On Degeneracy of Lower Envelopes of Algebraic Surfaces

Kimikazu Kato *

Abstract

We analyze degeneracy of lower envelopes of algebraic surfaces. We focus on the cases omitted in the existing complexity analysis of lower envelopes [Halperin and Sharir 1993], and re-define the degeneracy from the viewpoint of the adjacency structure and the number of connected components. We also define badness of degeneracy from such viewpoint and show how bad the degeneracy can be. This research is intended to contribute to a robust geometrical computation in the tolerant model.

1 Introduction

The importance of non-Euclidean Voronoi diagrams is arising. For example, a power diagram (weighted Voronoi diagram) is used to analyze the structure of the protein [7]. The angular Voronoi diagram [2] and the aspect ratio Voronoi diagram [1] are proposed as a tool for analysis of algorithms concerning computer vision. Among non-Euclidean Voronoi diagrams, the algebraic Voronoi diagram is one of the most important classes. Here "algebraic" means that the boundaries are expressed as polynomials. Actually, the examples enumerated above are all algebraic.

Concerning lower envelopes of algebraic surfaces, Halperin and Sharir [3] showed an upper bound of computational complexity. By analyzing the combinatorial structure, they showed the upper bound is $O(n^{2+\varepsilon})$ for any ε. However it is under the assumption that the surfaces are in a *general position*. It means that the discussion about degenerate cases is intentionally avoided.

From the viewpoint of real world applications, analysis of the degeneracy leads to robustness of computation. In such systems as computer-aided design (CAD) and geography information system (GIS), which deal with algebraic curves and surfaces, the tolerant model is employed inside. In the tolerant model, in order to handle the numerical error of computation, two points are assumed as a same point if their distance is within a given fixed small number. For example, the crossing point of two general spline curves cannot be expressed rigidly with double precision coordinate values. In the tolerant model, even if three curves are conceptually assumed to meet at one point, there might be a small region surrounded by the curves in reality. That kind of unexpected topological situation is prone to errors, and thus, for a robust computation, it is important to enumerate all the topological situations in advance.

In this paper, we define the degeneracy of a lower envelope of algebraic surfaces and analyze the worst case of degeneracy. As for a Euclidean Voronoi diagram, Sugihara and Iri [6] invented a robust algorithm even for a degenerate case. Although a similar robust algorithm can be a ultimate goal, an algorithm is not discussed in the present paper. We will just classify possible topological situation around degenerate point. That will directly be helpful for an enumeration of test cases for software which uses algebraic objects.

As a start-up of this line of research, Muta and the author [5] showed classification of singularities of the angular Voronoi diagram. It was considered that singularities were key point to analyze the degeneracy of the algebraic Voronoi diagram and classification of singularities was a first step to discuss the degeneracy. However, in a general context, the current paper shows an evil degeneracy may occur even without a singularity.

The rest of paper is organized as follows. In Sect. 2, we give some definition of terms and define degeneracy. We analyze the worst cases of degeneracy in Sect. 3. Then we conclude in Sect. 4.

2 Preliminaries

In the paper by Halperin and Sharir [3], they considered patches of algebraic surfaces to define a lower envelope and the algebraic surfaces used there can be general; i.e. a surface is defined by $f(x,y,z) = 0$ where f in polynomial in x, y, z. However, in this paper, we restrict the domain. We only consider surfaces expressed as $z = f(x,y)$ and assume that a surface is not a patch; i.e. the surface contains no boundary condition and is expressed by $z = f(x,y)$ alone.

Definition 1 (Lower envelope of surfaces) *For a given set of surfaces in the xyz-space, the projection of a set of lowest parts of $\mathcal{S}$ in z-direction, into xy-plane is called a* lower envelope *of $\mathcal{S}$. In other words, when a set of surfaces $\mathcal{S}$ is given as*

$$\mathcal{S} = \left\{ S_i \;\middle|\; S_i = \left\{ (x,y,z) \;\middle|\; z = f_i(x,y) \right\} \right\}, \qquad (1)$$

*Nihon Unisys, Ltd. kimikazu.kato@unisys.co.jp

then the lower envelope $\mathcal{D}$ of $\mathcal{S}$ is defined as

$$\mathcal{D} = \Big\{ D_i \;\Big|\; D_i = \big\{(x,y) \mid f_i(x,y) < f_j(x,y) \text{ for } \forall j\big\},\ D_i \neq \emptyset \Big\}. \tag{2}$$

In this paper, we only focus on the case when f_i's are all polynomials with coefficients in $\mathbb{R}$. In such a case, we simply say $\mathcal{D}$ is a diagram of $\{f_i\}$, or an algebraic diagram if not mentioning f_i's. To introduce a computational error model to a lower envelope, we consider a *parameterized* algebraic diagram defined as follows.

Definition 2 *A parameterized algebraic diagram $\mathcal{D}_\varepsilon$ is a lower envelope of a set of algebraic surfaces*

$$\mathcal{S}_\varepsilon = \Big\{ S_{\varepsilon i} \;\Big|\; S_{\varepsilon i} = \big\{(x,y,z) \mid z = f_i(x,y,\varepsilon)\big\} \Big\}, \tag{3}$$

where f_i is polynomial in x, y, ε. In other words, $\mathcal{D}_\varepsilon$ is defined as

$$\mathcal{D}_\varepsilon = \Big\{ D_{\varepsilon i} \;\Big|\; D_{\varepsilon i} = \big\{(x,y) \mid f_i(x,y,\varepsilon) < f_j(x,y,\varepsilon) \text{ for } \forall j\big\},\ D_{\varepsilon i} \neq \emptyset \Big\}, \tag{4}$$

for a set of polynomials $f_i(x,y,\varepsilon)$

For a small ε, D_ε emulates a behavior of a diagram with a numerical error. When we say "give a perturbation to $\mathcal{D}_\varepsilon$", it means "consider $\mathcal{D}_\varepsilon$ for a sufficiently small ε."

In short, a diagram is called degenerate if its perturbation gives a change of its topological structure. Then, what is a "change of topological structure?" In the Euclidean Voronoi diagram, it only means the adjacency structure of regions. However, in the algebraic diagram, there is another case to consider: the number of connected components of a region might change.

Before defining the degeneracy, we define an adjacent graph of a algebraic diagram as follows.

Definition 3 *Suppose that a diagram $\mathcal{D}$ is defined by polynomials $f_1, \ldots, f_n$. Then, an adjacent graph $G = (V, E)$ of a diagram $\mathcal{D}$ is an undirected graph whose vertex set is $V = \{1, \ldots, n\}$ and whose edge set is $E = \big\{(i,j) \mid D_i \text{ is adjacent to } D_j \text{ via a curve}\big\}$.*

Now we define two types of degeneracy as follows.

Type I The case when the adjacency graph changes with a perturbation.

Type II The case when the number of connected components of a region changes with a perturbation.

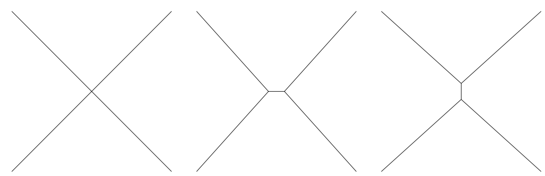
The Type I degeneracy is a generalization of degeneracy of Euclidean Voronoi diagram. Type II only appears in an algebraic diagram. Note that, since we do not consider patches of surfaces, other degenerate cases mentioned in [3] – the case when two boundary meet, singular point of a surface lies on a boundary of another patch, etc. – are omitted intentionally.

The followings are examples of degeneracy.

Example 1 *When polynomials are given as*

$$f_1 = (x-1-\varepsilon)^2 + y^2,\ f_2 = (x+1+\varepsilon)^2 + y^2,$$
$$f_3 = x^2 + (y-1+\varepsilon)^2,\ f_4 = x^2 + (y+1-\varepsilon)^2, \tag{5}$$

then $\mathcal{D}_0$ is shown as Fig. 1a, which is a degenerate case of a Euclidean Voronoi diagram. By giving ε a perturbation, its topological structure changes as in Fig. 1b ($\varepsilon > 0$) or Fig. 1c ($\varepsilon < 0$). This is an example of Type I degeneracy. Actually, the adjacency graph is changed as ε perturbs.



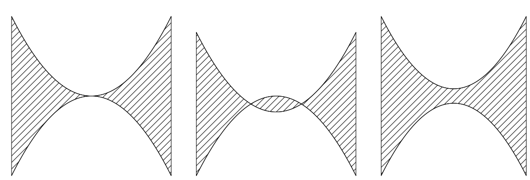
(a) $\varepsilon = 0$ (b) $\varepsilon > 0$ (c) $\varepsilon < 0$

Figure 1: Example of Type I degeneracy

Example 2 *Suppose that polynomials are given as*

$$f_1 = (y - x^2)(y + x^2 - \varepsilon),\ f_2 = 0. \tag{6}$$

Then, when $\varepsilon = 0$, the region $D_1 = \big\{(x,y) \mid f_1(x,y) < f_2(x,y))\big\}$ has two connected components (Fig. 2a). By giving ε a perturbation, the number of its connected components becomes three for $\varepsilon > 0$ (Fig. 2b), and becomes one for $\varepsilon < 0$ (Fig. 2c).



(a) $\varepsilon = 0$ (b) $\varepsilon > 0$ (c) $\varepsilon < 0$

Figure 2: Example of Type II degeneracy

3 Analysis of degeneracy

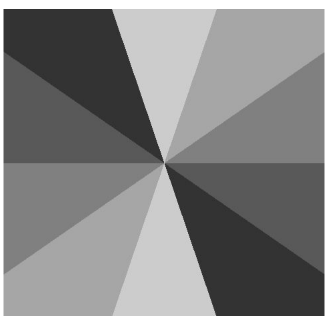
For the Euclidean Voronoi diagram, the adjacency graph of the regions can only be a planar graph, but for the algebraic diagram, it can become arbitrary.

Since adjacency graph can be arbitrary for the algebraic diagram, the thinkable worst case is that a cyclic graph becomes a perfect graph. The following shows it really happens.

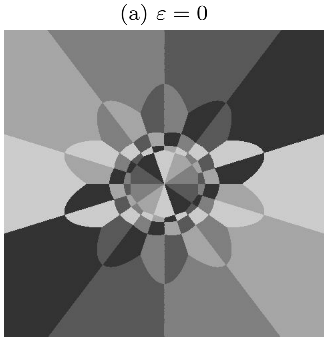
Theorem 1 *Consider a lower-envelope defined by*

$$f_i(x,y,\varepsilon) = \left(x\cos\frac{i\pi}{n} - y\sin\frac{i\pi}{n}\right)^2 + \varepsilon\left(x\sin\frac{i\pi}{n} + y\cos\frac{i\pi}{n}\right)^4 - \left(x\sin\frac{i\pi}{n} + y\cos\frac{i\pi}{n}\right)^2. \quad (7)$$

Then its adjacency graph changes from a cyclic graph to a perfect graph as ε *changes from* $\varepsilon = 0$ *to* $\varepsilon > 0$ *(Fig. 3).*



(a) $\varepsilon = 0$



(b) $\varepsilon > 0$

Figure 3: Example of a worst case of Type I

Proof. When $\varepsilon = 0$, by substituting $(r\cos\theta, r\sin\theta)$ for (x,y), f_i can be expressed as

$$f_i(r\cos\theta, r\sin\theta, 0) = -r\cos\left(2\theta - \frac{2i\pi}{n}\right), \quad (8)$$

Thus

$$f_i < f_j \text{ for } \forall j \quad (9)$$

$$\Leftrightarrow \cos\left(2\theta - \frac{2i\pi}{n}\right) > \cos\left(2\theta - \frac{2j\pi}{n}\right) \text{ for } \forall j \quad (10)$$

$$\Leftrightarrow \frac{(2i-1)\pi}{2n} < \theta < \frac{(2i+1)\pi}{2n} \text{ or } \pi + \frac{(2i-1)\pi}{2n} < \theta < \pi + \frac{(2i+1)\pi}{2n}. \quad (11)$$

Define the region dominated by f_i as R_i, then the inequality above means R_i is adjacent to R_{i+1} for $i = 0, \ldots, n-2$, and R_{n-1} is adjacent to R_0. Thus the adjacency graph for f_i's is cyclic.

As for $\varepsilon > 0$, we will only consider the area sufficiently near to the line $y = 0$. By expanding the inequality $f_i(x,0) > f_j(x,0)$, we obtain

$$\cos\frac{2i\pi}{n} + \varepsilon x^2\sin^4\frac{i\pi}{n} > \cos\frac{2j\pi}{n} + \varepsilon x^2\sin^4\frac{j\pi}{n}, \quad (12)$$

Regard R_i and R_{n-i} as the same region, and denote it by R'_i (i.e. $R'_i = R_i \cup R_{n-i}$). Then by the formula above, the diagram around $y = 0$ is equivalent to the lower envelope of $g_i(x) = \cos\frac{2i\pi}{n} + \varepsilon x^2\sin^4\frac{i\pi}{n}$. Since the graph of $y = g_i(x)$ appears as Fig.4, R'_i is proved to be adjacent to R'_{i+1} for $i = 0, \ldots, \lfloor n/2 \rfloor$.

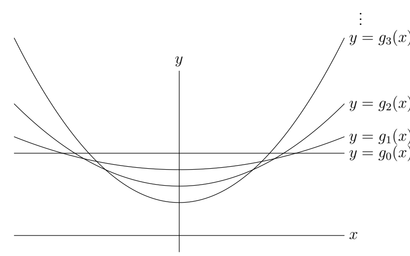


Figure 4: The graph of g_i's

Then, we will analyze the component of R'_i. Because

$$f_i(x,y) - f_{n-i}(x,y) = 8xy\cos\frac{i\pi}{n}\sin\frac{i\pi}{n}\left(\varepsilon\left(x^2\cos^2\frac{i\pi}{n} + y^2\sin^2\frac{i\pi}{n}\right) - 1\right), \quad (13)$$

in the area $x > 0$, the dominance in R'_i depends on the sign of y and $\varepsilon\left(x^2\cos^2\frac{i\pi}{n} + y^2\sin^2\frac{i\pi}{n}\right) - 1$. Consider the equation $\varepsilon\left(x^2\cos^2\frac{i\pi}{n}\right) - 1$ which is given by substituting 0 for y in the latter formula, and denote its solution by α. Then, when defining $\beta_i > 0$ as the solution of the equation

$$\cos\frac{2i\pi}{n} + \varepsilon x^2\sin^4\frac{i\pi}{n} = \cos\frac{2(i+1)\pi}{n} + \varepsilon x^2\sin^4\frac{(i+1)\pi}{n}, \quad (14)$$

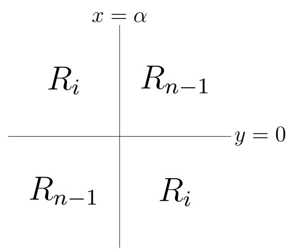


Figure 5: The component of R'_i

R_1	R_{n-1}	R_2	R_{n-2}	$\cdots$	R_{n-i+1}	R_i	R_{n-i}	$\cdots$
R_{n-1}	R_1	R_{n-2}	R_2	$\cdots$	R_{i-1}	R_{n-i}	R_i	$\cdots$

Figure 6: Diagram around $y = 0$

$\cdots$	$R_{[n-i+j+1]}$	R_i	$R_{[n-i+j]}$	$\cdots$
$\cdots$	R_{i-1}	$R_{[n-i+j]}$	R_i	$\cdots$

Figure 7: Diagram around $y = 0$ after a rotation

With some simple calculation, it can be proved that $\beta_{i-1} < \alpha < \beta_i$. Since β_i is the boundary between R'_i and R'_{i+1}, the boundary of R_i and R_{n-i} appears in both $y > 0$ part and $y < 0$ part in R'_i. Thus the component of R'_i appears as Fig. 5.

Overall coloring around $y = 0$ in the area $x > 0$ is shown as Fig. 6, and it shows R_i is adjacent to R_{n-i+1} and R_{n-i}. By transferring (x, y) by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\frac{j\pi}{n} & \sin\frac{j\pi}{n} \\ -\sin\frac{j\pi}{n} & \cos\frac{j\pi}{n} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (j = 0, \cdots, n-1),$$

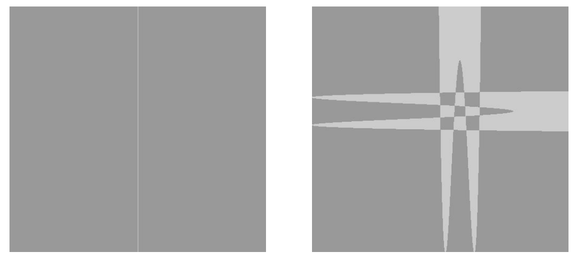
and applying the same discussion as above, the diagram becomes Fig. 7 (where $[m] = m \mod n$), and thus R_i is adjacent to $R_{[n-1+j]}$ and $R_{[n-i+j+1]}$. This means all the possible combination of R_i and R_j appears as adjacent regions. □

Now we will consider the worst case for Type II. The more largely the number of crossing points of boundaries changes, the worst we can say it is. How much the number of crossing points can change? Here, because of the difficulty of analysis on general cases, we show some examples only under the assumption that the lower-envelope is expressed by two polynomials, and f_1 is degree d and $f_2 = 0$.

Under that condition, Bézout's theorem explained bellow tells that the thinkable worst case is when f_1 can be factorized into two polynomials whose degrees are

(a) $\varepsilon = 0$ (b) $\varepsilon > 0$

Figure 8: Example of a worst case of Type II; when $d \mod 4 = 2$



(a) $\varepsilon = 0$ (b) $\varepsilon > 0$

Figure 9: Example of a worst case of Type II; when $d \mod 4 = 0$

1. $\frac{d}{2}$ and $\frac{d}{2}$ for even d; and
2. $\frac{d-1}{2}$ and $\frac{d+1}{2}$ for odd d.

Theorem 2 (Bézout, see [4] for example) *In the two dimensional projective space $\mathbb{P}^2_K$ for an algebraically closed field K, suppose that two polynomials f and g whose degree are m and l respectively are given. Then the number of crossing point of $f = 0$ and $g = 0$ is ml including the multiplicity.*

The following example shows it really achieves the thinkable maximum number of self crossing points.

Example 3 *Consider the lower-envelope given by*

$$f_1 = \left[\varepsilon^{l_1} y - \prod_{i=0}^{l_1}(x - i\varepsilon)\right]\left[\varepsilon^{l_2} x - \prod_{i=0}^{l_2}(y - i\varepsilon)\right],\ f_2 = 0, \tag{15}$$

where

$$\begin{cases} l_1 = l_2 = \frac{d}{2} & (d \text{ is even}) \\ l_1 = \frac{d-1}{2}, l_2 = \frac{d+1}{2} & (d \text{ is odd}) \end{cases} \tag{16}$$

Then, the number of crossing points of boundaries changes from 1 to $\frac{d^2}{4}$ (for even d) or $\frac{d^2-1}{4}$ (for odd d) as ε changes from 0 to $\varepsilon > 0$.

Typical picture of this example for even d is shown in Fig. 8 and Fig. 9. Note that for even $d/2$ and $\varepsilon = 0$, the region dominated by f_2 is empty and a boundary appears as $y = 0$ (see Fig. 9a).

4 Conclusion

We have defined degeneracy of a lower envelop of algebraic surfaces from the viewpoint of a perturbation of topological structures, and have shown some examples which are considered as the worst cases. The examples shown might seem to be too artificial because it is under a very general setting and especially the error model employed here is too arbitrary from the practical point of view, but we showed possibility of evilness in computation of the algebraic diagram. It is an important step toward a robust implementation of a application which utilizes geometric objects. Furthermore, we shed a light on a combinatorial aspect of a perturbation of a set of algebraic surfaces.

Acknowledgements

This research is supported by Promotion program for Reducing global Environmental loaD through ICT innovation (PREDICT), Ministry of Internal Affairs and Communications of Japan.

References

[1] T. Asano. Aspect-ratio voronoi diagram and its complexity bounds. *Information Processing Letters*, 105(1. 31):26–31, 2007.

[2] T. Asano, N. Katoh, N. Tamaki, and T. Tokuyama. Angular Voronoi diagram with applications. In *Proceedings of 3rd International Symposium on Voronoi Diagram in Science and Engineering*, pages 32–39, Banff, Canada, 2006.

[3] D. Halperin and M. Sharir. New bounds for lower envelopes in three dimensions, with applications to visibility in terrains. In *SCG '93: Proceedings of the Ninth Annual Symposium on Computational Geometry*, pages 11–18, New York, NY, USA, 1993. ACM.

[4] R. Hartshorn. *Algebraic Geometry*. Springer–Verlag, 1977.

[5] H. Muta and K. Kato. Degeneracy of angular Voronoi diagram. In *Proceedings of 4th International Symposium on Voronoi Diagram in Science and Engineering*, pages 288–293, Wales, UK, 2007.

[6] K. Sugihara and M. Iri. A robust topology-oriented incremental algorithm for Voronoi diagrams. In *International Journal of Computational Geometry and Applications*, volume 4, pages 179–228, 1994.

[7] A. Varshney, F. Brooks, and D. Richardson. Defining computing and visualizing molecular interfaces. *IEEE Visualization*, 95:36–43, 1995.