

Computing the Straight Skeleton of a Monotone Polygon in $O(n \log n)$ Time

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Abstract

The straight skeleton of a simple polygon is defined as the trace of the vertices when the initial polygon is shrunk in self-parallel manner [2]. In this paper, we propose a simple algorithm for drawing the straight skeleton of a monotone polygon. The time and space complexities of our algorithm are $O(n \log n)$ and $O(n)$ respectively.

1 Introduction

Skeleton like structure is often used for the description of basic topological characteristics of a 2D object, and have numerous applications in image processing, computer vision, solid modeling, mesh generation, to name a few [3, 11]. The medial axis or the skeleton of a simple polygon P is the locus of the centers of all the circles inside P that touch the boundary of P in two or more points. The medial axis of an arbitrary simple polygon consists of line segments and parabolic arcs, where each parabolic arc corresponds to a reflex vertex. It can also be viewed as the Voronoi diagram whose sites are the edges and vertices of the polygon boundary. The medial axis is basically the one dimensional representation of a simple polygonal object. In spite of many uses of medial axis, its curved arcs have been considered to be a shortcoming in different applications.

Straight skeleton for a simple polygon has been introduced in [2]. It is solely made of straight line segments which are pieces of angular bisectors of polygonal edges. Unlike the medial axis, the straight skeleton of an arbitrary simple polygon is not the Voronoi diagram of its edges.

The definition of the straight skeleton can be generalized to (i) arbitrary planar straight line graphs [1], which has applications in planar motion planning, and (ii) in 3D polyhedra, which has applications to solid modeling. Some nice applications of the straight skeleton are also demonstrated in computational origami [6, 7]

Although the medial axis of a simple polygon can be constructed in linear time [4], the fastest known algorithm for the straight skeleton runs much slower [9].

Table 1, lists the time and space bounds of various algorithms. Here n is the total number of vertices and r is the number of reflex (non-convex) vertices of the polygon; ϵ is an arbitrarily small positive constant.

Cheng and Vigneron [5] established connection between straight skeleton and motorcycle graph, and presented an $O(n\sqrt{n} \log n)$ time algorithm for computing motorcycle graph. They also proposed a randomized algorithm for computing the straight skeleton of a simple polygon, which runs in $O(n \log^2 n)$ time after computing motorcycle graph induced by reflex vertices of the polygon. Combining these two algorithms they proposed a randomized algorithm for computing the straight skeleton of a simple polygon with r reflex vertices which runs in $O(n \log^2 n + r^{3/2} \log r)$ expected time.

2 The straight skeleton and its properties

The straight skeleton of a simple polygon P is denoted by $S(P)$. The $S(P)$ is defined using an appropriate shrinking process for P . The boundary of P is contracted towards its interior in a self-parallel manner, and at the same speed for all edges. Each vertex of P moves along the angular bisector of its incident edges. This situation continues as long as the boundary does not change topologically. Then a new node of $S(P)$ is created. The shrinking process continues in the resulting polygon(s). The process terminates when the area of all the resulting polygons become zero. For the demonstration of the shrinking process, the reader may refer to [2]. It gives a hierarchy of nested polygons. Each polygon in the hierarchy is obtained after any one of the following three events.

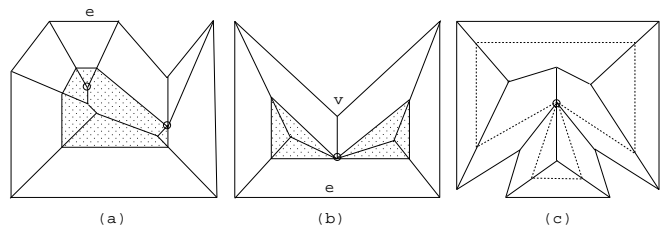


Figure 1: Illustration of (a) Edge-event, (b) split-event, and (c) vertex-event

edge-event: An edge e of P vanishes due to shrinking. This event is denoted by $EE(e)$. After the occurrence of $EE(e)$, the edges incident to the two end-points

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Table 1: Time and space complexity of earlier algorithms.

Time	Space	Reference
$O(n^2 \log n)$	$O(n)$	[1]
$O(nr \log n)$	$O(nr)$	[2, 9]
$O(n \log n + nr \log(n/r))$	$O(nr)$	[1, 8, 9]
$O(n \log n + nr + r^2 \log r)$	$O(n)$	[1, 9]
$O(n \log n + nr)$	$O(n + r^2)$	[1, 8, 9]
$O(n \log n + nr)$	$O(n)$	[10]
$O(n^{1+\epsilon} + n^{8/11+\epsilon} r^{9/11+\epsilon})$	$O(n^{1+\epsilon} + n^{8/11+\epsilon} r^{9/11+\epsilon})$	[9]
$O(n \log^2 n + r^{3/2} \log r)^*$	$O(n)$	[5]

* indicates randomized algorithm

of e become adjacent (see Fig. 1(a)); thus the number of edges in the resulting polygon is reduced by one.

split-event: An edge is split by a reflex vertex. In other words, a reflex vertex touches this edge during the shrinking process. This event is denoted by $SE(v, e)$, where v and e are the participating reflex vertex and edge of P respectively. After the occurrence of $SE(v, e)$, the polygon P is split into two sub-polygons of non-zero area (see Fig. 1(b)).

We will use the terms arcs and nodes for denoting the objects that form the boundary of the skeleton, in order to distinguish them from the objects forming the polygon, which will be called edges and vertices. Bisector pieces are called *arcs*, and their endpoints which are not vertices of P are called *nodes* of $S(P)$. Each edge/split-event introduces a node of degree three in $S(P)$. The angle(s) incident at the new vertex in the resulting polygon(s) is/are always convex.

In the degenerate cases, $S(P)$ may have nodes of degree greater than three, introduced by the occurrence of simultaneous events at the same location. In most cases, we can handle these events one at a time using standard perturbation techniques. This replaces a high-degree node with several nodes of degree three, connected by zero-length arcs.

vertex-event: It occurs when two or more vertices (but no edge) reach the same point simultaneously (See Fig. 1(c)). Unlike edge/split-event, a vertex-event may introduce a new reflex vertex into the shrunken polygon, although the total number of reflex vertices always decreases (See Fig. 1(c)).

Proposition 1 ([1]) *The straight skeleton $S(P)$ of P satisfies: (a) $S(P)$ is a tree with n leaf nodes corresponding to the vertices of P , less than or equal to $(n-1)$ non-leaf nodes and less than or equal to $(2n-3)$ arcs, (b) it partitions P into n connected faces; each face is a monotone polygon, and (c) in the boundary of each face, exactly one edge of the P appears.*

Proposition 2 *Let e_1 and e_2 be two edges incident on a vertex v . Let v' be the position of v at an instant of time during the shrinking process (before v vanishes*

due to edge/split-event). If r is the Euclidean distance $dist(v, v')$ between v and v' , ϕ is the angle incident at v inside the polygon, and χ is the distance traveled by e_1 and e_2 in self-parallel manner to meet v' , then $\chi = r \sin(\frac{\phi}{2})$, and is referred to as shrinking parameter at that instant of time.

The node of $S(P)$ generated by an edge-event $EE(e)$ is the point (ξ) of intersection of the bisectors of the angles of P incident to the end points of e .

A reflex vertex v may need to be tested with several edges to identify the exact edge e which participates in the split-event (with v). Let v be tested with the edges e_1, e_2, \dots, e_k and the corresponding event points be respectively $\xi_1, \xi_2, \dots, \xi_k$. The split-event $SE(v, e_i)$ may occur if $dist(v, \xi_i) \leq dist(v, \xi_j)$ for all $j = 1, 2, \dots, k$. Lemma 6 says that a split-event may disappear due to an edge-event during shrinking process.

3 The straight skeleton of a monotone polygon

A simple polygon P is said to be monotone with respect to a line L if its intersection (if any) with any line perpendicular to L is connected. Without loss of generality, we may assume that the line L is the Y -axis. The left (resp. right) monotone chain of P is the boundary of P from v_t to v_b in anti-clockwise (resp. clockwise) order where v_t and v_b are the vertices of P having maximum and minimum y -coordinate respectively. In the following lemmata we investigate the effect of monotonicity of a polygon in connection with the computation of its straight skeleton. All the proof of the lemmata of this paper are in Appendix.

Lemma 3 *For any split-event in a monotone polygon, the reflex vertex and the edge being split always belong to different monotone chains.*

Lemma 4 *Let v_1 and v_2 be two reflex vertices belonging to the same monotone chain such that v_1 and v_2 participates in the split-events $SE(v_1, e_1)$ and $SE(v_2, e_2)$ respectively. If v_1 is above v_2 (i.e. having larger y -coordinate value), then either e_1 and e_2 are the same edge, or e_1 is above e_2 .*

Lemma 5 *During the shrinking process of a monotone polygon, a vertex-event with the bisectors of two angles belonging to the same chain can not happen.*

Lemma 6 *The shrinking process never introduces a new reflex vertex.*

Lemma 7 *If ξ is the event point of $SE(v, e)$ then $d(\xi, \ell(e)) = \min\{d(\xi, \ell(f)); f \in P\}$, where $\ell(x)$ denote the line containing edge x and $d(\xi, \ell(f))$ is the perpendicular distance from ξ to $\ell(f)$.*

4 Algorithm

We create two arrays A_L and A_R with the vertices (and hence the edges) in the left and right monotone chains in a top to bottom order. In addition, we store the reflex vertices of the left and right chains in two arrays, namely REF_L and REF_R , in top to bottom order. The output of this algorithm is the skeleton tree $S(P)$ of the polygon P . We use a priority queue Q that stores the edge- and split-events; Q is maintained in the form of a min-heap in increasing order of the shrinking parameters of these events. Since a vertex-event can be considered as a split-event also, we need not have to identify the vertex-events separately. During the execution of the algorithm, we maintain a variable Δ which indicates the amount of shrinking of the polygon P up to the current instant of time.

An edge-event $EE(e)$ is stored in Q in the form of a tuple $(EE', e, \xi_1, \xi_2, \chi)$, where ξ_1 and ξ_2 are the two nodes of $S(P)$. When a node ξ is created in $S(P)$ at the time of processing $EE(e)$, ξ_1 and ξ_2 are connected with ξ . Initially, ξ_1 and ξ_2 contain the two end-points of the edge e in its original position¹. Similarly, a split-event $SE(v, e)$ is stored in Q in the form of a tuple (SE', v, e, ξ, χ) , where both v and e point to their own presence in A_L and A_R , and ξ is the node in $S(P)$ corresponding to the original position of the reflex vertex v . Each edge (resp. reflex vertex) in A_L and A_R also points to its corresponding edge-event (resp. split-event) in Q . Note that, after each edge-/split-event during the shrinking process, the edges of P get modified. But, we do not make these changes in A_L and A_R .

We also attach a height balanced binary tree $T(e)$ with each edge e in A_L and A_R . The reason is that the same edge may participate in different split-events with different vertices. Initially $T(e)$ contains e only. During the execution of a $SE(v, e)$, e is split into two parts. Each of these parts may further be split while executing another split-event $SE(v', e)$. These pieces of e are stored in $T(e)$.

¹If e is a polygonal edge, then ξ_1 and ξ_2 are the two end-points of e . If e is generated by a edge- or a split-event at a point ξ^* (say), then coordinate of ξ^* is stored as the corresponding end-point of e .

The execution starts by computing $EE(e)$ for all edges e of P , and storing them in Q . We now explain the method of computing the split-events corresponding to the reflex vertices of P .

Identification of split-events: In order to identify the edge e associated to a split-event $SE(v, e)$ corresponding to a reflex vertex v , we may need to inspect several edges of the opposite chain (see Lemma 3). Let $\{v_1, v_2, \dots, v_k\}$ be the set of reflex vertices in REF_L from top to bottom. First we compute the split-event corresponding to the vertex v_i , $i = \frac{1+k}{2}$, using Lemma 7 in $O(n)$ time. Let split-event be $SE(v_i, e_j)$, and it partitions A_R in two parts A'_R and A''_R (say). We include appropriate portion of e_j in both A'_R and A''_R . After computation of $SE(v_i, e_j)$, the edge participating to the split-event of a reflex vertex in $\{v_1, v_2, \dots, v_{i-1}\}$ will be a member in A'_R , and the edge participating to the split-event of a reflex vertex in $\{v_{i+1}, v_{i+2}, \dots, v_k\}$ will be a member in A''_R (see Lemma 4). Again, we find the split-event for the vertices $v_{i'}$ and $v_{i''}$ where $i' = \frac{i}{2}$ and $i'' = \frac{i+1+k}{2}$ in $O(n)$ time and so on. Thus, the time complexity for computing all the split-events is $O(n \log n)$ (see Lemmata 4 and 7). The same method is followed to compute the split-events for the reflex vertices in REF_R . All these split-events are stored in Q . Note that, a split-event generated at the beginning or later stage during the shrinking process, may not occur during the further shrinking process. Because, a reflex vertex may be destroyed after processing an edge-event (see Lemma 6). If such a situation takes place, the split-event corresponding to that vertex is deleted from Q .

Processing of edge/split-events: All the vertices of P are the leaf nodes in $S(P)$. Initially, Q contains the edge-events corresponding to all the edges in P and split-events corresponding to all the reflex vertices in P , Δ is set to 0. We choose an event having minimum shrinking parameter χ from Q , and perform one of the following procedures as stated below. The process continues until Q becomes empty. Observations 1 and 2, say that while processing an event some existing events will no longer be valid. These are to be deleted from Q . Also some events in Q may need to be modified during the shrinking process. The pointers attached to the vertices and edges of P in A_L and A_R help in accessing those events in Q .

Observation 1 *An edge-event $EE(e)$ for an edge e may not take place during the shrinking process. This happens if the shrinking parameter of a split-event $SE(v, e)$ for some reflex vertex v and the edge e is smaller than that of $EE(e)$. In this case, after $SE(v, e)$ the shrunken edge e is split into two pieces e_1 and e_2 , and they lie in different sub-polygons. Here, after the occurrence of $SE(v, e)$, we delete $EE(e)$ from Q , and store the new events $EE(e_1)$ and $EE(e_2)$ in Q .*

Observation 2 A split-event $SE(v, e)$ for a reflex vertex v may not take place during the shrinking process. This situation arises if one of the edges e' attached to v may disappear due to $EE(e')$ prior to $SE(v, e)$.

Processing of an edge-event $EE(e)$: The tuple attached to $EE(e)$ is $(EE', e, \xi_1, \xi_2, \chi)$, where $e = [\alpha, \beta] \in A_L$; $e_1 = [\alpha, \gamma]$ and $e_2 = [\beta, \delta]$ are two edges adjacent to e . We compute the event point ξ , and create a node of $S(P)$ at ξ ; join ξ with ξ_1 and ξ_2 to create two arcs of $S(P)$. The edge e is deleted from A_L , and the event $EE(e)$ is deleted from Q . If e_1 and e_2 changes to $e'_1 = [\xi, \gamma']$ and $e'_2 = [\xi, \delta']$ respectively after $EE(e)$, and χ_1 and χ_2 are the shrinking parameter of $EE(e'_1)$ and $EE(e'_2)$ respectively, then the shrinking parameter of $EE(e_1)$ and $EE(e_2)$ in Q are updated to $\Delta + \chi_1$ and $\Delta + \chi_2$ respectively. The ξ_1 parameter of $EE(e_1)$ and ξ_2 parameter of $EE(e_2)$ are also updated with ξ . Finally, Δ is updated with $\Delta + \chi$.

If the angle incident at one of the end-points, say α of $e = [\alpha, \beta]$ is reflex, then the corresponding split-event may be deleted from Q (see Lemma 6). Note that, if an edge-event $EE(e)$ occurs then the edge e is not involved in any split-event (see Lemma 7).

Processing of a split-event $SE(v, e)$: The tuple attached to $SE(v, e)$ is (SE', v, e, ξ_1, χ) . We create a node of $S(P)$ at the event point ξ , join ξ with ξ_1 , and then delete the event $SE(v, e)$ from Q . Next, we compute the perpendicular projection η of ξ on the edge e , and search in the tree $T(e)$ attached to the edge e to find the segment e^* of e that contains η . The root of $T(e)$ is reached using the pointer stored in the e field of the tuple. Let e_1^* and e_2^* be the two pieces of e^* generated after the split-event $SE(v, e)$, and e'_1 and e'_2 be the shrunken form of e_1^* and e_2^* after $SE(v, e)$. If χ_1 and χ_2 are the shrinking parameter of $EE(e'_1)$ and $EE(e'_2)$ respectively, then the shrinking parameter of $EE(e_1^*)$ and $EE(e_2^*)$ are $\Delta + \chi_1$ and $\Delta + \chi_2$ respectively. We now delete $EE(e^*)$ from Q , and insert two new events $EE(e_1^*)$ and $EE(e_2^*)$. If the tuple attached to $EE(e^*)$ is $(EE', e^*, p_1, p_2, \chi)$, then the tuple of $EE(e_1^*)$ and $EE(e_2^*)$ will be $(EE', e_1^*, p_1, \xi, \Delta + \chi_1)$ and $(EE', e_2^*, \xi, p_2, \Delta + \chi_2)$ respectively. Finally, we delete e^* and insert e_1^* , e_2^* in $T(e)$. Finally, Δ is updated with $\Delta + \chi$.

Note that, at the time of processing a split-event $SE(v, e)$, we do not need to update the existing split-event involving e in the queue, because the other split-events automatically detect its proper position on e (see identification procedure of split-event in Section 4).

4.1 Complexity analysis

Computation of all the edge-events of P takes $O(n)$ time. From the discussions in identification of split-events in Section 4, the split-events for all reflex vertices in P , and all the vertex-events (if any) can be com-

puted in $O(n \log n)$ time. As initially we have generated $O(n)$ events, the creation of the initial heap Q needs $O(n)$ time. While processing an $EE(e)$ or a $SE(v, e)$, at most two edge-events can be generated. In addition a $O(\log n)$ time search in $T(e)$ is required while processing a $SE(v, e)$. Moreover, the processing of an event also includes deleting at most two old events from Q and inserting at most two new events in Q , which requires $O(\log n)$ time. Since, the total number of internal nodes in the skeleton tree $S(P)$ is $O(n)$, the total number of events processed is also $O(n)$. Thus, the time complexity of our algorithm is $O(n \log n)$ in the worst case. Apart from storing the polygon during the shrinking process and the output skeleton tree $S(P)$, we also use (i) a priority queue Q for storing the event points, and (ii) the height balanced binary tree attached to each edge of P . The space required for both these data structures depends on the number of edge-events and split-events, which is $O(n)$ in the worst case.

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Appendix

Proof of Lemma 3: If possible let the split-event $SE(v, e)$ occurs during the shrinking process with shrinking parameter χ where vertex v and edge e belongs to the same monotone chain. Without loss of generality, we can assume that $SE(v, e)$ is the first event that occurs during the shrinking process, otherwise it is sufficient to prove the lemma for the polygon having the vertex v' instead of v and edge e' instead of e just before the shrinking parameter χ . Now we investigate the position of edge e_1 (just above the edge e in the chain) when shrinking parameter is χ . Since the chain is monotone with respect to Y -axis, the new position of e_1 is above the new position of e which leads to a contradiction that split-event $SE(v, e)$ is the first event.

Proof of Lemma 4: If both v_1 and v_2 cause split-events, then either these split-events will occur simultaneously or one will occur before the other. Now we consider the following three cases. We assume that, v_1 and v_2 are in the right chain, and e_1 and e_2 are in the left chain.

v_1 causes split-event earlier than v_2 : By Proposition 1(b), the two sub-polygons generated due to the split-event are monotone. Since v_2 lies below v_1 in the right monotone chain, after the split-event $SE(v_1, e_1)$ the vertex v'_2 (the new position of vertex v_2) will belong to the lower sub-polygon (see Fig. 2(a)). Since the sub-polygons have disjoint interior, if v_2 splits an edge, it must belong to the lower sub-polygon. Hence, if v_1 causes a split-event earlier than v_2 , then v_2 cannot split an edge above the one split by v_1 .

v_2 causes split-event earlier than v_1 : Following a similar argument as in the earlier case, v_1 will belong to the upper sub-polygon and hence cannot split an edge below the one split by v_2 .

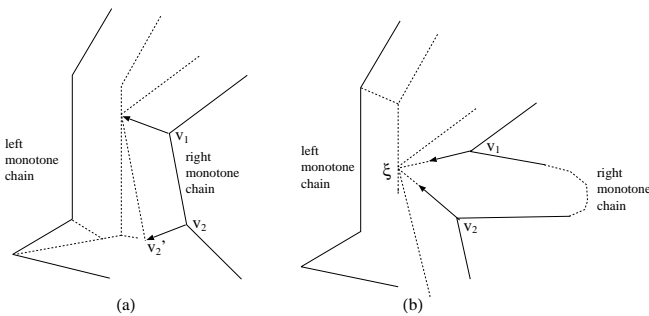


Figure 2: Proof of Lemma 4

Both the split-events occur simultaneously: If the bisector of reflex vertex v_2 splits an edge above the one split by v_1 then the bisector of the angle incident at v_2 must intercept the bisector of the angle incident at v_1 before the split-event happens. Thus, the edges incident on v_1 are different from the edges incident on v_2 (see Fig. 2(b)). Let ξ be the point where the simultaneous split-events occur. The point ξ must lie on the bisectors of the angles incident to both v_1 and v_2 .

Note that, for a valid split-event at ξ , only one edge can be split as shown in Fig. 2(b). The reason is that, if both $SE(v_1, e_1)$ and $SE(v_2, e_2)$ happen at the same point ξ , it implies that ξ is on the common end-point of e_1 and e_2 after shrinking. Thus, ξ appears on the bisectors of three angles (i) incident at v_1 , (ii) incident at v_2 and (iii) the angle between e_1 and e_2 . Thus, a vertex-event takes place at ξ . Lemma 5, stated below, says that such a vertex-event is impossible to happen during the shrinking process of a monotone polygon.

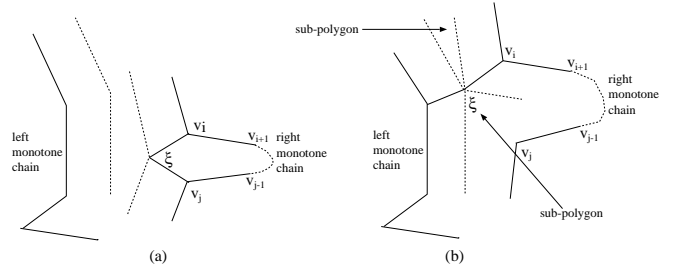


Figure 3: Proof of Lemma 5

Proof of Lemma 5: Consider a pair of vertices v_i and v_j in the same monotone chain, where $[v_i, v_j]$ is not an edge of P , and the bisectors of the angles incident at v_i and v_j form a vertex-event at point ξ (see Fig. 3). This implies, after the commencement of this event, the polygon P is split into two polygons. But, as P is monotone, the edge $v_i v_{i+1}$ will move below the edge $v_{j-1} v_j$ during the shrinking process before v_i and v_j reaches ξ . This leads to a contradiction.

Proof of Lemma 6: The topology of the polygon may be changed due to an edge-event or a split-event or a vertex-event. Below we consider each of these events separately.

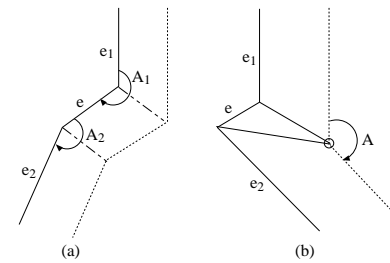


Figure 4: Proof of Lemma 6

Let us consider the time instant when $EE(e)$ happens for an edge e . Let e_1 and e_2 be the two edges adjacent to e . Let A_1 be the angle between the edges e_1 and e and A_2 be the angle between the edges e_2 and e . Since $EE(e)$ happens, A_1 and A_2 are not both reflex. If A_1 and A_2 are both non-reflex, then after the edge-event $EE(e)$, the new angle A between the edges e_1 and e_2 in the shrunken polygon is also non-reflex. So, we need only to consider the case where one of A_1 and A_2 is reflex and the other one is non-reflex. Without loss of generality, say A_1 is reflex. Note that, if $A_2 \geq 180^\circ - A_1$

then the edge-event $EE(e)$ does not take place (see Fig. 4(a)). If $A_2 < 180^\circ - A_1$, then after the edge-event $EE(e)$ the angle A becomes non-reflex (see Fig. 4(b)). Thus, an edge-event does not introduce any new reflex vertex.

After a split/vertex-event, the original polygon is split into two disjoint monotone sub-polygons at the event point (say ξ). In each of the resulting sub-polygons, the newly created vertex ξ is the common end-point of the upper and lower monotone chains. In other words, the angle incident at ξ inside both the sub-polygons is non-reflex. Thus, for a monotone polygon, a split/vertex-event never introduces a new reflex vertex in the resulting polygons.

Proof of Lemma 7: Consider two arbitrary edges e and f of the polygon. Let e' and f' are the new positions of e and f respectively for the shrinking parameter $\chi = \chi'$. In the shrinking process each edge of the polygon moves in parallel fashion and $d(e, e') = d(f, f')$, where $d(g, h)$ denotes the perpendicular distance between the lines $\ell(g)$ and $\ell(h)$. Let α be a point on the perpendicular bisector of the angle at v inside the polygon and C_α be a circle with radius $dist(\alpha, v)$ and center at α . We proceed along the perpendicular bisector of the angle at v inside the polygon. As soon as the circle C_ξ (for a point ξ) touches an edge of the polygon, a split-event takes place at ξ with shrinking parameter $\chi = dist(v, \xi)$ provided that the edge-event $EE(e, \chi')$ has not already occurred for some $\chi' < \chi$.