Triangulating a System of Disks

Daniel Peterseim**

Abstract
A generalization of Delaunay triangulations applicable to some system of disks is introduced. In this subdivision of the convex hull of the union of disks, the classical role of vertices is assumed by the given disks; neighboring disks are connected by channel-like objects which assume the classical role of edges. The generalized Delaunay triangulation is derived by performing a limiting process of classical Delaunay triangulations with respect to a convergent sequence of polygonal approximations of the system of disks. We comment on duality with respect to certain Voronoi diagrams.

The present paper is a revised version of the conference contribution [10].

Acknowledgment. The author would like to thank three anonymous referees for their valuable comments and advice concerning the manuscript and beyond.

1 Introduction
Let $\mathcal{B}$ be a system (finite set) of closed disks in the plane. We shall identify every $B = \{x \in \mathbb{R}^2 : \text{dist}(x, c_B) \leq r_B\} \in \mathcal{B}$ by its center $c_B \in \mathbb{R}^2$ and its radius $r_B > 0$; $\text{dist}(\cdot, \cdot)$ being the Euclidean distance in $\mathbb{R}^2$. We assume that the disks are pairwise disjoint, i.e. $B_1 \cap B_2 = \emptyset$ for all $B_1, B_2 \in \mathcal{B}$.

The restriction to the two dimensional setting conduces to keep the presentation of the basic concept as simple and clear as possible. The setting yet has an interesting practical application. Consider the disks to be cross sections of fibers in fiber-reinforced composite materials such as fiber glass, for example. It is an important task of computational mechanics to derive insight about effective material properties, e.g. transport, mechanical, and electromagnetic properties. If the considered fiber composite is unidirectional, then effective material properties are typically modeled by partial differential equations on some cross section which justifies the value of our geometric model.

The solution of such partial differential equations is challenging due to the highly complicated geometry represented by the disks. Considering, e.g., the stationary heat equation $\text{div} c(x) \nabla u(x) = f(x)$ in a bounded domain $\Omega$ with given temperature at the boundary, the coefficient $c$ takes significantly different values in $\Omega \setminus \cup \mathcal{B}$ and $\cup \mathcal{B}$. The use of standard finite element methods requires the jumps of the coefficient, i.e. the disks, to be resolved by the underlying computational mesh, which easily becomes extremely expensive.

The aim of this paper is to describe an efficient and problem adapted subdivision of the convex hull of the union of disks $\text{conv}(\cup \mathcal{B})$ to be used within special finite element methods. The desired subdivision will turn out to be a generalization of the classical Delaunay triangulation [4]: Given a set of points $S \subset \mathbb{R}^2$, the classical Delaunay triangulation is a set of (closed) triangles $\mathcal{D}(S)$ determined by the classical Delaunay criterion saying that the open circumdisk of any triangle must not contain any elements of $S$. The Delaunay triangulation $\mathcal{D}(S)$ is not unique if $S$ contains 4 points that are co-circular. We cure this instance of geometric degeneracy by considering $\mathcal{D}(S)$ as a subdivision into (closed) cyclic polygons with 3 or more vertices such that a strict Delaunay criterion is fulfilled: The (unique) closed circumdisk of each cyclic polygon does not contain any vertices of $S$ excepts its own ones.

A generalized concept of triangulation, in which the point set $S$ is replaced by some system of multidimensional convex objects, is introduced in [9]. There, generalized Delaunay triangulations are derived either by exploiting duality with respect to certain (additively weighted) Voronoi diagrams (see Section 4) or by generalizing the aforementioned Delaunay criterion. In this paper we provide a different (self-contained) construction, in which the generalized triangulation is derived as the limit of certain classical Delaunay triangulations related to polygonal approximations of the disks (see Section 2). The subdivision of the geometry is the basis of new finite element spaces as indicated briefly in Section 3.

Notation. We use capital letters $A, B, C, \ldots$ to indicate sets, bold letters $x, y, z, \ldots$ indicate points in $\mathbb{R}^2$. Systems of sets are denoted by calligraphic capital letters $\mathcal{B}, \mathcal{D}, \ldots$. For systems of sets $\mathcal{B}$ we denote the union of its elements by $\cup \mathcal{B} := \bigcup_{B \in \mathcal{B}} B$. Given a set $X \subset \mathbb{R}^2$ we denote its closure by $\text{cl}(X)$, its relative interior by $\text{relint}(X)$, and its boundary by $\text{bnd}(X)$.

2 Generalized Delaunay Triangulation
Consider polygonal approximations $\mathcal{B}^n$, $n \in \mathbb{N}$, of $\mathcal{B}$ in which the disks are approximated by certain regu-
lar polygons. More precisely, $B^n := \{B^n : B \in B\}$, where $B^n$ is a regular, i.e. equiangular and equilateral, polygon with $2^n$ vertices $V(B^n)$ located on $\text{bnd}(B)$ (the circumcircle of $B^n$). By $V(B^n) := \bigcup_{B \in B} V(B^n)$ we denote the set of vertices of $B^n$. For technical purposes the polygonal approximations $B^n$ are assumed to be nested, i.e. $V(B^n) \subset V(B^{n+1})$ for all $n \in \mathbb{N}$. The classical Delaunay triangulation with respect to the point set $V(B^n)$ is denoted by $\mathcal{D}^n := \mathcal{D}(V(B^n))$. Fig. 1 shows some set of disks (a), the corresponding polygonal approximations, and triangulations for $n$ being 3 (b) and 5 (c).

We observe that for $n$ tending to infinity, new structures evolve, namely channel-like connections between neighboring disks.

To investigate this process let us fix some disk $B \in \mathcal{B}$, and some point $x \in \partial B$. Consider the sets

$$\omega^n_x := \{ T \in \mathcal{D}^n \setminus B^n : x \in T \}, \ n \in \mathbb{N},$$

of cyclic polygons containing $x$. The circumdisks of the elements of $\omega^n_x$ form the sets

$$C^n_x := \bigcup \{ C_T : T \in \omega^n_x \}, \ n \in \mathbb{N},$$

where $C_T$ denotes the (closed) circumdisk of a cyclic polygon $T$. Due to the Delaunay property, $C^n_x$ converges, as $n \to \infty$, to the maximal (w.r.t. to the diameter) disk $C_x \subset \mathbb{R}^2 \setminus \bigcup B$ which is tangential to $B$ in $x$. Note that $C_x$ is unbounded (a shifted halfspace) if and only if $x \in \text{bnd}(\text{conv}(\bigcup B))$. The disk $C_x$ tangentially intersects other disks of $B$ besides $B$. These (finitely many) intersection points $C_x \cap \bigcup B$ span a unique cyclic polygon $T_x := \text{conv}(C_x \cap \bigcup B)$. Considering disks as cyclic polygons with infinitely many vertices, a subdivision of $\text{conv}(\bigcup B)$ into cyclic polygons is given by

$$\mathcal{D}^\infty := B \cup \{ T_x : x \in \text{bnd}(\bigcup B) \}.$$  \hfill(1)

By construction the above subdivision is strictly Delaunay. However, $\mathcal{D}^\infty$ has infinitely many elements which is not desired from a practical point of view.

The desired (finite) subdivision is derived by taking the quotient modulo of the mapping $\mathcal{B}(\cdot) : \mathcal{D}^\infty \to \mathcal{P}(\mathcal{B})$,

$$\mathcal{B}(T) := \{ B \in \mathcal{B} : V(T) \cap B \neq \emptyset \} \text{ for } T \in \mathcal{D}^\infty.$$

Let us say that $T \in \mathcal{D}^\infty$ connects a subset of disks $A \subset \mathcal{B}$ if $\mathcal{B}(T) = A$. $\mathcal{B}(\cdot)$ maps the cyclic polygons of $\mathcal{D}^\infty$ into the power set of $\mathcal{B}$, which is finite. The generalized Delaunay triangulation is given by the preimages of ensembles of disks under $\mathcal{B}(\cdot)$. In this context we distinguish 3 characteristic classes of cyclic polygons:

1. Cyclic polygons that connect only a single disk, i.e. either the disks themselves or single points in $\text{relint}(\text{bnd}(\bigcup B) \cap \text{conv}(\bigcup B))$. We neglect the latter points and refer to the disks as generalized vertices.

2. Cyclic polygons that connect exactly two disks. The union of polygons that connect a certain pair of disks is denoted as a generalized edge. More pre-
smooth solutions to the conductivity problem can be approximated by elements of $S$ up to an error that is proportional to maximal distance between neighboring disks (generalized vertices) as in the classical case of a triangulated point set. Further valuable properties such as a discrete maximum principle which holds for $S^n$ [3] are preserved in the reduced limit space $S$. A detailed derivation and analysis of such methods will be provided in [11].

4 Generalized Voronoi-Delaunay Duality

We want to outline the relationship between generalized Delaunay triangulations and certain Voronoi diagrams. Consider again the circumcircles $C_{x}$, $x \in bnd(\cup B)$, of the elements of the (infinite) Delaunay triangulation $\mathcal{D}^\infty$ (see (1)) and note that $C_{x} = C_{y}$ for all $y \in C_{x} \cap (\cup B)$. Moreover, the center of $C_{x}$ is equidistant to all $y \in C_{x} \cap (\cup B)$. Thus, the union of the circumcircle centers $C_{x}$, $x \in bnd(\cup B)$, defines the Voronoi diagram with respect to the system of disks $B$, i.e. the circumcircle centers form curves which tessellate the plane into regions reflecting proximity with respect to the disks in $B$. The latter Voronoi tessellation is well known as the additively weighted Voronoi tessellation. We refer to [5, Section 4.5.3], and references therein; a visualization is given in Fig. 3(a,b). The relation between the generalized Delaunay triangulation and the Voronoi diagram with respect to the system of disks can be regarded as a generalization of the classical straight line duality (see, e.g., [12]) between the Delaunay triangulation and the Voronoi tessellation [13] with respect to a set of points. There are fast algorithms (proportional to $\text{card}(B)$ up to logarithmic factors) available for Voronoi diagrams with respect to a system of disks [6, 8, 7]. These algorithms, by duality, provide also the combinatorial structure of the generalized Delaunay triangulations. The precise geometric objects, especially the generalized edges, can be represented by their extreme points. The latter can be computed (locally and parallelized) by projecting Voronoi vertices onto the associated disks.

Note that, if the disks in $B$ are of equal size, then the Voronoi tessellation with respect to $B$ and the Voronoi tessellation with respect to disk centers coincide; see Fig. 3(a). The generalized Delaunay triangulation, also known as triangle-neck partition in the special case of equally sized disks [2], and the classical Delaunay triangulation of the disk centers are combinatorially equivalent, i.e. they connect the same vertices. Moreover the cyclic polygons of the generalized triangulation are isotropically scaled versions of their classical counterparts. If the radii of the disks tend to zero we recover the classical Delaunay triangulation of the midpoints. If the disks are not of equal size, the generalized Delaunay triangulation is not induced by the Delaunay triangulation of the disk centers since combinatorial changes

3 Finite Element Methods

Generalized Delaunay triangulations are a suitable geometric basis for finite element simulations of effective properties of composite materials. New finite element spaces can be derived in a similar manner as the triangulation itself, i.e. by first taking the limit of classical spaces with respect to the approximative triangulations $\mathcal{D}^n$ and then choosing a finite subspace appropriate for the problem to be solved. Considering, e.g., heat conductivity in a fiber composite with perfectly conducting cylindrical fibers, a suitable discrete space with respect to every disk (due to perfect conductivity) and then requiring the shape functions to be constant and cyclic polygons connecting 3 or more disks.

$$\mathcal{E} := \{E_{B_1, B_2} \neq \emptyset : B_1 \neq B_2 \in B\},$$

and cyclic polygons connecting 3 or more disks.

$$\mathcal{T} := \left\{T \in \mathcal{D}^\infty \mid B(T) = \{B_1, B_2, \ldots, B_k\}, B_1 \neq \ldots \neq B_k \in B, k \geq 3 \right\}.$$ 

In contrast to classical triangulations, the consideration of vertices and edges as genuine elements of the subdivision is essential to ensure that the union of all elements indeed covers $\text{conv}(\cup B)$. As in the classical setting intersections of distinct elements are of lower dimension (at most 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{A generalized nodal basis function taking value 1 in a certain disk and 0 in all others.}
\end{figure}
might appear as it can be observed in Fig. 3(b). Similarly, the Voronoi tessellation with respect to $B$ and the Voronoi tessellation with respect to the disk centers do not coincide. In addition, Fig. 3(c) illustrates that Voronoi edges, i.e. 1-dimensional intersections of neighboring Voronoi cells, might not be connected [8] which leads to multiple connectivity of the corresponding dual generalized Delaunay edge.

5 Conclusion

We have introduced a subdivision of the convex hull of a union of disks which contains the disks themselves as elements. Further elements are simple geometric objects, i.e. generalized edges and cyclic polygons. The number of elements of the subdivision is of order $\text{card}(B)$ which is minimal in comparison to the descripational complexity of $B$.

The new generalized triangulations inspire the design of new finite element methods perfectly fitted to the difficulties and requirements originating from the complicated geometries of random composite materials. Our approach is not restricted to systems of disks; generalizations to multidimensional systems of convex sets are straightforward.

References