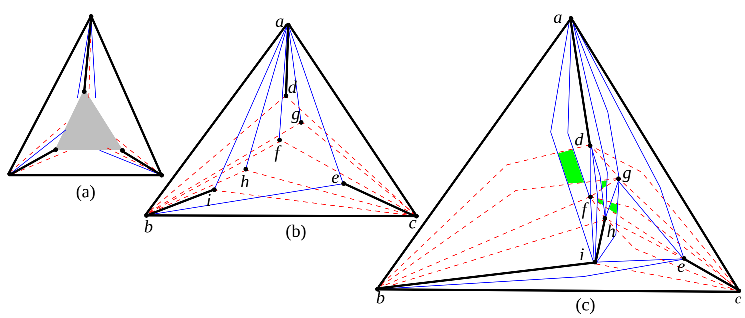

Fig. 1. (a) Illustration for the shared edges (bold). (b) Initial point set. (c) A geometric thickness two representation Γ of K'_9, where the planar layers are shown in red (dashed) and blue (thin). Black edges can belong to either red or blue layer. Free quadrangles are shown in green (shaded). Some edges are drawn with bends for clarity.

Hutchinson et al. [14, Theorem 6] proved that if any geometric thickness two graph with $6n-18$ edges exists, then the convex hull of its geometric thickness two representation must be a triangle. This representation is equivalent to the union of two plane triangulations that share at least six common edges, i.e., the three outer edges and the other three edges are adjacent to the three outervertices, as shown in black in Figure 1(a). Since each triangulation contains

$3n-6$ edges, the upper bound of $2(3n-6)-6=6n-18$ follows. These properties of an edge maximal geometric thickness two representation motivated us to examine pairs of triangulations that create many edge crossings while drawn simultaneously. In particular, we first created a set of points interior to the convex hull such that addition of straight line segments from each interior point to the three points on the convex hull creates two plane drawings that, while drawn simultaneously, contain a crossing in all but the six common edges. Figure 1(b) illustrates such a scenario. We then tried to extend each of these two planar drawings to a triangulation by adding new edges such that every new edge crosses at least one initial edge. We found multiple distinct point sets for which all but one newly added edge cross at least one initial edge, resulting in multiple distinct geometric thickness two representations with $2(3n-6)-7=6n-19$ edges. For example, see Figure 1(c), where the underlying graph is K_9'.

Let Γ be a geometric thickness two representation. A triangle in Γ is *empty* if it contains exactly three vertices on its boundary, but does not contain any vertex in its proper interior, e.g., the triangle Δghi in Figure 1(d). A *quadrangle in Γ is free* if it is created by the intersection of two empty triangles but does not contain any vertices of Γ, as shown in Figure 1(d) in green.

Theorem 1. *For each $n \geq 9$, there exists a geometric thickness two graph with n vertices and $6n-19$ edges that contains K_9 minus an edge as a subgraph. For each $n \geq 11$, there exists a geometric thickness two graph with $6n-19$ edges that does not contain K_8.*

Proof. Since K_9' has a geometric thickness two representation, as shown in Figure 1(c), the claim holds for $n=9$. We now claim that given an n-vertex geometric thickness two representation with $6n-19$ edges that contains a free quadrangle, one can construct a geometric thickness two representation with $n+1$ vertices and $6(n+1)-19$ edges. One can verify this claim as follows. Place a new vertex p interior to the free quadrangle. Since a free quadrangle is the intersection of two empty triangles, one can add three straight line edges from p to the three vertices of each empty triangle such that the new drawing in each layer remains planar, as shown in Figure 2(a). Since the number of vertices increases by one, and the number of edges increases by six, the resulting geometric thickness two representation must have $6n-19+6=6(n+1)-19$ edges.

Observe that there are at least four free quadrangles in the geometric thickness two representation of K_9', as shown in Figure 1(d). Therefore, for each $i, 9 \leq i \leq 12$, we can construct a geometric thickness two representation Γ_i with i vertices and $6i-19$ edges that contain at least one free quadrangle. We use these four geometric thickness two representations as the base case, and assume inductively that for each $9 \leq i < n$ there exists a geometric thickness two representation Γ_i with i vertices and $6i-19$ edges that contains at least one free quadrangle. We now construct a geometric thickness two representation with n vertices and $6n-19$ edges that contains a free quadrangle. By induction, Γ_{n-4} has a free quadrangle. We add four vertices to this quadrangle and complete the triangulation in each planar layer by adding 24 new edges in total, as shown in

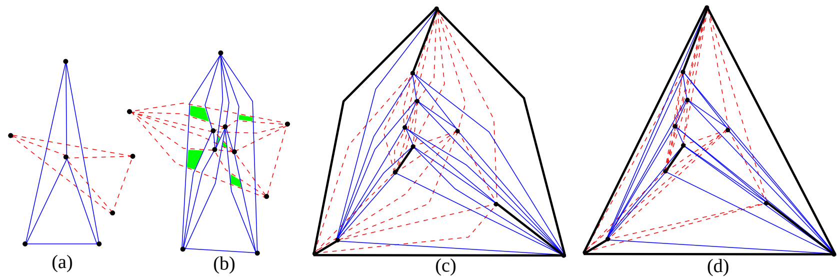

Fig. 2. (a)–(b) Adding vertices to a geometric thickness two drawing. (c)–(d) A graph with 11 vertices, 47 edges and geometric thickness two that does not contain K_8.

Figure 2(b). Consequently, the new geometric thickness two representation Γ_n contains $6(n-4)-19+24 = 6n-19$ edges. Since the newly added edges create new free quadrangles, Γ_n also contains a free quadrangle.

The existence of such graphs that do not contain K_8 is proved using the graph illustrated in Figure 2(c). The details are omitted due to space constraints. □

3 Thickness Two Graphs with $\overline{\theta}(G) \geq 3$

We enumerate all possible geometric thickness two drawings of K_9' using Aichholzer et al.'s [1] point-set order-type database. Figure 3 illustrates all the three different configurations of nine points that support geometric thickness two drawings of K_9'. It might initially appear that Figures 3(a) and (b) are the same. However, observe that g lies on the left half-plane of (d, e) in Figure 3(a) and on the right-half plane of (d, e) in Figure 3(b). We enumerated these geometric thickness two representations by performing the steps S_1 and S_2 below for every point-set order-type P that consists of nine points.

S_1. Construct a straight-line drawing Γ of K_9 on P.
S_2. For every edge e^* in Γ, execute the following.
- Delete e^* and test whether the proper intersection graph[1] determined by the remaining straight lines is 2-colorable. If the graph is 2-colorable, then Γ is a geometric thickness two representation of K_9'.
- Reinsert e^* in Γ.

Let $\Gamma_i, 1 \leq i \leq 3$, be the drawings of K_9' depicted in Figures 3(a)–(c), respectively. The seven black edges in each of these drawings do not contain any crossing, i.e., these edges are shared in both triangulations. By E_i and E_i' we denote the set of all edges, and the set of black edges in Γ_i, respectively. Let $E_i'' = E_i \setminus E_i'$. We verifty that the partition of the edges of E_i'' into red and blue is unique by checking that the proper intersection graph G_i of E_i'' is connected.

[1] Each vertex in a *proper intersection graph G of a set of straight line segments* corresponds to a distinct line, and two vertices of G are adjacent if and only the corresponding straight lines properly cross.

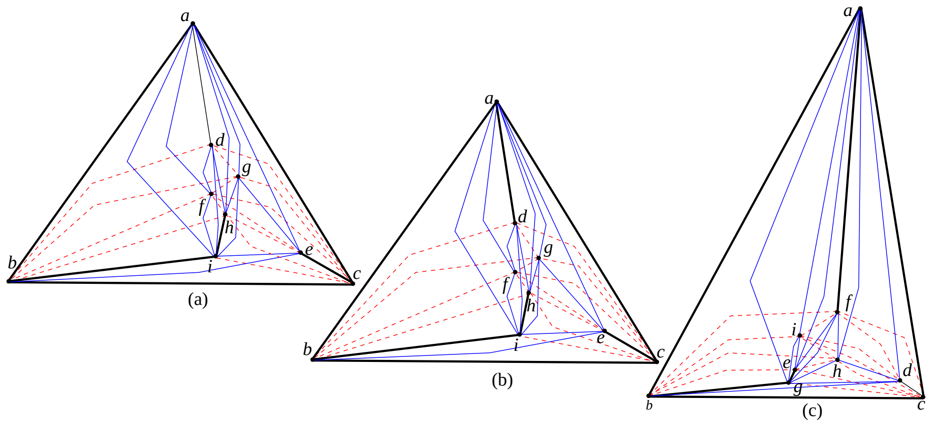

Fig. 3. (a)–(c) Geometric thickness two representations of K_9', where $K_9' = K_9 \setminus (d, e)$. Edges are drawn with polylines for clarity.

Fact 1 *Let Γ be a geometric thickness two representation of K_9'. Then the partition of the straight-line edges of Γ, except the seven edges that do not contain any proper crossing, into two planar layers is unique.*

We now categorize the vertices of a K_9' into two types: *unsaturated* (vertices of degree 7), and *saturated* (vertices of degree 8). The vertices d and e of Figures 3(a)–(c) are unsaturated, and all other vertices are saturated vertices. Take a new vertex and make it adjacent to the two unsaturated vertices and any five saturated vertices of a K_9'. Let the resulting graph with 10 vertices be G_s. The following theorem shows that $\overline{\theta}(G_s) = 3 > \theta(G_s) = 2$, whose proof is omitted due to space constraint. The idea of the proof is first to show a thickness two representation of G_s, and then to show that G_s contains a vertex v that is not straight-line visible to all of its neighbors in any geometric thickness two representation of $G_s \setminus v$. Finally, the proof shows that for every graph G with at most 9 vertices, $\overline{\theta}(G) = \theta(G)$.

Theorem 2. *The smallest graph G (with respect to the number of vertices) such that $\overline{\theta}(G) = 3 > \theta(G) = 2$ contains 10 vertices.*

4 Geometric Thickness Two Graph Recognition

Our proof that testing whether $\overline{\theta}(G) \leq 2$ is NP-hard is inspired by a technique of [13]. We reduce the 3SAT problem that given a CNF-system with a set U of variables and a collection C of clauses over U, where each clause consists of exactly three literals, asks whether there is a satisfying truth assignment for U.

Given an instance $I(U, C)$ of 3SAT, we construct a graph G such that there exists a satisfying truth assignment for U if and only if there exists a geometric thickness two representation of G. Before describing the construction of G, we observe some properties of the geometric thickness two representations of K_9'.

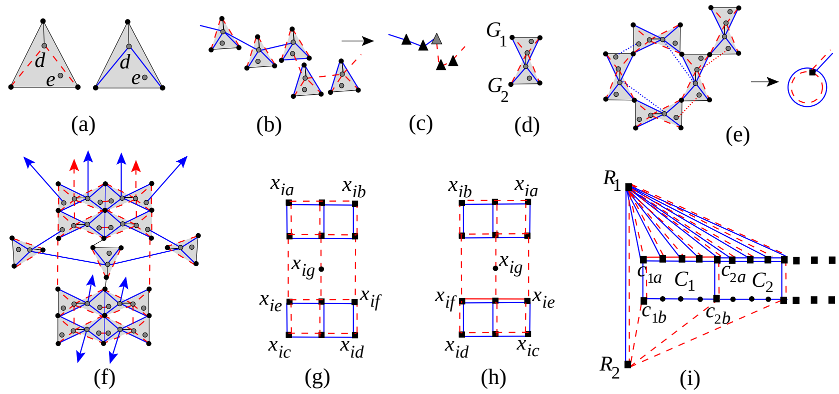

Fig. 4. (a) Hypothetical geometric thickness two representations of K_9'. (b)–(c) A path through the unsaturated vertices, and its hypothetical representation. (d)–(e) A geometric thickness two representation with 5 copies of Γ_H, and its hypothetical representation. Note that the order (inside or outside) among the red (dashed) and blue (solid) lines are not important. (f) A literal gadget, the arrows denote possible connections with other gadgets. (g)–(h) Hypothetical representation of a literal gadget, when the literal is (g) true, and (h) false. (i) Illustration for clause gadgets.

By Fact 1, observe that every geometric thickness two drawing of K_9' can be denoted by one of the two hypothetical representations shown in Figure 4(a). Each black (respectively, gray) vertex of Figure 4(a) is a saturated (respectively, unsaturated) vertex[2]. We denote the two planar layers of a drawing by the *red layer* L_r (containing the red edges) and *blue layer* L_b (containing the blue edges). Each black edge can be assigned an arbitrary layer unless it is crossed by some other edge. Observe from Figure 3 that if the unsaturated vertex d is surrounded by a blue (respectively, red) triangle, then the other unsaturated vertex e is surrounded by a red (respectively, blue) triangle. Therefore, if we create a path connecting the unsaturated vertices of several copies of K_9', as shown in Figure 4(b), then the edges of that path must be of same color. Although here we require the copies of K_9' to be non-overlapping and non-nesting, this will not be significant for our reduction. In the hypothetical representation, we denote each K_9' with either a black triangle (if its incident edges are of same color), or a gray triangle (if its incident edges are of different colors).

Let G_1 and G_2 be two distinct copies of K_9'. Let s_i and u_i, be a saturated and an unsaturated vertex of G_i, $1 \le i \le 2$, respectively. Let H be the graph that is obtained by merging s_1 and u_1 with u_2 and s_2, respectively, and then removing the resulting multi-edges (if any). Observe that a geometric thickness two representation of H can be constructed by taking two copies of the drawing of Figure 3(a), and then placing one copy on top of the other copy by rotating it such that the two drawings share the edge (s_i, u_i). Figure 4(d) illustrates a

[2] As defined in Section 3, a vertex v is saturated if it has degree 8, and unsaturated if it has degree 7.

hypothetical geometric thickness two drawing Γ_H of H. Examining every candidate pair $((s_1, u_1), (s_2, u_2))$ in the drawing of Figures 3(a)–(c) we can observe that in every geometric thickness two drawing of H, the vertex $u_1(=s_2)$ must lie on the convex hull of the drawing of G_1. Similarly, the vertex $u_2(=s_1)$ must lie on the convex hull of the drawing of G_2. Consequently, H can be represented by Γ_H. Figure 4(e) shows how to connect several copies of Γ_H to create a geometric thickness two representation where no two Γ_H properly cross, and then shows its hypothetical representation. We sometimes use a black square to illustrate the connection between two different drawings, as shown in Figure 4(e).

Construction of G. Assume that $I(U, C)$ contains l literals and t clauses. For every literal $x_i, 1 \leq i \leq l$, construct a literal gadget Γ_{x_i} as depicted in Figure 4(f). Figure 4(g) is a simplified representation of the literal gadget, which will correspond to the value true. On the other hand, Figure 4(h) (i.e., the mirror embedding of Figure 4(f)) will correspond to the value false. We call the vertex x_{ig} the *central vertex* of Γ_{x_i}. By the *lower-half* of Γ_{x_i} we denote the subgraph induced by x_{ic}, x_{id}, x_{ie} and x_{if}. The vertices of Γ_{x_i} that are not in the lower-half, induce the *upper-half*. Construct the clause gadgets as shown in Figure 4(i). Observe that the vertices R_1 and R_2 are incident to a set of rectangles, where each rectangle (i.e., *clause box*) B_r corresponds to a clause $C_r, 1 \leq r \leq t$. The top, left and right sides of B_r constitutes a chain of Γ_H, and the bottom side is a path of three vertices. We merge the central vertices of the three literals of C_r with a distinct vertex of the bottom side of B_r. Let E_b be the set that consists of the edges on the bottom side of the clause boxes. The construction of the clause gadget ensures that the edges of E_b lie on the same planar layer, w.l.o.g., on blue layer L_b. Then the clause boxes force the edges (c_{rb}, R_2) to lie on the other planar layer, i.e., the red layer L_r. For each literal gadget Γ_l, we add an edge between R_2 and the gadget such that the edge is forced to lie on L_r. Similarly, for each literal gadget Γ_l, add an edge between the top side of the clause box and the gadget such that the edge is forced to lie on L_b.

We now add some edges among the literal gadgets that correspond to the same literal. For every literal x_i, we order its literal gadgets according to their appearance in different clauses. Let $l_1, l_2, \ldots, l_{t'}$ be the literal gadgets that correspond to the literal x_i. Then for each index $q, 1 \leq q < t'$, we add an edge between the vertex x_{ie} (respectively, x_{if}) of l_q and the vertex x_{ic} (respectively, x_{id}) of l_{q+1}. We denote all these edges by E_l. Figure 5(a) shows how the edges in E_l forces the corresponding literal gadgets to have the same truth value.

Finally, we add a few more edges among the literal gadgets that belong to the same clause. Let x_i, x_j, x_k be the three literals of C_r. We then add a path between x_{ib} and x_{ja} that contains three unsaturated vertices of two copies of K'_9, as shown in Figure 5(b). Similarly, we add a path between x_{ka} and x_{jb} that contains three unsaturated vertices of two copies of K'_9.

Theorem 3. *It is NP-hard to determine whether the geometric thickness of an arbitrary graph is at most two.*

Proof. Given an instance $I(U, C)$ of 3SAT, we first construct the corresponding graph G and then prove that $I(U, C)$ has a satisfying truth assignment if and

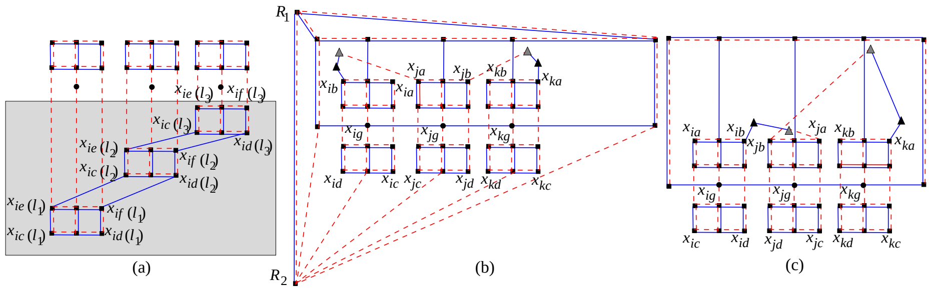

Fig. 5. (a) Literal gadgets that correspond to the same literal, which is true. (b)–(c) Two gadgets: (b) x_j is true, x_i, x_k are false, (c) x_i is true, x_j, x_k are false.

only if G has a geometric thickness two representation. The proof is similar to the hardness proof for simultaneous straight-line embedding of two planar graphs [13]; thus we give only an outline of the proof.

Assume first that $I(U, C)$ is satisfiable. We now construct a geometric thickness two representation of G. We draw the clause gadgets as shown in Figure 4(i). Then for each literal, we assign a horizontal region and draw its corresponding gadgets as shown in Figure 5. Finally, for each clause $C_r = (x_i \vee x_j \vee x_k)$, we draw the paths between x_{ib} and x_{ja}, and x_{ka} and x_{jb} such that no two edges on the same layer cross, as follows. Observe that at least one literal in C_r is true. If the literal in the middle, i.e., x_j, is true, then we draw these paths as shown in Figure 5(b). Observe that we can adapt this drawing for the case when one of x_i and x_k, or both are true. Similarly, if the literal x_i or x_j is true, w.l.o.g., x_i, then we draw these paths as shown in Figure 5(c). Observe that we can adapt this drawing for the case when one of x_j and x_k, or both are true.

Assume now that G has a geometric thickness two representation. Observe that the graph induced by the edges in L_r in Figure 4(i) is a subdivision of a triconnected planar graph. Consequently, by a theorem of Whitney [22], such a graph has a unique combinatorial embedding up to homomorphisms of the plane. We choose the planar embedding such that the edge (R_1, R_2) lies on the outerface and the clause boxes obtain the same order as depicted in Figure 4(i). Observe that upper-halves of the three literal gadgets of each clause are forced to lie inside the corresponding clause box. Hence the paths between x_{ib} and x_{ja}, and x_{ka} and x_{jb} must be drawn inside the clause box. Consequently, at least one of the literal gadget must correspond to true in each clause box, otherwise, there must be a crossing in the same planar layer. Since the edges in E_l forces the literal gadgets corresponding to the same literal to have consistent embeddings, we find a satisfying truth assignment for $I(U, C)$. □

5 NP-hardness of Colorability

In this section we show the NP-hardness of coloring a graph with geometric thickness t with $4t-1$ colors. By $I(G,T,C)$ we denote the problem of coloring a graph G with C colors, where $\overline{\theta}(G) \leq T$. We first introduce a few definitions. A *join* between two graphs is an operation that given two graphs, adds all possible edges that connect the vertices of one graph with the vertices of the other graph. By $\mathcal{G}_t$ we denote a class of thickness t graphs that satisfies the following conditions: (1) $\mathcal{G}_1$ is the class of planar graphs. (2) If $t{>}1$, then $\mathcal{G}_t$ consists of the graphs obtained by taking a join of K_2 and G, where $G \in \mathcal{G}_{t-1}$.

Observe that $\overline{\theta}(\mathcal{G}_t) \leq t$. We now have the following lemma, whose proof is omitted due to space constraints.

Lemma 1. *It is NP-hard to color an arbitrary graph $G \in \mathcal{G}_t$ with $2t+1$ colors.*

We use Lemma 1 to prove the NP-hardness of coloring geometric thickness t graphs with $4t-1$ colors. We employ induction on t. If $t=1$, then coloring a planar graph (i.e., $t=1$) with $4t-1=3$ colors is NP-hard [15]. We now assume inductively that for any $t' < t$, it is NP-hard to color a geometric thickness t' graph with $4t'{-}1$ colors. To prove the hardness of coloring a geometric thickness t graph with $4t{-}1$ colors, we reduce the hardness of coloring a geometric thickness $t-1$ graph with $2(t-1)+1$ colors. Given an instance $I(G, t-1, 2(t-1)+1)$, we construct a graph H such that $\overline{\theta}(H) \leq t$ and H is $4t-1$ colorable if and only if G is $2(t-1)+1$ colorable.

Let the number of vertices in G be n. Take n copies $H_1, H_2, \ldots, H_n$ of K_{2t}, and join each vertex of G with a distinct H_i, $1 \leq i \leq n$. Finally, take a copy H' of K_{2t-1} and join it with every H_i. Let the resulting graph be $H(G,t)$. To prove that $\overline{\theta}(H(G,t)) = t$, we first review a construction of Dillencourt et al. [7] that gives a thickness t representation of K_{4t}. They proved that the $4t$ vertices of a K_{4t} can be arranged in two rings of $2t$ vertices each, an outer ring and an inner ring, such that it can be embedded using exactly t planar layers. The vertices of the inner ring are arranged to form a regular $2t$-gon. For each pair of diametrically opposite vertices, a zigzag path is constructed as illustrated in Figure 6(a). This path has exactly one diagonal connecting diametrically opposite points (i.e., the diagonal connecting the two gray points in the figure.) The union of these zigzag paths, taken over all t pairs of diametrically opposite vertices, contains all the edges of K_{2t} in the inner ring, as shown in Figure 6(b). For each pair of diametrically opposite vertices, we can draw rays from each vertex of the zigzag path, in two opposite directions, so that none of the rays crosses any edge of the zigzag path. These rays, in each direction, meet at a common point (e.g., p or q) forming the outer ring, as shown in Figure 6(c).

Lemma 2. $\overline{\theta}(H(G,t)) \leq t$*, where $t > 1$ and $G \in \mathcal{G}_{t-1}$.*

Proof. We compute a thickness t representation of $H(G,t)$, as follows. Since $G \in \mathcal{G}_{t-1}$, $\overline{\theta}(G){\leq}t{-}1$. Take a thickness $t{-}1$ representation of G and rotate it (if necessary) such that no two vertices lie on the same vertical line. Let the

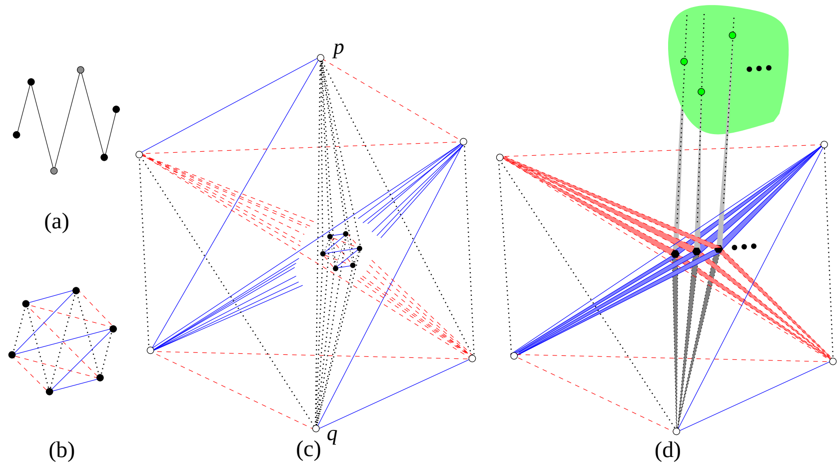

Fig. 6. (a)–(c) Dillencourt et al.'s construction [7]. (a) A zigzag path in the inner ring. (b) K_{2t}, where t=3. (c) K_{4t}, where t=3. (d) The geometric thickness two representation of $H(G, t)$, where t=3. Each subgraph H_i is determined by an inner ring. The vertices of G are in the green region.

resulting drawing be Γ. Now construct an outer ring as in Dillencourt et al.'s construction [7], and delete a vertex from the ring to obtain a thickness t drawing of H', as shown in Figure 6(d). For each H_i, construct an inner ring that lies along the vertical line determined by its corresponding vertex in Γ. Figure 6(d) shows this correspondence with dotted lines. All the edges that connect the vertices of G with the vertices of H_i (i.e., the edges in the light-gray region) lie in the t-th layer. Note that the inner rings must be scaled down small enough such that these edges do not create any edge crossing in the t-th layer. □

Theorem 4. *It is NP-hard to color an arbitrary geometric thickness t graph with $4t - 1$ colors.*

Proof (Outline). If $t = 1$, then coloring a planar graph (i.e., $t = 1$) with $4t-1 = 3$ colors is NP-hard [15]. Assume now that $t > 1$. Given an instance $I(G, t-1, 2(t-1)+1)$, where $G \in \mathcal{G}_{t-1}$, we construct a corresponding graph $H(G, t)$. We prove that G is $2(t-1)+1$ colorable if and only if $H(G, t)$ is $4t - 1$ colorable. The details are omitted due to space constraints. □

Acknowledgements. We thank the anonymous reviewers for their suggestions to improve the presentation of the paper. The third author received financial support from the Faculty of Graduate Studies at the University of Manitoba for partial travel expenses to present this paper.

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