Constrained k-center and Movement to Independence

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Abstract

We obtain hardness results and approximation algorithms for two related geometric problems involving movement. The first is a constrained variant of the k-center problem, arising from a geometric client-server problem. The second is the problem of moving points towards an independent set.

1 Introduction

Given a set S of n points in the plane, the k-center problem is to find k congruent disks of minimum radius r that cover S [1, p. 276]. We study the following constrained variant of the k-center problem:

Constrained k-center

Instance: A set $P = \{p_1, \ldots, p_n\}$ of n black points and a set $Q = \{q_1, \ldots, q_k\}$ of k red points in the plane. P and Q are not necessarily disjoint.

Problem: Find a set $\mathcal{D} = \{D_1, \ldots, D_k\}$ of k disks *constrained* to the set $Q = \{q_1, \ldots, q_k\}$ of k red points (that is, for $1 \le j \le k$, the disk D_j contains the corresponding red point q_j) such that all points in P are covered by the union of the disks in $\mathcal{D}$, and the maximum radius of the disks in $\mathcal{D}$ is minimized.

The problem Constrained k-center is the geometric version of a movement problem originally proposed by Demaine et al. [3] in the graph-theoretical setting: Given a connected graph G in which some vertices are occupied by clients and some vertices are occupied by servers, the problem Facility-location Movement is that of moving both the clients and the servers in the graph until each client occupies the same vertex as some server, such that the maximum movement of a client or a server is minimized; here the distance is the path length in the graph. The authors [3] observed that a 2-approximation can be achieved simply by keeping each server at its original location and moving each client to its nearest server. Friggstad and Salavatipour [7] showed that this simple 2-approximation is in fact best possible: Unless P = NP, Facility-location Movement is NP-hard to approximate within $2 - \varepsilon$ for any constant $\varepsilon > 0$.

Here we focus on the geometric version, where the clients and servers are points in the Euclidean plane (or more generally, in $\mathbb{R}^d$), and the movement is measured as the Euclidean distance, rather than the number of edges of a path in the graph. The task is to determine a movement of the clients and servers, so that in the end, each client coincides with some server, and the maximum movement is minimized.

Let P be the set of clients, and Q be the set of servers, where $|P| = n$ and $|Q| = k$. Usually k is much smaller than n. Let us first observe that our Constrained k-center problem is essentially the same as the Facility-location Movement problem. Indeed, consider an optimal solution to the Facility-location Movement problem with maximum movement λ. Then the disks of radius λ centered at the server locations after the movement cover all clients and servers at their original locations. Conversely, consider a set of disks, say of radius λ, in an optimal solution to Constrained k-center. Then moving the clients and the server contained in each disk (with ties broken arbitrarily) to its center, gives a solution to the Facility-location Movement problem with the maximum movement at most λ.

The afore-mentioned 2-approximation works in this setting as follows. Let d denote the maximum black-red (client-server) distance obtained by assigning each black point to its closest red point. Let OPT denote an optimal solution and ALG denote the solution returned by the algorithm. Then clearly

$$\text{OPT} \ge \frac{d}{2}, \text{ and ALG} = d, \tag{1}$$

and the ratio 2 immediately follows. It is worth observing that the algorithm which keeps fixed each red point achieves ratio 2 even on the line: place two red points at 0 and $2+\varepsilon$, and two black points at $1+\varepsilon$ and $3+\varepsilon$. Then $\text{OPT} = (1+\varepsilon)/2$, while $\text{ALG} = 1$ (this tight example can be easily extended for a larger number of points).

We first show that the approximation lower bound for the problem remains close to 2 already for the planar variant.

Theorem 1 Constrained k-center *in the plane is NP-hard to approximate within* 1.8279.

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On the other hand, we have the following positive result showing that constant approximations for CONSTRAINED k-CENTER can be obtained by a fixed parameter tractable algorithm [8] with k as the parameter.

Theorem 2 *For any given $\varepsilon > 0$, there exists a $(1+\varepsilon)$-approximation algorithm for* CONSTRAINED k-CENTER *in the plane that runs in $O(\varepsilon^{-2k} \cdot n)$ time. Moreover, there exist: a 1.87-approximation algorithm that runs in $O(3^k k \cdot n)$ time, a 1.71-approximation algorithm that runs in $O(4^k k \cdot n)$ time, and a 1.61-approximation algorithm that runs in $O(5^k k \cdot n)$ time.*

In the second part of the paper, we study another movement problem proposed by Demaine et al. [3]:

MOVEMENT TO INDEPENDENCE

Instance: A set $P = \{p_1, \dots, p_n\}$ of n points in $\mathbb{R}^d$, and a threshold distance Δ.

Problem: Find a set $Q = \{q_1, \dots, q_n\}$ of n (target) points in $\mathbb{R}^d$, one point $q_i \in Q$ for each point $p_i \in P$, such that the minimum pairwise distance $\min_{i,j} |q_i q_j|$ among the points in Q is at least Δ, and that the maximum movement $\max_i |p_i q_i|$ from any point $p_i \in P$ to the corresponding target point $q_i \in Q$ is minimized.

There is a natural connection between MOVEMENT TO INDEPENDENCE and the dispersion problem in a set of congruent disks. The problem of *dispersion* in a given set of disks is that of selecting n points, one in each disk, such that the minimum inter-point distance is maximized.

DISPERSION IN CONGRUENT DISKS

Instance: A set $\{D_1, \dots, D_n\}$ of n congruent disks.

Problem: Find a set $Q = \{q_1, \dots, q_n\}$ of n points, one point $q_i \in Q$ in each disk D_i, such that the minimum pairwise distance $\min_{i,j} |q_i q_j|$ among the points in Q is maximized.

The dispersion problem was introduced by Fiala et al. [6] in a more general setting as "systems of distant representatives", generalizing the classic problem "systems of distinct representatives". See also [2, 4]. Fiala et al. [6] showed that dispersion in unit disks is NP-hard. As a corollary we obtain

Theorem 3 MOVEMENT TO INDEPENDENCE *in the plane (and in higher dimensions) is NP-hard.*

Let $P = \{p_1, \dots, p_n\}$ be a set of n points in $\mathbb{R}^2$. Denote by $\mathrm{OPT}(x)$ the minimum maximum movement for the instance (P, x) of the problem MOVEMENT TO INDEPENDENCE in $\mathbb{R}^2$ (that is, the value of the optimal solution to this instance). Demaine et al. [3] presented a polynomial-time algorithm for MOVEMENT TO INDEPENDENCE on an instance $(P, 1)$ with maximum movement at most $\mathrm{OPT}(1) + 1 + \frac{1}{\sqrt{3}}$. Their algorithm moves the points to the grid points of an equilateral triangular lattice of unit side. By a scaling argument, this algorithm can be turned into an algorithm for (P, x) for any $x > 0$, with maximum movement at most $\mathrm{OPT}(x) + (1 + \frac{1}{\sqrt{3}})x$. We have the following complementary result:

Theorem 4 *There exists a polynomial-time approximation algorithm for* MOVEMENT TO INDEPENDENCE *in the plane that moves any given set P of n points in $\mathbb{R}^2$ to another set Q of n points in $\mathbb{R}^2$, with a maximum movement no more than the minimum maximum movement necessary for a threshold distance of 1, and such that the minimum pairwise distance among the points in Q is at least $c = \frac{1}{3+2/\sqrt{3}} = 0.24\dots$.*

2 The Two Problems on the Line and on a Closed Curve

As a warm-up exercise, we first study the two problems CONSTRAINED k-CENTER and MOVEMENT TO INDEPENDENCE on the line and on a closed curve. The *distance* between two points on a closed curve is the length of the shorter subcurve determined by the two points. In these two settings, both problems can be solved exactly in polynomial time.

Proposition 1 *There exists an exact algorithm running in $O((n+k)\log(n+k))$ time for* CONSTRAINED k-CENTER *on the line.*

Proof. Observe that there exists an optimal solution consisting of a set of k disjoint intervals $I_j = [u_j, v_j]$, $1 \le j \le k$, such that $u_j, v_j \in P \cup Q$ and $q_j \in I_j$ for $j = 1, \dots, k$. We next show that such a solution can be computed in $O(n \log n)$ time by dynamic programming.

Order the $n+k$ points in $P \cup Q$ from left to right with indices $1, \dots, n+k$; in case of ties, put the red points before black points. Let $s_1, \dots, s_k$ be the indices of the k red points, $1 \le s_1 < \dots < s_k \le n+k$. Partition the list $P \cup Q$ of $n+k$ points into $k+1$ contiguous sublists $L_0, L_1, \dots, L_k$ such that, for $1 \le j \le k$, the red point s_j is the first point in L_j (the sublist L_0 contains no red points). For each point i in $P \cup Q$, $1 \le i \le n+k$, denote by $j[i]$ the index j, $0 \le j \le k$, such that the point i is in the sublist L_j. For each point i such that $1 \le j[i] \le k$, denote by $D[i]$ the minimum interval length of $j = j[i]$ intervals, constrained to the red points $s_1, \dots, s_j$, that cover the points in $P \cup Q$ from 1 to i.

Denote by $dist(i_1, i_2)$ the distance between two points with indices i_1 and i_2 in $P \cup Q$. The dynamic programming algorithm has the following base case for each

$i \in L_1$,

$$D[i] = \mathrm{d}ist(1, i),$$

and the following recurrence for each $i \in L_j$, $j = 2, \ldots, k$,

$$D[i] = \min_{t \in L_{j-1}} \max \big\{ D[t], \, \mathrm{d}ist(t + 1, i) \big\}.$$

Note that $D[t]$ is an increasing function of t for $t \in L_{j-1}$, and that $\mathrm{d}ist(t + 1, i)$ is a decreasing function of t for $1 \leq t < i$. Thus, by a binary search, we can compute $D[i]$ for each $i \in L_j$ in $O\big(\log(|L_{j-1}| + 1)\big)$ time, for increasing values of j from 2 to k. The desired entry is $D[n + k]$. The overall running time is clearly $O((n + k) \log(n + k))$. □

Proposition 2 *There exists an exact algorithm running in $O(\frac{1}{k}(n+k)^2 \log(n+k))$ time for* Constrained k-center *on a closed curve.*

Proof. A closed curve containing n black points and k red points has a subcurve containing at most n/k black points between two red points. For each pair of consecutive points on this subsurve, we can cut the curve between the pair and obtain an instance of the problem Constrained k-center on a line. There are at most $(n + k)/k$ such instances on a line, and each of them can be solved exactly in $O((n + k) \log(n + k))$ time by Proposition 1. The overall optimal solution for these instances on a line is an optimal solution for the original instance on a closed curve. □

Proposition 3 *There exists a polynomial-time exact algorithm based on linear-programming for* Movement to Independence *on the line.*

Proof. Sort the points in P by increasing x-coordinates $a_1 \leq a_2 \leq \ldots \leq a_n$. Observe that in an optimal solution, no two points in P need to swap their order. Let $x_1 \leq x_2 \leq \ldots \leq x_n$ be the new x-coordinates of the points after the move (the ith point moves from a_i to x_i). Computing an optimal solution amounts to solving the following linear program with the n variables x_i and $3n - 1$ constraints:

$$\begin{aligned} &\text{minimize} \quad z \qquad \text{(LP1)} \\ &\text{subject to} \quad \begin{cases} x_{i+1} - x_i \geq \Delta, & 1 \leq i \leq n-1 \\ x_i - a_i \leq z, & 1 \leq i \leq n \\ -x_i + a_i \leq z, & 1 \leq i \leq n \end{cases} \end{aligned}$$

□

While Movement to Independence on the line is always feasible, this is not the case for the new variant on a closed curve. Let γ be a closed curve of length $L = |\gamma|$. Obviously, Movement to Independence on γ admits a solution if and only if $L \geq n\Delta$. We show next that an exact solution can still be found via linear-programming.

Proposition 4 *There exists a polynomial-time exact algorithm based on linear-programming for* Movement to Independence *on a closed curve.*

Proof. For simplicity, we can assume that γ is drawn in the plane as a circle centered at the origin, and that the input points (in P) are numbered counterclockwise, as $1, \ldots, n$ on γ, and their initial positions (γ-coordinates) are $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n < L$. Refer to the point whose γ-coordinate is 0 as the *origin* of γ. As in the proof of Proposition 3, it is crucial to observe that there exists an optimal solution such that the circular order of the points in P on γ remains the same. Moreover, OPT $\leq L/2$, since the distance on γ from any (input) point to any other point is at most $L/2$.

Let $x_i \in [0, L]$, $i = 1, \ldots, n$, be the new γ-coordinates of the points after an optimal move (the ith point moves from a_i to x_i). Note that these coordinates uniquely identify the movement of the points, since OPT $\leq L/2$. Observe also that in an optimal solution, *not all* the points move in the same direction on γ (clockwise or counterclockwise): indeed, assuming such a move, the smallest of the moves can be canceled out from each move, with a strict decrease in the optimal solution, which would be a contradiction.

Consider an optimal solution O such that the circular order of the points in P on γ remains the same. Assign a $+$ sign to each point that moves counterclockwise in O, and a $-$ sign to each point that moves clockwise in O (points that do not move can be assigned any sign arbitrarily). By our previous observation, we can assume that not all signs are the same, and consequently, there exist two adjacent points on γ with opposite signs moving away from each other in the optimal solution O. We pick a new origin of γ between these two points (or coincident with one of them), and find an optimal movement for the points, subject to the constraint that no point crosses the origin of γ. We do this for all n pairs of adjacent points on γ.

It remains to show that these n cases can be implemented as n linear programs with the n variables x_i, where each LP has $O(n)$ constraints. Fix a pair of adjacent points as described above. Assume that $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n < L$ are the new γ-coordinates of the n points on γ. These coordinates can be computed in $O(n)$ time as they implement a simple circular shift. An optimal movement for the points amounts to solving the following linear program:

$$\text{minimize} \quad z \qquad \text{(LP2)}$$

$$\text{subject to} \begin{cases} x_1 \geq 0, \\ x_n \leq L, \\ x_{i+1} - x_i \geq \Delta, & 1 \leq i \leq n-1 \\ x_1 + L - x_n \geq \Delta, \\ x_i - a_i \leq z, & 1 \leq i \leq n \\ -x_i + a_i \leq z, & 1 \leq i \leq n \end{cases}$$

The algorithm first checks whether the feasibility condition is met, and assuming it is, it solves the n linear programs (one for each adjacent pair of points on γ) and then selects the LP whose solution gives the overall minimum z. □

3 NP-hardness of Constrained k-center

Proof of Theorem 1. We show that CONSTRAINED k-CENTER is NP-hard by a reduction from the NP-hard problem PLANAR-3SAT [9]. A reduction for k-center based on similar ideas, however from another problem PLANAR-VERTEX-COVER, appears in [5]. Let (V, C, G) be a PLANAR-3SAT instance, which consists of a set V of n boolean variables, a set C of m clauses that are disjunctions of three literals, and a planar embedding G of the bipartite graph with a vertex for each variable and each clause, and with an edge connecting a variable to a clause if and only if a literal of the variable occurs in the clause. We will construct a CONSTRAINED k-CENTER instance consisting of a gadget for each variable, clause, and literal.

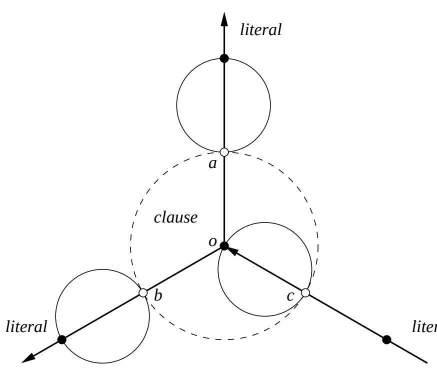

Figure 1: Connection between three literals in a clause. The three red points a, b, c (drawn as empty circles) come from three different literals, and are placed at the vertices of an equilateral triangle inscribed in a circle centered at the shared black point o; $|oa| = |ob| = |oc| = 2$. The black point o in the clause gadget is covered only if at least one of the three literals is true. In this example, the literal of c is true, and the literals of a and b are false.

We now describe our construction. The gadget for each clause is a single black point. The gadget for each variable is a closed chain of alternating black and red points. The gadget for each literal is an open chain of alternating black and red points, with a black point at one end and a red point at the other end. The clause and variable gadgets model the vertices of the planar graph. The literal gadgets model the edges: each literal gadget is connected to the corresponding clause gadget at the end with a red point, and to the corresponding variable gadget at the end with a black point. We illustrate in Figure 1 the connection between the gadgets of a clause and its three literals, and in Figure 2 the connection between the gadgets of a literal and its variable. The distance between consecutive black and red points in each variable or clause gadget is exactly 2 (which is the diameter of unit-radius disk) except at the junctions where a literal gadget is connected to a variable gadget (between c and e in Figure 2).

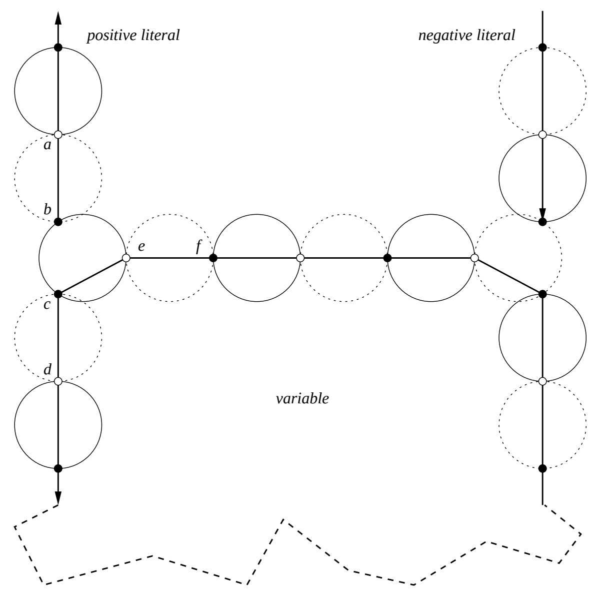

Figure 2: Connection between a variable and its literals. The four points d, c, e, f are part of a variable gadget; the two points a, b are part of a literal gadget; a, b, c, d are collinear; b, c, e are on a circle of unit radius; $ef \perp ad$; $|ab| = |cd| = |ef| = 2$, $|ac| = |bd| = |bf| = |cf|$. Red points are drawn as small empty circles. Large solid and dotted circles (of unit radius) correspond to true and false assignments, respectively.

Write $x = |bc|/2$ for the configuration in Figure 2. Then $|ac| = |bd| = 2 + 2x$, and

$$|bf| = |cf| = \sqrt{\left(\sqrt{1 - x^2} + 3\right)^2 + x^2}.$$

Let x be the solution to the equation

$$2 + 2x = \sqrt{\left(\sqrt{1 - x^2} + 3\right)^2 + x^2},$$

and let $y = 1 + x$. Then $|ac| = |bd| = |bf| = |cf| = 2y$, and y is the solution to the following quartic equation

$$4y^4 - 11y^2 - 18y + 25 = 0.$$

A calculation shows that $y = 1.8279\ldots$. Assume that $1 \le r < y$ in the following. It is easy to check that for any such r, a disk of radius r that contains a red point can contain at most one black point in our construction, except at the junction between each literal and its variable, where a disk may contain the red point e and the two black points b and c as in Figure 2. Now set the parameter k to the number of red points in the construction. Then the Planar-3SAT formula is satisfiable if and only if the Constrained k-center instance has a feasible solution with k disks of radius r. The slackness in the disk radius r implies that Constrained k-center is NP-hard to approximate within $y - \varepsilon$ for any constant $\varepsilon > 0$. □

4 Approximation for Constrained k-center

Proof of Theorem 2. The idea of our approximation algorithm is very simple, namely to enumerate the approximate positions of an optimal constrained disk cover. Fix an optimal solution $O = \{\Omega_1, \ldots, \Omega_k\}$. Suppose that the red point q_j is covered by a disk Ω_j of radius r^* in the solution O. Then the center c_j of Ω_j is contained in a disk D_j of radius r^* centered at q_j.

It is well-known that a disk of radius 1 can be covered by three smaller disks of radii $\frac{\sqrt{3}}{2}$, whose centers form an equilateral triangle, as shown in Figure 3. Now place three points around the red point q_j in an equilateral triangle formation (in some arbitrary orientation) such that the distance from q_j to each point is $\frac{1}{2}r^*$. Hence the disk D_j is covered by three smaller disks, E_{j1}, E_{j2}, E_{j3} of radius $\frac{\sqrt{3}}{2}r^*$ centered at the three points. Recall that c_j is contained in D_j, so it is covered by one of the disks E_{j1}, E_{j2}, E_{j3}. Let $F_{j1} \supset E_{j1}$, $F_{j2} \supset E_{j2}$, $F_{j3} \supset E_{j3}$, be three larger concentric disks of radius $(\frac{\sqrt{3}}{2}+1)r^*$. Since Ω_j has radius r^*, it is covered by one of the larger disks F_{j1}, F_{j2}, F_{j3} of radius $(\frac{\sqrt{3}}{2}+1)r^*$. So all black points covered by Ω_j are also covered by one of these three larger disks.

By the preceding observation, given any candidate radius r, we can either find a feasible solution of k disks of radii $(\frac{\sqrt{3}}{2}+1)r$ by enumerating one of three possible disks for each red point and testing the black points for containment, all in $O(3^k k \cdot n)$ time, or decide (correctly) that there is no feasible solution with radius r. By (1), we can find a radius $\bar{r}$ such that $\frac{1}{2}r^* \le \bar{r} \le r^*$ in $O(kn)$ time. Then, by a binary search in the range $[\bar{r}, 2\bar{r}]$, we can obtain a $(\frac{\sqrt{3}}{2}+1+\varepsilon)$-approximation in $O(3^k k \cdot \log \frac{1}{\varepsilon} \cdot n)$ time, which is linear in n for any constants k and ε.

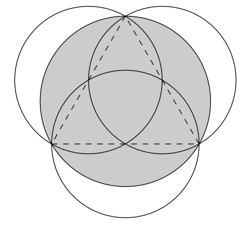

Figure 3: Covering a disk of radius 1 by three smaller disks of radius $\sqrt{3}/2$. The three sides of the equilateral triangle inscribed in the unit-radius disk are the diameters of the three smaller disks. The distance from the center of the unit-radius disk to the center of each smaller disk is $1/2$.

In particular, since $\frac{\sqrt{3}}{2} + 1 = 1.8660\ldots$, we have a 1.87-approximation algorithm that runs in $O(3^k k \cdot n)$ time.

Similarly, a disk of unit radius can be covered by four disks of radius $\sqrt{2}/2 = 0.707\ldots$, and we get a 1.71-approximation in $O(4^k k \cdot n)$ time. By an old result of Neville [10], a disk of unit radius can be covered by five disks of radius $0.609383\ldots$, and we get a 1.61-approximation in $O(5^k k \cdot n)$ time. To obtain a finer approximation, note that a disk of radius 1 can be covered by $O(\varepsilon^{-2})$ disks of radius ε. Our algorithm can be obviously generalized to obtain a $(1+\varepsilon)$-approximation in $O(\varepsilon^{-2k} \cdot n)$ time. □

5 Movement to Independence: NP-hardness and Approximation

Proof of Theorem 3. To verify the NP-hardness, we make a reduction from the problem Dispersion in congruent disks in a set $\mathcal{D}$ of disks of radius λ via the following claim, whose proof is immediate from the definitions of the two problems.

Claim. Dispersion in unit disks in a set $\mathcal{D}$ of disks of radius λ is feasible if and only if Movement to Independence for the center points of the disks in $\mathcal{D}$ can be attained with a maximum movement at most λ.

This concludes the proof of Theorem 3. □

Proof of Theorem 4. Let δ be the minimum pairwise distance $\min_{i,j} |p_i p_j|$ of the points in P. We claim that if $\delta \le x \le 1$, then

$$\mathrm{OPT}(1) \ge \mathrm{OPT}(x) + (1-x)/2.$$

To see that the claim is true, imagine all points move from start configuration to target configuration with the same speed as in an optimal solution for OPT(1). Pause the points as soon as their minimum pairwise distance is x. Then the movement is at least $\mathrm{OPT}(x)$ before the pause and at least $(1-x)/2$ after the pause.

Let $c = \frac{1}{3+2/\sqrt{3}} = 0.24\ldots$. We now give an algorithm that moves the points to minimum pairwise distance at least c using maximum movement at most OPT(1). Consider two cases:

(1) $\delta \geq c$ Stay put.

(2) $\delta < c$ Use the algorithm of Demaine et al. [3] with a smaller grid of size $x = c$. Then the maximum movement is at most

$$\mathrm{OPT}(c) + \left(1 + \frac{1}{\sqrt{3}}\right) c \leq \mathrm{OPT}(1) - \frac{(1-c)}{2} + \left(1 + \frac{1}{\sqrt{3}}\right) c = \mathrm{OPT}(1). \qquad \square$$

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