Watchman tours for polygons with holes

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Abstract. A watchman tour in a polygonal domain (for short, polygon) is a closed curve such that every point in the polygon is visible from at least one point of the tour. The problem of finding a shortest watchman tour is NP-hard for polygons with holes. We show that the length of a minimum watchman tour in a polygon P with k holes is $O(\text{per}(P) + \sqrt{k} \cdot \text{diam}(P))$, where per$(P)$ and diam(P) denote the perimeter and the diameter of P, respectively. Apart from the multiplicative constant, this bound is tight in the worst case. A watchman tour of this length can be computed in $O(n \log n)$ time, where n is the total number of vertices. We generalize our results to watchman tours in polyhedra with holes in 3-space. We obtain an upper bound $O(\text{per}(P) + \sqrt{k \cdot \text{per}(P) \cdot \text{diam}(P)} + k^{2/3} \cdot \text{diam}(P))$, which is again tight in the worst case.

1 Introduction

Visibility and art gallery problems with stationary guards (watchmen) have been studied extensively since the early 1980s [15]. Mobile guards have been also considered soon after, see *e.g.* [7, 8, 14]. A *watchman tour* in a polygonal domain (polygon, for short) is a tour (i.e., closed curve) inside the polygon such that every point in the polygon is visible from some point along the tour. Two points in a polygon are visible to each other if the line segment between them lies in the polygon.

The watchman tour problem asks for a watchman tour of minimum length [2, 13]. The problem has a polynomial time solution for simple polygons with n vertices (and no holes). Tan [18] gave an $O(n^5)$-time algorithm improving an earlier $O(n^6)$-time algorithm by Carlsson *et al.* [5]. Other variants are discussed in [1, 13, 14]. In contrast, computing a shortest watchman tour in a polygon with holes is known to be NP-hard [7]. In Section 3, we revisit the old NP-hardness proof by Chin and Ntafos [7] and make some necessary clarifications.

Our main result is a tight worst-case upper bound for the minimum length of a watchman tour in a polygon with holes. Our upper bound depends on three parameters of a polygon P: the number of holes, $k = k(P)$, the diameter, diam(P), and the perimeter, per(P). The perimeter of P is the total length of the boundary of P (including the boundary of the holes). In Section 2 we prove:

Theorem 1 *The minimum length of a watchman tour for a polygon P with k holes is $O(\text{per}(P) + \sqrt{k} \cdot \text{diam}(P))$. This bound cannot be improved for polygons with $\text{per}(P) > c\, \text{diam}(P)$ for any fixed $c > 2$. A watchman tour of this length can be computed in $O(n \log n)$ time, where n is the total number of vertices of P.*

We have $\text{per}(P) > 2 \cdot \text{diam}(P)$ for every polygon P. If, however, per(P) is very close to $2 \cdot \text{diam}(P)$, then the polygon is long and skinny, and the above upper bound is no longer tight up to constant factors.

Theorem 1 generalizes to polyhedra, possibly with holes and handles, in three dimensions. The boundary of a polyhedron is composed of piecewise linear manifolds. We define the perimeter per(P) of a polyhedron P in 3-space as the total length of all edges of P.

Theorem 2 *The minimum length of a watchman tour for a polyhedron P in 3-space with k holes is at most $O(\text{per}(P) + \sqrt{k \cdot \text{per}(P) \cdot \text{diam}(P)} + k^{2/3} \cdot \text{diam}(P))$. This bound cannot be improved for polyhedra with $\text{per}(P) > c\, \text{diam}(P)$ for any fixed $c > 3$. A watchman tour of this length can be computed in $O((nk)^{4/5+\delta})$ expected time for any $\delta > 0$, where n is the total number of vertices, edges, and faces of P.*

Our results give a partial answer to a question of Nilsson [14], posed in his PhD thesis: "Is it possible to find approximative solutions to guarding problems, with good worst case bounds?"

2 Bounds on the length of an optimal tour

In this section we prove Theorems 1 and 2. By a classical result of Few [10], the shortest path through k points in the unit square $[0, 1]^2$ has (Euclidean) length at most $\sqrt{2k}+7/4$. Few also proved that the minimum spanning tree of these points has length at most $\sqrt{k} + 7/4$. Both upper bounds are constructive. For constructing a spanning path, he lays out about $\sqrt{k}$ equidistant horizontal lines, and then visits the points layer by layer, with the path alternating directions along the horizontal strips.

The current best lower bound for the length of such a path is also due to Few: it is $\left(\frac{4}{3}\right)^{1/4} \sqrt{k} - o(\sqrt{k})$,

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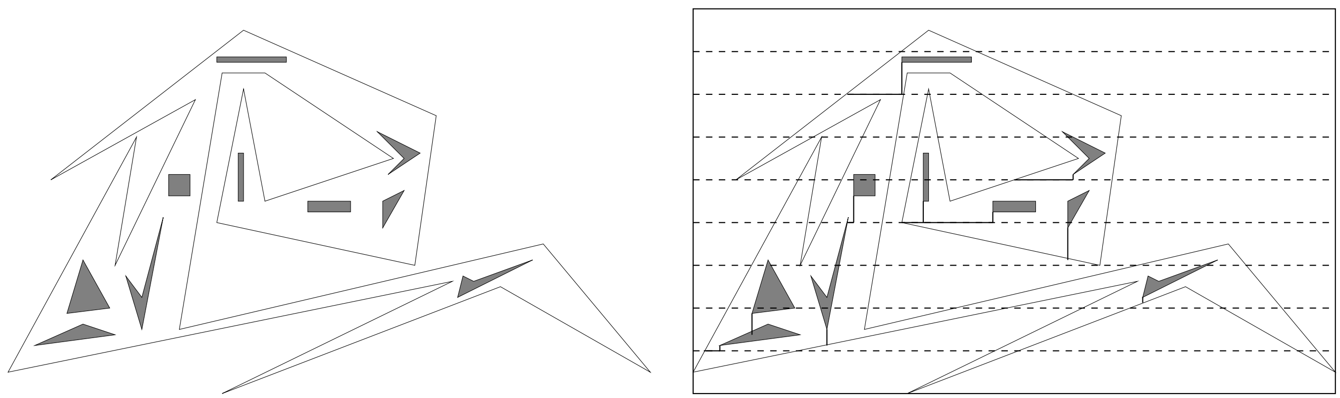

Figure 1: Left: a polygon with $k = 10$ holes. Right: Connecting the circuits ∂H_i, $i = 0, \ldots, k$ by adding connectors.

where $(4/3)^{1/4} = 1.075\ldots$. In a related problem, Chung and Graham [9] showed that the length of the shortest *Steiner* tree through k points in the unit square is at most $0.995\sqrt{k}$ (they gave details for an improvement to $0.99995\sqrt{k}$ only). In every dimension $d \geq 3$, Few showed that the maximum length of a shortest path through k points in the unit cube is $\Theta(k^{1-1/d})$.

Algorithm for constructing a watchman tour. Observe that for a simple polygon P *without* holes, ∂P is a watchman tour of length $\text{per}(P)$. Let now P be a polygon with $k \geq 1$ holes, $H_1, \ldots, H_k$. For convenience, denote by H_0 the unbounded hole defined by the exterior of P. The boundary of P consists of $k+1$ pairwise disjoint circuits $\partial P = \bigcup_{i=0}^{k} \partial H_i$. We have $\text{per}(P) = \sum_{i=0}^{k} |\partial H_i|$, where the vertical bars $|\cdot|$ stand for the Euclidean length.

Our algorithm works as follows. Compute an axis-aligned bounding box B of the polygon P whose longest side is of length at most $\text{diam}(P)$. We augment the disjoint union of circuits ∂H_i, $i = 0, \ldots, k$, with at most $2k$ line segments and possibly (at most $3k$) new vertices to a connected graph G. The new segments added are called *connectors*. The connectors are either vertical or horizontal. We replace each connector by double edges and obtain a multi-graph G' where every vertex has even degree. The watchman tour we construct, W, is an arbitrary Eulerian tour in G', which traverses the entire boundary of P (including the hole boundaries) once, and traverses every connector twice. It is easy to verify that each point p in the interior of P is seen from some point along the tour W: Take an arbitrary line through p and consider its first intersection with the boundary ∂P of P. Since W traverses ∂P, p is visible from some point on $\partial P \subset W$.

It remains to explain how to draw the connectors and to bound their total length. As in Few's method, subdivide the bounding box B into horizontal strips by a *raster* of at most $\sqrt{k}$ equidistant horizontal lines such that consecutive raster lines are at $\text{diam}(P)/\sqrt{k}$ distance apart. We construct the connectors in two phases (refer to Fig. 1). In the first phase from a lowest point (vertex) of each interior hole H_i, drop a vertical ray ℓ_i downwards until it hits the outer boundary, the boundary of another hole, or a horizontal raster line. Let v_i denote this vertical segment, and let p_i be its lower endpoint. If point p_i is in the interior of P, then it lies on one of the raster lines. In the second phase from every point p_i lying in the interior of P, draw a horizontal ray leftwards until it hits the outer boundary, the boundary of another hole, or another point p_j, $j \neq i$. Let h_i denote this horizontal segment. If p_i is already on the boundary of another hole or on ∂H_0, then h_i is an empty segment (i.e., not needed).

It is clear that by adding at most $2k$ (horizontal and vertical) connectors $h_i \cup v_i$, we obtain a connected graph G containing all the circuits ∂H_i, $i = 0, \ldots, k$.

Upper bound. The total length of the horizontal raster lines is at most $\text{diam}(P) \cdot \sqrt{k}$, so the total length of the (at most k) horizontal connectors does not exceed this bound. There are k vertical connectors, each of length at most $\text{diam}(P)/\sqrt{k}$. Hence their total length is also bounded by $k \cdot \text{diam}(P)/\sqrt{k} = \text{diam}(P) \cdot \sqrt{k}$. Consequently, the total length of W is

$$|W| = |\partial H_0| + \sum_{i=1}^{k} (|\partial H_i| + 2|v_i| + 2|h_i|) = O(\text{per}(P) + \sqrt{k} \cdot \text{diam}(P)). \quad (1)$$

Algorithm description and analysis. Let n denote the total number of vertices of P. The bounding box B and the raster lines can be computed in $O(n)$ time. (We do not compute the full arrangement of the raster lines and P, which may have up to $\Theta(n\sqrt{k})$ vertices.) The set of connectors, henceforth the graph G can be computed by a standard line-sweep algorithm [3, 4] in $O(n \log n)$ time. Sweep a horizontal line ℓ top-down. For every position of ℓ, we maintain in sorted order its intersection points with the vertices and edges of P and with the vertical connectors v_i. This order changes only if ℓ passes through a vertex of P or a point p_i, or if ℓ coincides with a raster line. So there are at most $n+k+\sqrt{k} \leq 3n$ events overall. When the sweep line ℓ coincides with one of the raster lines, we can find the closest intersection point in $\ell \cap \partial P$ to the left of each $p_i \in \ell$ in $O(\log n)$ time.

Observe that the graph G, as well as the multigraph G' have $O(n)$ edges each. Once G' is constructed, computing an Eulerian tour of G' takes $O(n)$ time. Hence the total time taken by the algorithm is $O(n \log n) + O(n) = O(n \log n)$.

Lower bound. We now show that our upper bound for the tour length in (1) is tight in the worst case for every $k \geq 0$ and $\text{per}(P) > c \cdot \text{diam}(P)$, where $c > 2$ is a fixed constant. We may assume w.l.o.g. that $\text{diam}(P) = 1$. We construct a polygon lying in a disk D of unit diameter. If $\text{per}(P) > 2\sqrt{2}$, then let the outer boundary H_0 of P be a square inscribed in D combined with a long and narrow zig-zag "snake" of total edge length $\text{per}(P) - 2\sqrt{2}$ and very small width $0 < \varepsilon \ll 1$ (Fig. 2). The snake lies in D such that the diameter of H_0 is 1. If $c < \text{per}(P) \leq 2\sqrt{2}$, then let H_0 be a rhombus of diameter 1 and side length $\frac{1}{4}(\text{per}(P) - \varepsilon)$ for a small $0 < \varepsilon \ll c-2$. In both cases, we have $\text{per}(H_0) = \text{per}(P) - \varepsilon$, and H_0 contains a square of side length $\Omega(1)$.

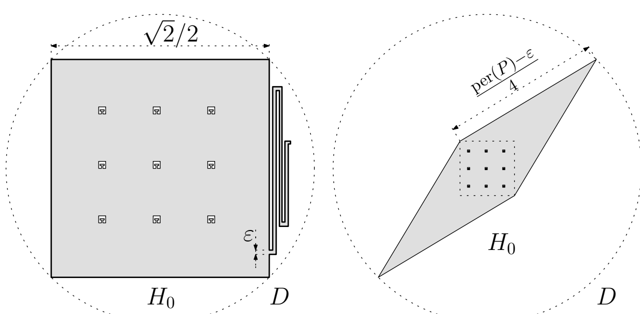

Figure 2: Lower bound constructions for the cases $\text{per}(P) > 2\sqrt{2} \cdot \text{diam}(P)$ and $2 \cdot \text{diam}(P) < \text{per}(P) \leq 2\sqrt{2} \cdot \text{diam}(P)$.

Arrange k small holes in a grid-like pattern in a maximal inscribed square of H_0. Each hole has $O(1)$ vertices, ε/k perimeter, and a small hidden "cave" that can be seen only by entering it; see *e.g.*, Fig. 3(right). By Few's result, the length of the shortest watchman tour that visits the caves in all holes is $\Omega(\sqrt{k})$. If $\text{per}(P) > 2\sqrt{2}$, the length of any walk from the bottom of the zig-zag snake to one of the furthest caves is $\Omega(\text{per}(P))$. We conclude that in both cases the length of the shortest watchman tour for P is

$$\Omega(\text{per}(P) + \sqrt{k}) = \Omega(\text{per}(P) + \sqrt{k} \cdot \text{diam}(P)),$$

as required. This completes the proof of Theorem 1.

Generalization to 3-dimensions. A polyhedron (possibly with holes) in 3-space is a piecewise-linear 3-manifold with boundary. Let $\text{per}(P)$ denote the total length of the edges of a polyhedron P. Note that every point p in the interior of P sees at least one point on some edge of P. Indeed, consider an arbitrary plane h containing p. The intersection $P \cap h$ is a collection of disjoint polygons (possibly with holes), one of which contains p. In a triangulation of this planar polygon, p lies in a triangle, and hence it sees the three vertices of the triangle. All three vertices are intersection points of h with some edges of P. It follows that a tour that traverses every edge of P is a watchman tour: i.e., every interior point of P is seen from some point of the tour.

Our algorithm for computing a watchman tour for P is analogous to the planar case. We augment the 1-skeleton of P to obtain a connected graph G. We then double some of the edges in G to make all vertex degrees even, and our watchman tour is an arbitrary Eulerian tour in this multi-graph.

Choose a lowest point w_i in each interior hole H_i, $i = 1, \ldots, k$. Compute an axis-aligned bounding box B of the polyhedron P of side length at most $\text{diam}(P)$. Subdivide B into horizontal strips by a *raster* of at most $k^{1/3}$ equidistant horizontal planes such that consecutive raster planes are at $\text{diam}(P)/k^{1/3}$ distance apart. Subdivide every strip by additional horizontal planes, if necessary, such that there are at most $k^{2/3}$ points w_i between consecutive horizontal planes. We have used at most $2k^{1/3}$ horizontal planes. From each w_i, drop a vertical line ℓ_i downwards until it hits the outer boundary, the boundary of another hole, or a horizontal plane. Let p_i be the lower endpoint of this vertical segment.

If point p_i is in the interior of P, then it lies on some horizontal plane. In each horizontal plane, we invoke our planar algorithm with $k' = k^{2/3}$ to construct connectors from every point p_i to the outer boundary or the boundary of another hole. The total length of the vertical connectors and of the horizontal connectors in the $O(k^{1/3})$ planes is bounded by

$$\begin{aligned} & O\left(k \cdot (\text{diam}(P)/k^{1/3}) + k^{1/3} \cdot (\sqrt{k^{2/3}} \cdot \text{diam}(P))\right) \\ & = O\left(k^{2/3} \cdot \text{diam}(P)\right). \qquad (2) \end{aligned}$$

For each interior hole H_i, we have computed a connector from a lowest vertex w_i to some point on the outer boundary ∂H_0 or the boundary of another hole. However, the endpoint of a connector may lie in the interior of a face. For every face f of P, let k_f denote the number of connector endpoints in the interior of f, with $\sum_f k_f \leq k$. In each face f, with $k_f \geq 1$, we construct a minimum spanning tree of the k_f connector endpoints in f and an arbitrary vertex of f. By Theorem 1, the length of an MST in a face f is $O(\sqrt{k_f} \cdot \text{diam}(f))$. The total length of these spanning trees is

$$\begin{aligned} & O\left(\sum_f \sqrt{k_f}\cdot \mathrm{diam}(f)\right) \\ = \ & O\left(\sqrt{\sum_f k_f}\sqrt{\sum_f \mathrm{diam}^2(f)}\right) \\ = \ & O\left(\sqrt{k}\sqrt{\mathrm{diam}(P)}\sqrt{\sum_f \mathrm{diam}(f)}\right) \\ = \ & O\left(\sqrt{k\cdot \mathrm{diam}(P)\cdot \mathrm{per}(P)}\right). \quad (3) \end{aligned}$$

In this chain of inequalities, we applied the Cauchy-Schwarz inequality, the bounds $\max_f \mathrm{diam}(f) \le \mathrm{diam}(P)$ and $\mathrm{diam}(f) \le \frac{1}{2}\mathrm{per}(f)$. We have $\sum_f \mathrm{diam}(f) \le \sum_f \frac{1}{2}\mathrm{per}(f) \le \mathrm{per}(P)$, since every edge is adjacent to exactly two faces.

By adding the term $\mathrm{per}(P)$ to the lengths in (2) and (3) the upper bound on the length of the tour in Theorem 2 follows.

All vertical segments of the connectors can be computed by a (randomized) batched ray shooting algorithm due to Pellegrini [17]. If the total number of vertices, edges and faces of P is n, then we can triangulate the faces of P into $O(n)$ triangles in $O(n\log n)$ time. Among $O(n)$ interior-disjoint triangles in 3-space, k batched ray shooting queries take $O((nk)^{4/5+\delta})$ expected time for any $\delta > 0$, where the constant of proportionality depends on δ. Similarly, we can compute all horizontal segments of the connectors simultaneously by the same algorithm in $O((nk)^{4/5+\delta})$ expected time. All minimum spanning trees over the $O(k)$ connector endpoints in the faces of P can be computed in $O(k\log n)$ time.

The lower bound constructions are similar to the planar case. We construct a polyhedron P with k holes and $\mathrm{per}(P) > c\cdot \mathrm{diam}(P)$ for a fixed $c > 3$. Note that $\mathrm{per}(P) > 3\cdot \mathrm{diam}(P)$ for every polyhedron, since the 1-skeleton of the outer boundary of P is a 3-connected graph, hence it contains at least three edge-disjoint paths between any two vertices. In particular, there are at least three edge-disjoint paths between two vertices at $\mathrm{diam}(P)$ distance apart, and at most one of these paths may have length $\mathrm{diam}(P)$. Hence $\mathrm{per}(P) > 3\cdot \mathrm{diam}(P)$.

We may assume without loss of generality that $\mathrm{diam}(P) = \Theta(1)$. We present three lower bound constructions, one for each term in the upper bound $O(\mathrm{per}(P) + \sqrt{k\cdot \mathrm{per}(P)} + k^{2/3})$.

Case 1: $c < \mathrm{per}(P) \le k^{1/3}$. In this case, $\max(\mathrm{per}(P), \sqrt{k\cdot \mathrm{per}(P)}, k^{2/3}) = k^{2/3}$. Let H_0 be a double pyramid obtained by gluing together two congruent pyramids, each with 1/2 height, and with a (small) triangular base. Arrange the holes in a grid-like pattern in a maximal inscribed cube in H_0 of side length $\Omega(1)$, each with a hidden "cave." By Few's result [10], a watchman tour that visits every cave has length $\Omega(k^{2/3})$.

Case 2: $k^{1/3} < \mathrm{per}(P) \le k$. In this case, $\max(\mathrm{per}(P), \sqrt{k\cdot \mathrm{per}(P)}, k^{2/3}) = \sqrt{k\cdot \mathrm{per}(P)}$. Let H_0 be composed of about $\Theta(\mathrm{per}(P))$ homothetic copies of a flat box of size $1\times 1\times \varepsilon$, for some small $\varepsilon > 0$, connected by a narrow corridor. Arrange about $\Theta(k/\mathrm{per}(P))$ holes, each with a hidden "cave," in a planar grid-like pattern in each flat box. By Few's result, the length of the watchman tour within each flat box is $\Omega(\sqrt{k/\mathrm{per}(P)})$, and so the total length is $\Omega(\sqrt{k\cdot \mathrm{per}(P)})$.

Case 3: $k < \mathrm{per}(P)$. In this case, $\max(\mathrm{per}(P), \sqrt{k\cdot \mathrm{per}(P)}, k^{2/3}) = \mathrm{per}(P)$. Let H_0 be a narrow zig-zag "snake" of diameter 1, built of a sequence of triangular prisms (and arbitrary holes of very small perimeter). A tour that visits both ends of the snake must have $\Omega(\mathrm{per}(P))$ length.

In all three cases, we have shown that there is a polyhedron P with k holes, $\mathrm{per}(P)$ perimeter and $\Theta(1)$ diameter such that the length of every watchman tour is $\Omega(\max(\mathrm{per}(P), \sqrt{k\cdot \mathrm{per}(P)}, k^{2/3})) = \Omega(\mathrm{per}(P) + \sqrt{k\cdot \mathrm{per}(P)} + k^{2/3})$, as required.

3 NP-hardness proof revisited

The Watchman Tour Problem (WTP) is the following.

> WTP: Given a polygon P with k polygonal holes, and a positive integer m, does there exist a watchman tour of total Euclidean length at most m?

The Geometric Traveling Salesman Problem (GTSP) [12, 16] is the following.

> GTSP: Given a set of n points in the plane, and a positive integer m, does there exists a tour of total length at most m that visits all the points?

It is known that GTSP is NP-hard with respect to both the L_1 and the L_2 metric [11, 16]. The NP-hardness proof in [6], and similarly that in [7] use a reduction from the Euclidean Geometric Salesman problem (L_2 metric) to the Watchman Tour Problem via a claim that relates the length of a solution for GTSP in the L_2 metric to the length of a solution for WTP: [6, Theorem 1, p. 25] and [7, Theorem 2.1, p. 40]. (Corollary 1 in [6] and Corollary 2.2 in [7] add even more to the confusion.) The correct reduction however is from GTSP in the L_1 metric. See Fig. 3.

Given set S of n lattice points, construct a polygon P as the slightly enlarged (grid) axis aligned rectangle of S. The holes are the cells of the grid, but only slightly

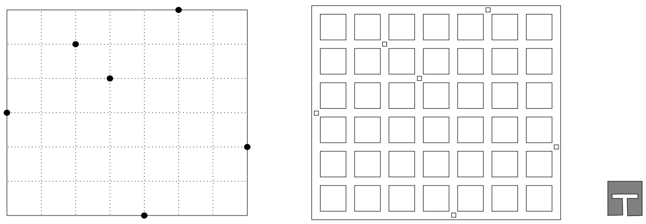

Figure 3: Left: point set S. Middle: polygon P with holes. Right: a small hole for each point in S.

smaller so that they are disjoint. In addition to these large (grid cell) holes, there are small holes corresponding to each point, each containing a "cave" that can only be seen by entering it. The holes are small enough so that they fit in the narrow corridors left by the big grid cell holes: the perimeter of each small hole is $1/80n$. The width of the corridors is $1/40n$.

The reduction, hence the NP-hardness, follows via the following claim, which is easy to verify:

Claim. For a positive integer m, there exists a tour of S of length at most m in the L_1 metric if and only if there exists a watchman tour of P of length at most $m + 0.1$ (in the usual L_2 metric).

Observe that the integrality requirement for m is crucial. Furthermore, no such claim holds if the length of the tour of the points in S is measured in the L_2 metric.

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