

Watchman tours for polygons with holes

Adrian Dumitrescu* Csaba D. Tóth†

Abstract. A watchman tour in a polygonal domain (for short, polygon) is a closed curve such that every point in the polygon is visible from at least one point of the tour. The problem of finding a shortest watchman tour is NP-hard for polygons with holes. We show that the length of a minimum watchman tour in a polygon P with k holes is $O(\text{per}(P) + \sqrt{k} \cdot \text{diam}(P))$, where $\text{per}(P)$ and $\text{diam}(P)$ denote the perimeter and the diameter of P , respectively. Apart from the multiplicative constant, this bound is tight in the worst case. A watchman tour of this length can be computed in $O(n \log n)$ time, where n is the total number of vertices. We generalize our results to watchman tours in polyhedra with holes in 3-space. We obtain an upper bound $O(\text{per}(P) + \sqrt{k \cdot \text{per}(P) \cdot \text{diam}(P)} + k^{2/3} \cdot \text{diam}(P))$, which is again tight in the worst case.

1 Introduction

Visibility and art gallery problems with stationary guards (watchmen) have been studied extensively since the early 1980s [15]. Mobile guards have been also considered soon after, see *e.g.* [7, 8, 14]. A *watchman tour* in a polygonal domain (polygon, for short) is a tour (i.e., closed curve) inside the polygon such that every point in the polygon is visible from some point along the tour. Two points in a polygon are visible to each other if the line segment between them lies in the polygon.

The watchman tour problem asks for a watchman tour of minimum length [2, 13]. The problem has a polynomial time solution for simple polygons with n vertices (and no holes). Tan [18] gave an $O(n^5)$ -time algorithm improving an earlier $O(n^6)$ -time algorithm by Carlsson *et al.* [5]. Other variants are discussed in [1, 13, 14]. In contrast, computing a shortest watchman tour in a polygon with holes is known to be NP-hard [7]. In Section 3, we revisit the old NP-hardness proof by Chin and Ntafos [7] and make some necessary clarifications.

Our main result is a tight worst-case upper bound for the minimum length of a watchman tour in a polygon with holes. Our upper bound depends on three parameters of a polygon P : the number of holes, $k = k(P)$, the diameter, $\text{diam}(P)$, and the perimeter, $\text{per}(P)$. The

perimeter of P is the total length of the boundary of P (including the boundary of the holes). In Section 2 we prove:

Theorem 1 *The minimum length of a watchman tour for a polygon P with k holes is $O(\text{per}(P) + \sqrt{k} \cdot \text{diam}(P))$. This bound cannot be improved for polygons with $\text{per}(P) > c \cdot \text{diam}(P)$ for any fixed $c > 2$. A watchman tour of this length can be computed in $O(n \log n)$ time, where n is the total number of vertices of P .*

We have $\text{per}(P) > 2 \cdot \text{diam}(P)$ for every polygon P . If, however, $\text{per}(P)$ is very close to $2 \cdot \text{diam}(P)$, then the polygon is long and skinny, and the above upper bound is no longer tight up to constant factors.

Theorem 1 generalizes to polyhedra, possibly with holes and handles, in three dimensions. The boundary of a polyhedron is composed of piecewise linear manifolds. We define the perimeter $\text{per}(P)$ of a polyhedron P in 3-space as the total length of all edges of P .

Theorem 2 *The minimum length of a watchman tour for a polyhedron P in 3-space with k holes is at most $O(\text{per}(P) + \sqrt{k \cdot \text{per}(P) \cdot \text{diam}(P)} + k^{2/3} \cdot \text{diam}(P))$. This bound cannot be improved for polyhedra with $\text{per}(P) > c \cdot \text{diam}(P)$ for any fixed $c > 3$. A watchman tour of this length can be computed in $O((nk)^{4/5+\delta})$ expected time for any $\delta > 0$, where n is the total number of vertices, edges, and faces of P .*

Our results give a partial answer to a question of Nilsson [14], posed in his PhD thesis: “Is it possible to find approximative solutions to guarding problems, with good worst case bounds?”

2 Bounds on the length of an optimal tour

In this section we prove Theorems 1 and 2. By a classical result of Few [10], the shortest path through k points in the unit square $[0, 1]^2$ has (Euclidean) length at most $\sqrt{2k} + 7/4$. Few also proved that the minimum spanning tree of these points has length at most $\sqrt{k} + 7/4$. Both upper bounds are constructive. For constructing a spanning path, he lays out about \sqrt{k} equidistant horizontal lines, and then visits the points layer by layer, with the path alternating directions along the horizontal strips.

The current best lower bound for the length of such a path is also due to Few: it is $(\frac{4}{3})^{1/4} \sqrt{k} - o(\sqrt{k})$,

*Department of Computer Science, University of Wisconsin–Milwaukee, WI 53201-0784, USA. Email: dumitres@uwm.edu. Supported in part by NSF CAREER grant CCF-0444188.

†Department of Mathematics and Statistics, University of Calgary, AB, Canada T2N 1N4. E-mail: cdtoth@ucalgary.ca. Supported in part by NSERC grant RGPIN 35586.

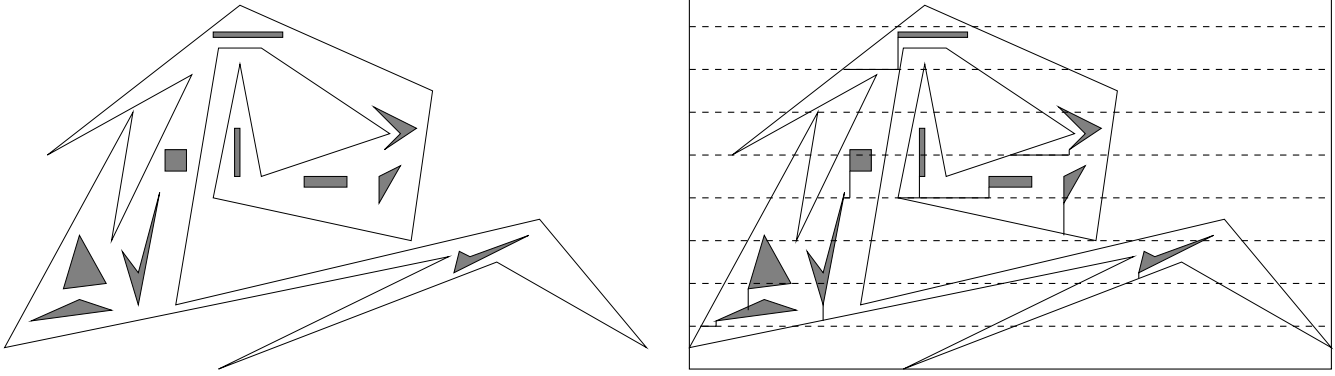


Figure 1: Left: a polygon with $k = 10$ holes. Right: Connecting the circuits $\partial H_i, i = 0, \dots, k$ by adding connectors.

where $(4/3)^{1/4} = 1.075 \dots$. In a related problem, Chung and Graham [9] showed that the length of the shortest Steiner tree through k points in the unit square is at most $0.995\sqrt{k}$ (they gave details for an improvement to $0.99995\sqrt{k}$ only). In every dimension $d \geq 3$, Few showed that the maximum length of a shortest path through k points in the unit cube is $\Theta(k^{1-1/d})$.

Algorithm for constructing a watchman tour. Observe that for a simple polygon P without holes, ∂P is a watchman tour of length $\text{per}(P)$. Let now P be a polygon with $k \geq 1$ holes, H_1, \dots, H_k . For convenience, denote by H_0 the unbounded hole defined by the exterior of P . The boundary of P consists of $k + 1$ pairwise disjoint circuits $\partial P = \bigcup_{i=0}^k \partial H_i$. We have $\text{per}(P) = \sum_{i=0}^k |\partial H_i|$, where the vertical bars $|\cdot|$ stand for the Euclidean length.

Our algorithm works as follows. Compute an axis-aligned bounding box B of the polygon P whose longest side is of length at most $\text{diam}(P)$. We augment the disjoint union of circuits $\partial H_i, i = 0, \dots, k$, with at most $2k$ line segments and possibly (at most $3k$) new vertices to a connected graph G . The new segments added are called *connectors*. The connectors are either vertical or horizontal. We replace each connector by double edges and obtain a multi-graph G' where every vertex has even degree. The watchman tour we construct, W , is an arbitrary Eulerian tour in G' , which traverses the entire boundary of P (including the hole boundaries) once, and traverses every connector twice. It is easy to verify that each point p in the interior of P is seen from some point along the tour W : Take an arbitrary line through p and consider its first intersection with the boundary ∂P of P . Since W traverses ∂P , p is visible from some point on $\partial P \subset W$.

It remains to explain how to draw the connectors and to bound their total length. As in Few's method, subdivide the bounding box B into horizontal strips by a *raster* of at most \sqrt{k} equidistant horizontal lines such that consecutive raster lines are at $\text{diam}(P)/\sqrt{k}$ dis-

tance apart. We construct the connectors in two phases (refer to Fig. 1). In the first phase from a lowest point (vertex) of each interior hole H_i , drop a vertical ray ℓ_i downwards until it hits the outer boundary, the boundary of another hole, or a horizontal raster line. Let v_i denote this vertical segment, and let p_i be its lower endpoint. If point p_i is in the interior of P , then it lies on one of the raster lines. In the second phase from every point p_i lying in the interior of P , draw a horizontal ray leftwards until it hits the outer boundary, the boundary of another hole, or another point $p_j, j \neq i$. Let h_i denote this horizontal segment. If p_i is already on the boundary of another hole or on ∂H_0 , then h_i is an empty segment (i.e., not needed).

It is clear that by adding at most $2k$ (horizontal and vertical) connectors $h_i \cup v_i$, we obtain a connected graph G containing all the circuits $\partial H_i, i = 0, \dots, k$.

Upper bound. The total length of the horizontal raster lines is at most $\text{diam}(P) \cdot \sqrt{k}$, so the total length of the (at most k) horizontal connectors does not exceed this bound. There are k vertical connectors, each of length at most $\text{diam}(P)/\sqrt{k}$. Hence their total length is also bounded by $k \cdot \text{diam}(P)/\sqrt{k} = \text{diam}(P) \cdot \sqrt{k}$. Consequently, the total length of W is

$$\begin{aligned}
 |W| &= |\partial H_0| + \sum_{i=1}^k (|\partial H_i| + 2|v_i| + 2|h_i|) \\
 &= O(\text{per}(P) + \sqrt{k} \cdot \text{diam}(P)). \tag{1}
 \end{aligned}$$

Algorithm description and analysis. Let n denote the total number of vertices of P . The bounding box B and the raster lines can be computed in $O(n)$ time. (We do not compute the full arrangement of the raster lines and P , which may have up to $\Theta(n\sqrt{k})$ vertices.) The set of connectors, henceforth the graph G can be computed by a standard line-sweep algorithm [3, 4] in $O(n \log n)$ time. Sweep a horizontal line ℓ top-down. For every position of ℓ , we maintain in sorted order its intersection

points with the vertices and edges of P and with the vertical connectors v_i . This order changes only if ℓ passes through a vertex of P or a point p_i , or if ℓ coincides with a raster line. So there are at most $n + k + \sqrt{k} \leq 3n$ events overall. When the sweep line ℓ coincides with one of the raster lines, we can find the closest intersection point in $\ell \cap \partial P$ to the left of each $p_i \in \ell$ in $O(\log n)$ time.

Observe that the graph G , as well as the multi-graph G' have $O(n)$ edges each. Once G' is constructed, computing an Eulerian tour of G' takes $O(n)$ time. Hence the total time taken by the algorithm is $O(n \log n) + O(n) = O(n \log n)$.

Lower bound. We now show that our upper bound for the tour length in (1) is tight in the worst case for every $k \geq 0$ and $\text{per}(P) > c \cdot \text{diam}(P)$, where $c > 2$ is a fixed constant. We may assume w.l.o.g. that $\text{diam}(P) = 1$. We construct a polygon lying in a disk D of unit diameter. If $\text{per}(P) > 2\sqrt{2}$, then let the outer boundary H_0 of P be a square inscribed in D combined with a long and narrow zig-zag “snake” of total edge length $\text{per}(P) - 2\sqrt{2}$ and very small width $0 < \varepsilon \ll 1$ (Fig. 2). The snake lies in D such that the diameter of H_0 is 1. If $c < \text{per}(P) \leq 2\sqrt{2}$, then let H_0 be a rhombus of diameter 1 and side length $\frac{1}{4}(\text{per}(P) - \varepsilon)$ for a small $0 < \varepsilon \ll c - 2$. In both cases, we have $\text{per}(H_0) = \text{per}(P) - \varepsilon$, and H_0 contains a square of side length $\Omega(1)$.

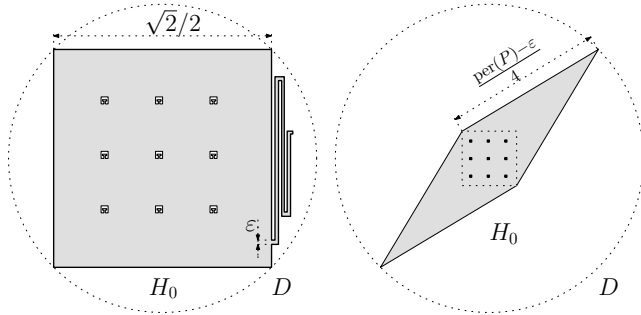


Figure 2: Lower bound constructions for the cases $\text{per}(P) > 2\sqrt{2} \cdot \text{diam}(P)$ and $2 \cdot \text{diam}(P) < \text{per}(P) \leq 2\sqrt{2} \cdot \text{diam}(P)$.

Arrange k small holes in a grid-like pattern in a maximal inscribed square of H_0 . Each hole has $O(1)$ vertices, ε/k perimeter, and a small hidden “cave” that can be seen only by entering it; see *e.g.*, Fig. 3(right). By Few’s result, the length of the shortest watchman tour that visits the caves in all holes is $\Omega(\sqrt{k})$. If $\text{per}(P) > 2\sqrt{2}$, the length of any walk from the bottom of the zig-zag snake to one of the furthest caves is $\Omega(\text{per}(P))$. We conclude that in both cases the length of the shortest watchman tour for P is

$$\Omega(\text{per}(P) + \sqrt{k}) = \Omega(\text{per}(P) + \sqrt{k} \cdot \text{diam}(P)),$$

as required. This completes the proof of Theorem 1.

Generalization to 3-dimensions. A polyhedron (possibly with holes) in 3-space is a piecewise-linear 3-manifold with boundary. Let $\text{per}(P)$ denote the total length of the edges of a polyhedron P . Note that every point p in the interior of P sees at least one point on some edge of P . Indeed, consider an arbitrary plane h containing p . The intersection $P \cap h$ is a collection of disjoint polygons (possibly with holes), one of which contains p . In a triangulation of this planar polygon, p lies in a triangle, and hence it sees the three vertices of the triangle. All three vertices are intersection points of h with some edges of P . It follows that a tour that traverses every edge of P is a watchman tour: *i.e.*, every interior point of P is seen from some point of the tour.

Our algorithm for computing a watchman tour for P is analogous to the planar case. We augment the 1-skeleton of P to obtain a connected graph G . We then double some of the edges in G to make all vertex degrees even, and our watchman tour is an arbitrary Eulerian tour in this multi-graph.

Choose a lowest point w_i in each interior hole H_i , $i = 1, \dots, k$. Compute an axis-aligned bounding box B of the polyhedron P of side length at most $\text{diam}(P)$. Subdivide B into horizontal strips by a *raster* of at most $k^{1/3}$ equidistant horizontal planes such that consecutive raster planes are at $\text{diam}(P)/k^{1/3}$ distance apart. Subdivide every strip by additional horizontal planes, if necessary, such that there are at most $k^{2/3}$ points w_i between consecutive horizontal planes. We have used at most $2k^{1/3}$ horizontal planes. From each w_i , drop a vertical line ℓ_i downwards until it hits the outer boundary, the boundary of another hole, or a horizontal plane. Let p_i be the lower endpoint of this vertical segment.

If point p_i is in the interior of P , then it lies on some horizontal plane. In each horizontal plane, we invoke our planar algorithm with $k' = k^{2/3}$ to construct connectors from every point p_i to the outer boundary or the boundary of another hole. The total length of the vertical connectors and of the horizontal connectors in the $O(k^{1/3})$ planes is bounded by

$$\begin{aligned} &O\left(k \cdot (\text{diam}(P)/k^{1/3}) + k^{1/3} \cdot (\sqrt{k^{2/3}} \cdot \text{diam}(P))\right) \\ &= O\left(k^{2/3} \cdot \text{diam}(P)\right). \end{aligned} \quad (2)$$

For each interior hole H_i , we have computed a connector from a lowest vertex w_i to some point on the outer boundary ∂H_0 or the boundary of another hole. However, the endpoint of a connector may lie in the interior of a face. For every face f of P , let k_f denote the number of connector endpoints in the interior of f , with $\sum_f k_f \leq k$. In each face f , with $k_f \geq 1$, we construct a minimum spanning tree of the k_f connector endpoints in f and an arbitrary vertex of f . By Theorem 1, the length of an MST in a face f is $O(\sqrt{k_f} \cdot \text{diam}(f))$. The

total length of these spanning trees is

$$\begin{aligned}
 & O\left(\sum_f \sqrt{k_f} \cdot \text{diam}(f)\right) \\
 = & O\left(\sqrt{\sum_f k_f} \sqrt{\sum_f \text{diam}^2(f)}\right) \\
 = & O\left(\sqrt{k} \sqrt{\text{diam}(P)} \sqrt{\sum_f \text{diam}(f)}\right) \\
 = & O\left(\sqrt{k \cdot \text{diam}(P) \cdot \text{per}(P)}\right). \tag{3}
 \end{aligned}$$

In this chain of inequalities, we applied the Cauchy-Schwarz inequality, the bounds $\max_f \text{diam}(f) \leq \text{diam}(P)$ and $\text{diam}(f) \leq \frac{1}{2} \text{per}(f)$. We have $\sum_f \text{diam}(f) \leq \sum_f \frac{1}{2} \text{per}(f) \leq \text{per}(P)$, since every edge is adjacent to exactly two faces.

By adding the term $\text{per}(P)$ to the lengths in (2) and (3) the upper bound on the length of the tour in Theorem 2 follows.

All vertical segments of the connectors can be computed by a (randomized) batched ray shooting algorithm due to Pellegrini [17]. If the total number of vertices, edges and faces of P is n , then we can triangulate the faces of P into $O(n)$ triangles in $O(n \log n)$ time. Among $O(n)$ interior-disjoint triangles in 3-space, k batched ray shooting queries take $O((nk)^{4/5+\delta})$ expected time for any $\delta > 0$, where the constant of proportionality depends on δ . Similarly, we can compute all horizontal segments of the connectors simultaneously by the same algorithm in $O((nk)^{4/5+\delta})$ expected time. All minimum spanning trees over the $O(k)$ connector endpoints in the faces of P can be computed in $O(k \log n)$ time.

The lower bound constructions are similar to the planar case. We construct a polyhedron P with k holes and $\text{per}(P) > c \cdot \text{diam}(P)$ for a fixed $c > 3$. Note that $\text{per}(P) > 3 \cdot \text{diam}(P)$ for every polyhedron, since the 1-skeleton of the outer boundary of P is a 3-connected graph, hence it contains at least three edge-disjoint paths between any two vertices. In particular, there are at least three edge-disjoint paths between two vertices at $\text{diam}(P)$ distance apart, and at most one of these paths may have length $\text{diam}(P)$. Hence $\text{per}(P) > 3 \cdot \text{diam}(P)$.

We may assume without loss of generality that $\text{diam}(P) = \Theta(1)$. We present three lower bound constructions, one for each term in the upper bound $O(\text{per}(P) + \sqrt{k \cdot \text{per}(P)} + k^{2/3})$.

Case 1: $c < \text{per}(P) \leq k^{1/3}$. In this case, $\max(\text{per}(P), \sqrt{k \cdot \text{per}(P)}, k^{2/3}) = k^{2/3}$. Let H_0 be a double pyramid obtained by gluing together two congruent pyramids, each with $1/2$ height, and with a (small) triangular base. Arrange the holes in a grid-like pattern in a maximal inscribed cube in H_0 of side

length $\Omega(1)$, each with a hidden ‘‘cave.’’ By Few’s result [10], a watchman tour that visits every cave has length $\Omega(k^{2/3})$.

Case 2: $k^{1/3} < \text{per}(P) \leq k$. In this case, $\max(\text{per}(P), \sqrt{k \cdot \text{per}(P)}, k^{2/3}) = \sqrt{k \cdot \text{per}(P)}$. Let H_0 be composed of about $\Theta(\text{per}(P))$ homothetic copies of a flat box of size $1 \times 1 \times \varepsilon$, for some small $\varepsilon > 0$, connected by a narrow corridor. Arrange about $\Theta(k/\text{per}(P))$ holes, each with a hidden ‘‘cave,’’ in a planar grid-like pattern in each flat box. By Few’s result, the length of the watchman tour within each flat box is $\Omega(\sqrt{k/\text{per}(P)})$, and so the total length is $\Omega(\sqrt{k \cdot \text{per}(P)})$.

Case 3: $k < \text{per}(P)$. In this case, $\max(\text{per}(P), \sqrt{k \cdot \text{per}(P)}, k^{2/3}) = \text{per}(P)$. Let H_0 be a narrow zig-zag ‘‘snake’’ of diameter 1, built of a sequence of triangular prisms (and arbitrary holes of very small perimeter). A tour that visits both ends of the snake must have $\Omega(\text{per}(P))$ length.

In all three cases, we have shown that there is a polyhedron P with k holes, $\text{per}(P)$ perimeter and $\Theta(1)$ diameter such that the length of every watchman tour is $\Omega(\max(\text{per}(P), \sqrt{k \cdot \text{per}(P)}, k^{2/3})) = \Omega(\text{per}(P) + \sqrt{k \cdot \text{per}(P)} + k^{2/3})$, as required.

3 NP-hardness proof revisited

The Watchman Tour Problem (WTP) is the following.

WTP: Given a polygon P with k polygonal holes, and a positive integer m , does there exist a watchman tour of total Euclidean length at most m ?

The Geometric Traveling Salesman Problem (GTSP) [12, 16] is the following.

GTSP: Given a set of n points in the plane, and a positive integer m , does there exist a tour of total length at most m that visits all the points?

It is known that GTSP is NP-hard with respect to both the L_1 and the L_2 metric [11, 16]. The NP-hardness proof in [6], and similarly that in [7] use a reduction from the Euclidean Geometric Salesman problem (L_2 metric) to the Watchman Tour Problem via a claim that relates the length of a solution for GTSP in the L_2 metric to the length of a solution for WTP: [6, Theorem 1, p. 25] and [7, Theorem 2.1, p. 40]. (Corollary 1 in [6] and Corollary 2.2 in [7] add even more to the confusion.) The correct reduction however is from GTSP in the L_1 metric. See Fig. 3.

Given set S of n lattice points, construct a polygon P as the slightly enlarged (grid) axis aligned rectangle of S . The holes are the cells of the grid, but only slightly

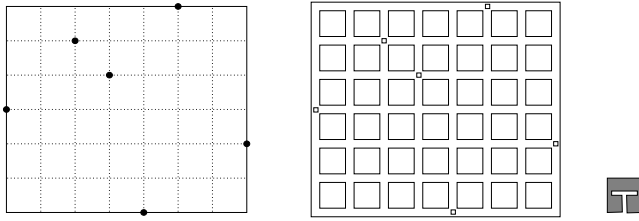


Figure 3: Left: point set S . Middle: polygon P with holes. Right: a small hole for each point in S .

smaller so that they are disjoint. In addition to these large (grid cell) holes, there are small holes corresponding to each point, each containing a “cave” that can only be seen by entering it. The holes are small enough so that they fit in the narrow corridors left by the big grid cell holes: the perimeter of each small hole is $1/80n$. The width of the corridors is $1/40n$.

The reduction, hence the NP-hardness, follows via the following claim, which is easy to verify:

Claim. For a positive integer m , there exists a tour of S of length at most m in the L_1 metric if and only if there exists a watchman tour of P of length at most $m + 0.1$ (in the usual L_2 metric).

Observe that the integrality requirement for m is crucial. Furthermore, no such claim holds if the length of the tour of the points in S is measured in the L_2 metric.

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