On Stretch Minimization Problem on Unit Strip Paper

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Abstract

For a given mountain-valley pattern of equidistant creases on a long strip paper, there are many folded states consistent with the pattern. Among these folded states, we like to fold a paper so that the number of the paper layers between each pair of hinged paper segments is minimized. We first formalize this problem as optimization problem. The complexity of the problem is not known. In this paper, we give partial results related to the problem. First, we show that the problem is well-defined even in a simple folding model. The simple folding model is the most primitive model of basic origami models, and hence the folding availability is very restricted. We show a universality theorem of the simple folding model for this problem. That is, every flat folded state consistent with any given pattern can be folded by a sequence of simple foldings. Next, we investigate the number of folded states consistent with a given pattern. For a given random mountain-valley pattern, the expected number of folded states consistent with the pattern is exponential. More precisely, the expected number $f(n)$ of folded states for a random mountain-valley pattern of length n is (1) $\Theta(1.65^n)$ from the experiments, and (2) between $\Omega(1.53^n)$ and $O(2^n)$ from theoretical bounds. The results say that a naive algorithm that checks all possible folded states for a given pattern to find an optimal folded state runs in an exponential time.

Keywords: linkage, pleat folding, rigid origami, universality theorem.

1 Introduction

What is the best way to fold an origami model? Origamists around the world struggle with this problem daily. Even if you have a good origami model with its crease pattern, this is not the end. To make the model, we have to search for clever, more accurate, or faster folding sequences and techniques. In this paper, we focus on the problem for accurate folding on a simple kind of one-dimensional creasing, where the piece of paper is a long rectangular strip, which can be abstracted into a line segment, and the creases uniformly subdivide the strip. A mountain-valley pattern is then simply a binary string over the alphabet $\{M, V\}$ (M for mountain, V for valley), which we call a *mountain-valley string*. Of particular interest in origami is the *pleat*, which alternates $MVMVMV\cdots$; see Figure 1. The pleat folding is quite unique in the sense that the folded state is unique [Asano et al. 10]. That is, there is only one unique folded state consistent with the string, and only the pleat folding has this property. In general, this is not the case. For example, for a string $MMVMMVMVVVV$, surprisingly, there are 100 distinct folded states consistent with this string. Among them, what is the *best* folded state? From the practical point of view, it seems better to decrease the number of paper layers between each pair of paper segments hinged at a crease as possible as we can. If we have many paper layers between the hinged papers, it becomes to be difficult to fold with accuracy, and if we have too many, we cannot fold any more. This is a typical problem we meet when we fold recent complex origami models (Figure 1).

Figure 1: Simple pleats and an origami angel with many pleats folded by Takashi Hojyo (reproduced with his kind permission).

For a folded state, we define a *stretch* at a crease i is the number of the paper layers between the papers hinged at the crease i. Then, we can consider two optimization problems as follows:

Input: A strip of paper of length $n+1$ with a mountain-valley string s in $\{M, V\}^n$.

Goal: Among the folded states consistent with s, we aim to find a folded state of unit length that (1) minimizes the maximum stretch of all stretches at

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each crease in the folded state, or (2) minimizes the total stretch of all stretches at each crease in the folded state.

We note that the minimization problem for the average stretch is equivalent to the second optimization problem (by dividing n). These two problems have different solutions in general. For example, among the 100 valid folded states of the string $MMVMMVMVVVV$, the minimum maximum stretch is 3, which is achieved by the folded state [4|3|2|5|6|0|1|7|9|11|10|8] (the details of this notation is described later), the minimum total stretch is 11 by the other state [4|3|2|0|1|5|6|7|9|11|10|8], and moreover, these solutions are unique for this string (they are checked by an exhaust search).

Here we state an open problem:

Open Problem: Determine the computational complexity of the minimization problems of the maximum/total stretch for a given string s in $\{M, V\}^n$.

We first show that the problem is well-defined even in a simple folding model. The simple folding model is one of basic origami models introduced by Arkin et al. [Arkin et al. 04]. We show that, even in the simple folding model, every folded state consistent with any given mountain-valley string can be folded. This universality theorem of the simple folding model is related to the one-dimensional flat folding problem [Demaine and O'Rourke 07, Section 12.1], and the locked chain problem, that has a long and rich history [Demaine and O'Rourke 07, Chapter 6].

The open problem seems to be NP-hard in general. We next prove this intuition by counting. For a given string, if the number of valid folded states for the string is not huge, a straightforward exhaust search can find the solutions efficiently. However, this is not the case. In this paper, we state the following negative results.

Theorem 1 *Let s be a mountain-valley string of length n taken uniformly at random, and $f(n)$ the expected number of folded states consistent with s. Then experimental results imply that $f(n) = \Theta(1.65^n)$. We also show the upper and lower bounds; $f(n) = \Omega(1.53^n)$ and $f(n) = O(2^n)$.*

The results guarantee that $f(n)$ is an exponential function, and hence the exhaust search approach has no hope in general.

Theorem 1 comes from more general counting problem:

Theorem 2 *Let $F(n)$ be the number of folded states of a paper of length $n+1$. Then experimental results imply that $F(n) = \Theta(3.3^n)$. We also have the upper and lower bounds: $F(n) = \Omega(3.06^n)$ and $F(n) = O(4^n)$.*

Theorem 1 says that a simple exhaust search runs in an exponential time in general. Unfortunately, we have no idea about the computational complexity of the optimization problems up to now.

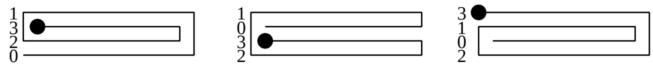

Figure 2: Three foldings for the mountain-valley string VVV.

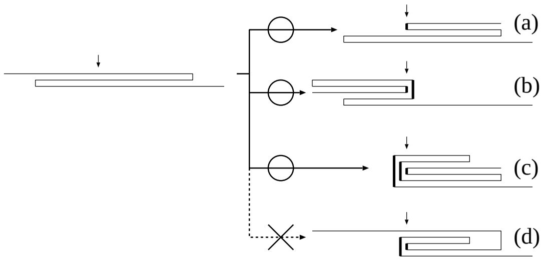

Figure 3: Simple folding model.

A part of Theorems 1 and 2 was presented as an oral talk at 5th international conference on Origami in science, mathematics, and education (5OSME) [Uehara 10]. All the results in this paper will be published in the future book that collects the works in 5OSME.

2 Preliminaries

The *paper strip* is a one-dimensional line with creases at every integer position. At first, the paper of length $n+1$ with the string of length n is placed at the interval $[0..n+1]$. The paper is *rigid* except the creases on the integer positions; that is, we can only fold the paper at these integer positions. At the end of the folding operations, all creases are folded, the paper becomes unit length, and the direction of each folded crease follows the letter (in $\{M, V\}$). That is, the ith letter of the mountain-valley string of length n indicates the final folded state of the crease at integer point i in $[1..n]$. We call each paper segment between i and $i+1$ the *ith segment*. Each final folded state can be represented by the ordering of the segments; for example, a pleat folding $MVMV$ is described by [0|1|2|3|4] or [4|3|2|1|0], and a crease string VVV produces [1|3|2|0], [1|0|3|2], [3|1|0|2], or their reverses (Figure 2). We distinguish between the left and right endpoints of the paper, but we sometimes ignore the reverse of one folded state since they are essentially the same. In fact, the sides of a folded state sometimes turn upside down when we fold all paper layers at a crease from right to left or from left to right.

We employ the *simple folding model* by Arkin et al. [Arkin et al. 04] (see also [Demaine and O'Rourke 07, Sec. 14.1] and [Cardinal et al. 09]). Precisely, each simple folding is the folding from a flat folded state to another flat

Figure 4: Two legal folded states for a string which cannot be exchanged by local simple (un)foldings.

folded state by the following operations: (1) put the flat (folded) paper (on the reverse side, if necessary) in a plane, (2) choose an integer point to fold, and (3) valley fold consecutive most inner paper layers at the crease. We note that a simple folding of some paper layers has a restriction that the most inner paper segments have to be folded. The other paper segments are fixed, and we cannot slip the paper segments into the other paper layers. In Figure 3, (a), (b), and (c) are simple foldings, but (d) is not allowed. A simple unfolding is defined by rewinding a simple folding; that is, we can *unfold* a folded state a to a folded state b if and only if a can be obtained from b by a simple folding. We note that a can be unfolded to one of several folded states; that is, a simple unfolding is not just a rewind of the *last* simple folding. (In a sense, conceptually, a simple unfolding can be seen as a simple folding. That is, they are the same operation that flips consecutive most inner paper layers at a crease.)

For a mountain-valley string s, we call a folded state *legal for s* if it is consistent with the string.

3 Universality of the simple folding model

In this section, we show that the simple folding model is strong enough to discuss the strip paper of equidistant creases. More precisely, we show that every legal folded state for any string can be made by a sequence of simple foldings.

We first observe that any string has a legal folded state:

Proposition 3 *For any given mountain-valley string s in $\{M, V\}^n$, there exists a legal folded state.*

Proof. We can fold a paper for any given string by using the idea of an "end fold" in [Demaine and O'Rourke 07, p. 192]; we make an end fold at the leftmost crease according to the string, glue it, and repeat it. After n foldings, we fold it into a unit length, and obtain a legal folded state for the string. □

Next we show that any legal folded state can be folded from the initial state by a sequence of simple foldings.

Theorem 4 *Let P be any legal folded state for a mountain-valley string s in $\{M, V\}^n$. Then P can be folded from the initial state by a sequence of simple foldings.*

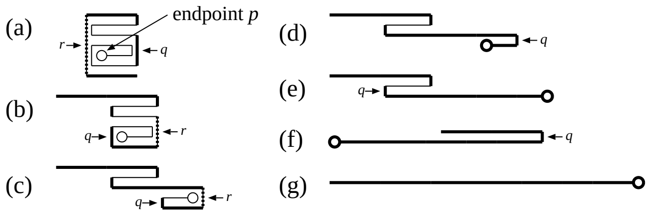

Figure 5: Simple (un)foldings.

Before proving Theorem 4, we comment on the claim of the theorem. One may think that Theorem 4 is "trivial". But it is not so trivial.

A typical counterexample for this intuition is shown in Figure 4; these two folded states are legal for the same mountain-valley string, but they cannot be exchanged by just local simple (un)foldings. (In fact, the left folded state is not so trivial to fold by a sequence of simple foldings.) This fact implies that folding of these states from the initial state requires some global strategy.

By definition of unfolding, a folded state P can be folded from the initial state by simple foldings if and only if P can be unfolded to the initial state. Hence, we prove Theorem 4 by showing how to unfold any folded state P to the initial state. This is strongly related to two well investigated problems in computational origami.

First, this is a kind of the "(un)locked chain problem in 2D" that has a long and rich history [Demaine and O'Rourke 07, Chapter 6]. It is known that there is no locked chain in 2D [Demaine and O'Rourke 07, Section 6.6]. However, this fact does not imply Theorem 4 since the operations are restricted to simple unfoldings in our theorem.

Second, our problem is also related to "one-dimensional flat foldings" [Demaine and O'Rourke 07, Section 12.1]. In this problem, we aim to determine if there exists a flat folded state for a given pattern on a strip paper. The known result says that we can find *one* flat folded state by repeating crimp folding and end folding if it exists. (In fact, Proposition 3 is a special case of this problem.) Hence, the known algorithm cannot construct a given *specified* folded state from the initial state. (In contrast with Theorem 4, this is not always possible for non-unit case; see Concluding Remarks.)

Thus, in a sense, our problem is more difficult than the above problems; the folded state is specified, and we can only use simple foldings to make it. On the other hand, all links in our "linkage" have unit length. Using this advantage, we can show the universality theorem for the unit strip paper in the simple folding model.

In the following proof, we do not need the fact that P is folded into unit length. (We only use that we can fold at every integer point.) Hence we prove the following stronger claim than Theorem 4:

Corollary 5 *Let P be any flat folded state of a paper of length $n+1$ such that every folded point is placed at an integer point in $[1..n]$ in the initial state. Then P can be folded from the initial state by a sequence of simple foldings which are made at each integer point. Moreover, the total number of simple (un)foldings is bounded above by $2n$.*

Proof. As mentioned above, we prove the claim by unfolding any folded state P to the initial state. Intuitively, we unbind the last segment and arrange the last consecutive segments in line. But before unbinding, we have to peel off the papers covering the last segment. To describe in detail, let p be the last endpoint of the paper, that will be placed at integer point $n+1$ in the initial state. We abuse the symbol P to denote the current flat folded state. We here define *visibility* of a point on P; a point is *visible* on P if and only if it appears on a surface of P. All visible points are drawn in thick lines in Figure 5. We note that a crease can be visible even if it is between two invisible segments (e.g., the crease point q in Figure 5(a) is a visible point of length 0). According to the visibility of the last endpoint p, we have two cases. (In the context of algorithm, we have two "phases".)

Case 1: The point p is not visible in the folded state P. (In Figure 5, (a), (b), and (c) are in this case.) Let q be the largest visible crease. That is, all points $r > q$ (including p) are invisible. We note that q can be flat. Let q' be the smallest folded crease with $q' > q$. If there is no folded crease greater than q, set $q' = p$.

We first suppose that the crease q is flat. Then, by the visibility of q, the papers on the visible side of q can be flipped by a simple folding (or a simple unfolding) at the crease point q'. Then, the largest visible crease is updated from q to $q' > q$.

Next we suppose that q is a folded crease. Without loss of generality, the crease $q+1$ is placed at left of q as in Figure 5(a). Then, the papers on the opposite side of $q-1$ with respect to the segment $q(= [q, q+1])$ covers the point $q+1$ but do not cover $q-1$ since q is visible. This fact implies that these papers can be flipped by a simple (un)folding at the crease point $q' \geq q+1$. (In Figure 5(a), the bottom paper is flipped by folding the crease r.)

In any case, the largest visible crease is updated from q to $q' > q$ by one (un)folding. We repeat this process until the point p becomes visible. The number of repeating is at most n, and hence the total number of (un)foldings in case 1 is at most n.

Case 2: The point p is visible in the folded state P. (In Figure 5, (d), (e), and (f) are in this case.) Let q be the largest folded crease. If q is not visible, since there is no folded crease between p and q, and p is visible, we can make q visible by just one simple (un)folding at the point q by using the same technique in case 1.

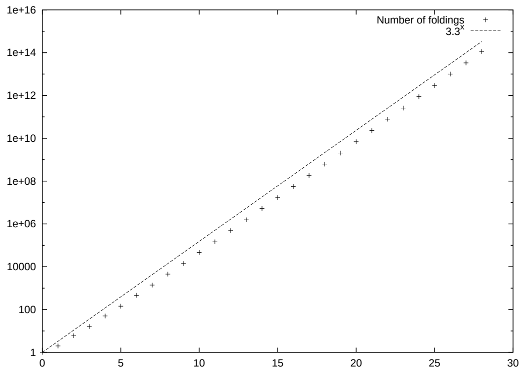

Figure 6: Experimental result of the number of folded states.

Now we can assume that all points in $[q, p]$ are visible. Moreover, these points can be seen from one side. (For example, suppose that $[q, r]$ are visible from top and $[r, p]$ are visible from bottom. In this case, since p is the endpoint of the paper, the paper becomes disconnected.) Hence we can unfold at the folded crease q and make it flat. This does not change the visibility of p. Thus we can repeat this process until the whole creases become flat. We can observe that these two (un)foldings (to make q visible if necessary, and to make q flat) can be done at once. Hence the total number of (un)foldings in this case is at most n.

From above arguments, we have Theorem 4 and Corollary 5. $\square$

By Proposition 3, the optimization problems are well-defined for any mountain-valley string. Moreover, by Theorem 4, the problem is worth considering on the simple folding model. Furthermore, if we have an optimal solution, it can be folded in linear time by Corollary 5.

4 The number of folded states

In this section, we will prove Theorems 1 and 2. Using Theorem 2, Theorem 1 follows easily. Hence we first focus on Theorem 2.

Recall that $F(n)$ is the number of folded states of a paper of length $n+1$. A simple algorithm can compute $F(n)$ in a straightforward way for small n, but we can find the correct values for larger n at "The On-Line Encyclopedia of Integer Sequences" with id:A000136[1]. (According to the site, this sequence is "the number of ways of folding a strip of n labeled stamps", which fits our problem.) Plotting the sequence, we have an experimental result $F(n) = \Theta(3.3^n)$ (Figure 6). Now we turn to the upper and lower bounds of $F(n)$.

Lemma 6 $F(n) = O(4^n)$.

[1] `http://www.research.att.com/~njas/sequences/A000136`

Proof. We first assume that n is even, say $n = 2k$, and each folded state of unit length is placed on the interval $[0..1]$. We see the relationship among the papers at the point 0. The papers should not be penetrated through each other. That is, at the point 0, k creases with one end (of the left end of the segment 0) make a nest structure. The number of nest structure with k pairs is given by the Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k} = \frac{(2k)!}{(k+1)!k!}$ (see, e.g., [Stanley 97]). Once the left end is connected to the right nest structure at the point 1, the paper order is automatically determined. The number of the possible connections of the left end to the right nest structure is k. Hence the number of folding ways can be bounded above by kC_kC_k.

Next we assume that n is odd, say $n = 2k+1$. Then, using the same argument, we have the upper bound $(k+1)C_{k+1}C_k$. Since $C_k \sim \frac{4^k}{k^{3/2}\pi} = O(4^k)$, the lemma follows. $\square$

In the proof, the connectivity of the paper is not counted in. To improve the upper bound, we have to consider the connectivity.

Lemma 7 $F(n) = \Omega(3.065^n)$.

Proof. We imagine folding the last k creases for some $k \ll n$. After folding the last k creases into unit length, we glue it, and obtain a paper of length $n-k+1$ with $n-k$ creases. Let $G(k)$ be the number of the folding ways of this last k creases under the constraint that the $(n-k)$th crease is not *covered*, which means the segments $(n-k-1)$ and $(n-k)$ are not separated by the other papers between $[n-k..n+1]$. Repeating this process, we have a lower bound: $F(n) > (G(k))^{\frac{n}{k}} = (G(k)^{\frac{1}{k}})^n$. This function $G(k)$ is also listed at "The On-Line Encyclopedia of Integer Sequences" with id:A000682[2]. (According to the site, this sequence is "the number of ways a semi-infinite directed curve can cross a straight line n times". This may not seem to fit our problem, but the semi-infinite directed curve corresponds to the paper strip itself, and the straight line corresponds to the point $(n-k+\frac{1}{2})$ on the paper strip.) Since the function $G(k)$ is a monotone increasing function for k, we use the largest value $G(43) = 830776205506531894760$ on the list, and obtain the lower bound $F(n) > (830776205506531894760^{\frac{1}{43}})^n = 3.06549^n$ for sufficiently large n. $\square$

By the experimental results listed on "The On-Line Encyclopedia of Integer Sequences" with Lemmas 6 and 7, we have Theorem 2 immediately.

Next we turn to Theorem 1. The number of mountain-valley strings of length n is 2^n. Hence, dividing the values in Theorem 2 by 2^n, we have Theorem 1.

[2]http://www.research.att.com/~njas/sequences/A000682

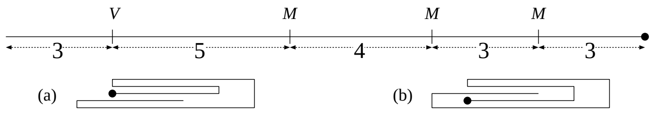

Figure 7: Two legal folded states; (a) is foldable by simple foldings, and (b) is not foldable by simple foldings.

5 Concluding Remarks

In this paper, we state an open problem that asks the computational complexity of the minimization problems of the maximum/total stretch of a strip paper with a given mountain-valley string. We first show that the problem is well-defined even in a simple folding model. That is, we show that any given folded state of a strip paper can be folded by a sequence of simple (un)foldings in Section 3. The proof of the universality theorem gives us a linear time algorithm that requires at most $2n$ (un)foldings. The improvement of this bound $2n$ to n remains to be open.

The universality theorem is related to the (un)locked chain problem in 2D [Demaine and O'Rourke 07, Chapter 6] and the one-dimensional flat folding problem [Demaine and O'Rourke 07, Section 12.1]. Extending the proof of Theorem 4, for the one-dimensional flat foldings, one might wonder if any specified legal folded state can be folded from the initial state even if we allow nonuniform intervals. However, this is not the case. In Figure 7, both of (a) and (b) are legal folded states for the above mountain-valley string $VMMM$. Although (a) is foldable by a sequence of simple foldings, (b) is not. In fact, (b) cannot be unfolded at all from this position by a simple unfolding. From the viewpoint of industry, the characterization of folded states that can be folded by a sequence of simple foldings seems to be a nice future work.

Although the computational complexity of the minimization problems are open, we show that the number of folding ways for a given mountain-valley string is exponential in general. Hence naive algorithms cannot solve the minimization problem. The author conjectures that finding an optimal folding is NP-complete, that is a future work.

Only the pleats have a unique (and optimal, of course) folded state. Hence, if a string is "close" to pleats, the number of folding ways may be small. However, this "closeness" is not necessarily clear. The string "$(\mathrm{MV})^{i-1}\mathrm{MM}(\mathrm{VM})^{i-1}$"[3] seems to be close to pleats except the center "MM". But the left pleats and right pleats are combined in any order to fold into unit length. As a result, this string has $\binom{2i}{i} \sim 2^i$ distinct legal folded states, which are also exponentially large. (This string

[3]Here we use the standard notation of string repetition; e.g., "$(\mathrm{MV})^3\mathrm{MM}(\mathrm{VM})^3$"="MVMVMVMMVMVMVM".

is called *shuffle pattern* and plays an important role to prove NP-completeness of a strongly restricted version of a puzzle *Kaboozle* [Asano et al. 10].) Hence the characterization of "close" strings to pleats is another further work.

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