Combinatorial Changes of Euclidean Minimum Spanning Tree of Moving Points in the Plane*

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Abstract

In this paper, we enumerate the number of combinatorial changes of the the Euclidean minimum spanning tree (*EMST*) of a set of n moving points in 2-dimensional space. We assume that the motion of the points in the plane, is defined by algebraic functions of maximum degree s of time. We prove an upper bound of $O(n^3\beta_{2s}(n^2))$ for the number of the combinatorial changes of the *EMST*, where $\beta_s(n)=\lambda_s(n)/n$ and $\lambda_s(n)$ is the maximum length of Davenport-Schinzel sequences of order s on n symbols which is nearly linear in n. This result is an $O(n)$ improvement over the previously trivial bound of $O(n^4)$.

1 Introduction

The minimum spanning tree for a weighted graph $G(V,E)$ is a connected sub-graph $G'(V,E')$ of G where sum of the weights of its edges is the minimum possible. For a set $P = \{p_1, p_2, \ldots, p_n\}$ of n points, we can construct a complete weighted graph with these points as its nodes and the weight of an edge is the Euclidean distance between its end points. Finding the minimum spanning tree for this graph is known as the Euclidean minimum spanning tree problem (*EMST* for short). This problem has many applications in geometry and graph theory and has been studied extensively before [2, 3].

We consider the kinetic version of the *EMST* problem in which the points are moving in the plane and we want to enumerate the total number of the combinatorial changes of the *EMST* during the motion. In this setting, the points are moving independently and we assume that the position of a point p_i at time t, denoted by $p_i(t)$, is defined by an algebraic function of maximum degree s. We are supposed to find an upper bound for the number of these changes from time $t = 0$ to time $t = \infty$.

For a restricted version of this problem in which the trajectory of each point is defined by a linear function of time, Katoh *et al.* [4] proved that the maximum number of the minimum spanning tree changes for L_1 and L_∞ metrics is $O(n^{5/2}\alpha(n))$ where $\alpha(n)$ is the inverse Ackerman's function. They obtained an upper bound of $O(n^3 2^{\alpha(n)})$ for the number of the combinatorial changes of the minimum spanning tree in the L_2 metric for the special case of linearly moving points (in terms of time). Basch *et al.* [1] presented an approximate method for $(1+\epsilon)$-*EMST* which considers $O(\epsilon^{-(d-1)}n^3)$ combinatorial changes in d-dimensional space when the points follow algebraic trajectories of fixed degree. However, for the case of our problem in which the points are moving according to algebraic functions in the plane and the distances are measured by the L_2 metric, the known upper and lower bounds of the number of combinatorial changes of the *EMST* is the trivial $O(n^4)$ and $\Omega(n^2)$ bounds.

In this paper, we show that the number of the combinatorial changes of the *EMST* for our setting is $O(n^3\beta_{2s}(n^2))$. Here, $\beta_s(n)$ is an extremely slowly growing function of n. Precisely, $\beta_s(n)=\frac{\lambda_s(n)}{n}$ where $\lambda_s(n)$ is the maximum length of Davenport-Schinzel sequences of order s on n symbols which is nearly linear in n. To be exact, $\lambda_s(n) = n\alpha(n)^{O(\alpha(n)^{s-3})}$, for $s > 3$ where $\alpha(n)$ is the inverse Ackerman's function[5].

2 Preliminaries

We first consider the problem of enumerating the number of changes on the *lower envelope* of a set of functions in the plane. Let $\mathcal{F}=\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be a collection of continuous polynomial functions. The lower envelope of $\mathcal{F}$ is defined as: $\mathcal{LE}_{\mathcal{F}}(x)=\min_{1\leq i\leq n} f_i(x)$. A *breakpoint* arises on the lower envelope whenever two functions f_i and f_j intersect on $\mathcal{LE}_{\mathcal{F}}$ (See Figure 1). In other words, the lower envelope is a sequence of partially defined functions connected at the breakpoints. The length of a lower envelope is the length of its sequence. We have the following theorem about the length of the lower envelope:

Theorem 1 *(Corollary 2.2 [5]) For any collection*

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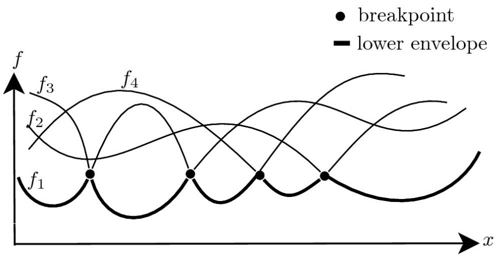

Figure 1: The lower envelope and its breakpoints

$\mathcal{F}$={$f_1, f_2, ..., f_n$} of n continuous, totally-defined, univariable functions, each pair of whose graphs intersect in at most s points, the length of the lower envelope sequence is at most $\lambda_s(n)$ where $\lambda_s(n) = n\beta_s(n)$ is the maximum length of a Davenport-Schinzel sequence of order s on n symbols.

In our usage, we need a slightly different version of the Theorem 1 to be applied on partial functions. Assume that a function $f_i \in \mathcal{F}$ is partial and it is undefined in range $I_j = (x_j, x'_j)$ and $f_i(x_j) > \mathcal{LE}_\mathcal{F}(x_j)$ and $f_i(x'_j) > \mathcal{LE}_\mathcal{F}(x'_j)$. This means that the discontinuity part of f_i occurs above the lower envelop. Figure 2 depicts this condition. While f_i is no longer continuous, we can not directly use the result of Theorem 1 to obtain an upper bound for the length of the lower envelope. It is fortunately easy to overcome this difficulty. While the start and the end of the discontinuity part of f_i lies above the lower envelope we can convert f_i to a continuous function f'_i in such a way that $\forall_{x \notin I_j} f'_i(x) = f_i(x)$ and $\forall_{x \in I_j} f'_i(x) > \mathcal{LE}_\mathcal{F}(x)$.

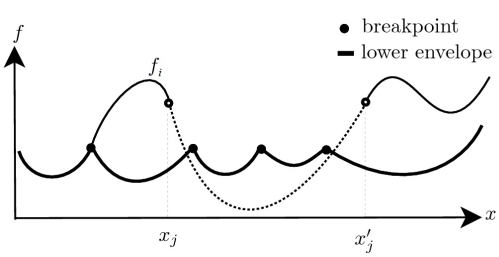

Figure 2: $I_j = (x_j, x'_j)$ of $f_i(x)$

If we use f'_i instead of f_i the lower envelope does not change and therefore its length is the same as before. Therefore, we can extend Theorem 1 to cover such functions. A function $f_i \in \mathcal{F}$ is defined to be $\mathcal{LE}_\mathcal{F}$-total if either it is total or for all ranges $I_j = (x_j, x'_j)$ where f_i is undefined we have $f_i(x_j) > \mathcal{LE}_\mathcal{F}(x_j)$ and $f_i(x'_j) > \mathcal{LE}_\mathcal{F}(x'_j)$. Similarly, f_i is defined to be $\mathcal{LE}_\mathcal{F}$-continuous if it is continuous in its domain. For partial functions continuity is not defined at the end points of its undefined ranges.

Now, we can rewrite Theorem 1 as follows:

Theorem 2 *For any collection $\mathcal{F}$={$f_1, f_2, ..., f_n$} of n $\mathcal{LE}_\mathcal{F}$-continuous, $\mathcal{LE}_\mathcal{F}$-total, univariable functions, each pair of whose graphs intersect in at most s points, the length of the lower envelope sequence is at most $\lambda_s(n)$ where $\lambda_s(n) = n\beta_s(n)$ is the maximum length of a Davenport-Schinzel sequence of order s on n symbols.*

3 Combinatorial Changes of the EMST

Let $\mathcal{E}(CG)$ and $\mathcal{E}(EMST)$ be respectively the set of edges of the complete graph and the edges of the $EMST$ of a set of moving points P in the plane and $path(p_i, p_j)$ be the simple path between p_i and p_j in the $EMST$.

After the initial computation of $\mathcal{E}(EMST)$ and during the motion, we must replace some edge $e_i \in \mathcal{E}(EMST)$ by another edge $e_j \in \mathcal{E}(CG) - \mathcal{E}(EMST)$. To enumerate the number of these changes we have to know such replacement candidate pairs. For each edge $p_ip_j \in \mathcal{E}(CG) - \mathcal{E}(EMST)$ there is a simple path $path(p_i, p_j)$ in the $EMST$ that connects p_i and p_j. Assume that p_sp_t has the maximum Euclidean length among edges of $path(p_i, p_j)$. Trivially, $|p_ip_j| > |p_sp_t|$ and if $|p_ip_j|$ gets to decrease while the points are moving it will be added to the $EMST$ just after the moment that its length reaches $|p_sp_t|$ (we assume that p_sp_t has still the maximum Euclidean length among the edges of $path(p_i, p_j)$). For such situations we say that p_ip_j is a *potential candidate* for p_sp_t. The set of all potential candidates of each edge $p_sp_t \in \mathcal{E}(EMST)$ is defined by $PK(p_sp_t)$.

According to the definition of $PK(p_sp_t)$, when the edge p_sp_t is removed from the $EMST$, $PK(p_sp_t)$ is no longer valid and we must build another $PK(p_ip_j)$ for the edge p_ip_j that has been inserted into the $EMST$ instead of p_sp_t. On the other hands, we always have $n-1$ edges in the $EMST$ and it is enough to have only $n-1$ set of PK's. We do this by labeling the edge of the $EMST$ by 1 to $n-1$ labels and for each label i we have PK_i which is the set of potential candidate edges $e_j \in \mathcal{E}(CG) - \mathcal{E}(EMST)$ that can take the position of the edge with label i in the $EMST$. The edge of label i in the $EMST$ is referred by $e(PK_i)$.

It is easy to see that $|e(PK_i)| \leq |e_{ij}|$ for all edges $e_{ij} \in PK_i$ and whenever $|e_{ij}|$ decrease to $|e(PK_i)|$ the edge $e(PK_i)$ is removed from the $EMST$ and e_{ij} takes its place and receives the label i just after the moment that $|e_{ij}|$ is smaller than $|e(PK_i)|$. We can describe this by the notion of the lower envelope as follows: assume that $g_i(t) = |e(PK_i)|^2$ and $f_{ij} = |e_{ij}|^2$ for all edges $e_{ij} \in PK_i$. Then, the lower envelope of $\mathcal{F}_i = \{f_{i1}, ..., f_{ik}\} \cup \{g_i\}$ defines the edges with the minimum length during the motion. It must be noted that $\mathcal{LE}_{\mathcal{F}_i}$ always lies on g_i and whenever another function f_{ij} intersects g_i its corresponding edge will be added to the $EMST$ and will be the edge $e(PK_i)$ and PK_i will be subject to some changes (insertion and deletion).

Therefore, an important notice about the PK_i and $\mathcal{F}_i$ sets is that these sets are not fixed and are subject to changes during the motion from time $t = 0$ to time $t = \infty$.

Using the above discussion we have the following Theorem about the number of the combinatorial changes of the *EMST* :

Theorem 3 *The total number of the combinatorial changes of the EMST of a set of n moving points in the plane is equal to the number of the breakpoints of the lower envelope of the functions $\mathcal{F}_i$.*

It is simple to show that the sets PK_i's are disjoint and therefore the $\mathcal{F}_i$'s sets of functions are disjoint. Moreover, $\bigcup_i (PK_i \cup e(PK_i)) = \mathcal{E}(CG)$ and therefore $|\bigcup_i \mathcal{F}_i| = O(n^2)$ which means that $|\mathcal{F}_i| = O(n^2)$.

Lemma 4 *Whenever an edge e is added or removed from a set $PK_i \cup e(PK_i)$ at time t, its corresponds function $f(t) = |e|^2$ lies above the lower envelop of $\mathcal{F}_i$ at this time.*

Proof. Assume that at time t an edge e is removed from $PK_i \cup e(PK_i)$ and it is inserted into $PK_j \cup e(PK_j)$. We know that for each PK_k the lower envelope $\mathcal{LE}_{\mathcal{F}k}$ lies on the function $g_k = |e(PK_k)|^2$. Therefore to prove the Lemma it is enough to show that at time $t - \epsilon$ we have $e \neq e(PK_i)$ and $e \neq e(PK_j)$ at time $t + \epsilon$. The correctness of these are directly derived from the definition of PK sets. An edge $p_k p_s$ at time t belongs to PK_i if and only if $e(PK_i) \in path(p_k p_s)$ at time t and has the maximum length. Therefore, if at time t we must remove e from $PK_i \cup e(PK_i)$ and add it to $PK_j \cup e(PK_j)$ we must have $e(PK_i) \in path(e)$, $e(PK_j) \in path(e)$ and $|e(PK_i)| < |e(PK_j)|$ at time $t + \epsilon$. While $e(PK_i)$ and $e(PK_j)$ belong to the *EMST*, e can not be equal to any one of these edges at time t. □

Using the results of the above Theorems, Lemma and discussions we have the following result:

Theorem 5 *For a set of n points moving in the plane according to some algebraic functions of maximum degree s, the number of the combinatorial changes of the EMST is $O(n^3\beta_{2s}(n^2))$.*

Proof. As the points are moving according to function of maximum degree s, the functions g_i and f_{ij} are defined by algebraic functions of maximum degree $2s$. Combining the results of Theorem 2 and Lemma 4 and the fact that $|\mathcal{F}_i| = O(n^2)$, the number of the breakpoints of the lower envelope of each $\mathcal{F}_i$ is $\lambda_{2s}(n^2)$. We have $n - 1$ sets of $\mathcal{F}_i$ corresponding to the $n - 1$ labels and therefore the total number of the breakpoints of the lower envelopes is $O(n\lambda_{2s}(n^2)) = O(n^3\beta_{2s}(n^2))$ which according to Theorem 3 is equal to the number of the combinatorial changes of the *EMST*. □

4 Conclusion

In this paper, we investigated the combinatorial changes of the minimum spanning tree of a set of moving points in L_2 metric. We proved an upper bound of $O(n^3\beta_{2s}(n^2))$ which is an improvement of $O(n)$ over the previously known bound.

Proving the tight bounds for the number of the changes of the *EMST* and extending it to higher dimensions are further directions of this research.

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