New ϵ-Net Constructions

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Abstract

In this paper, we give simple and intuitive constructions to obtain linear size ϵ-nets for α-fat wedges, translations and rotations of a quadrant and axis-parallel three-sided rectangles in $\mathbb{R}^2$. We also give new constructions using elementary geometry to obtain linear size weak ϵ-net for d-hypercubes and disks in $\mathbb{R}^2$.

1 Introduction

A set system H, also called hypergraph, is a pair $(X, \mathcal{F})$, where X is a finite set and $\mathcal{F}$ is a non-empty family of subsets of X. We restrict ourselves to geometric set systems $(X, \mathcal{F})$, where X is a set of points in $\mathbb{R}^2$ and $\mathcal{F}$ is family of subsets of X induced by geometric objects like wedges, quadrants, squares and disk.

For these set systems, we define ϵ-net as follows. A set $N \subseteq X \subseteq \mathbb{R}^2$ is called ϵ-net for $(X, \mathcal{F})$ if $N \cap S \neq \phi$ for all $S \in \mathcal{F}$ with $|S| \geq \epsilon|X|$. If $N \subseteq \mathbb{R}^2$, then it is called a weak ϵ-net for $(X, \mathcal{F})$.

Apart from the great theoretical importance they have in computational and combinatorial geometry, ϵ-nets have wide variety of applications in many geometric problems like hitting set, set cover, geometric partitions, range searching, etc. See [8] for a text book treatment of the topic. A central result in the theory of ϵ-nets called Epsilon-net theorem, due to Haussler and Welzl [6] states that, for set systems with bounded VC-dimension d, there exists an ϵ-net of size $O(\frac{d}{\epsilon}\log\frac{1}{\epsilon})$.

Linear size ϵ-nets exists for geometric objects like halfspaces in $\mathbb{R}^2$ and $\mathbb{R}^3$ [7, 9, 10], pseudo disks [7, 10]. Aronov et al. [2] show that $O(\frac{1}{\epsilon}\log\log\frac{1}{\epsilon})$ size ϵ-nets exist for axis-parallel rectangles. Recent result from Noga Alon [1] shows that there exist simple geometric set systems with VC-dimension two which do not admit linear size ϵ-nets. This result implies a (slightly) superlinear lower bound on the size of ϵ-nets for many geometric objects like lines, wedges and strips in $\mathbb{R}^2$ (or fat lines as referred in [1]), triangles, etc.

Weak ϵ-nets for convex objects (which have unbounded VC-dimension) have been studied in [3]. ϵ-nets have also been considered for the dual problem, where X is an arrangement of geometric objects like circles, squares, etc. and $\mathcal{F}$ is subsets of X induced by points. See [4] for more details.

1.1 Our results

In this paper, we give new constructions to get ϵ-nets for the following objects.

1) A simple construction to get an ϵ-net of size $O(\frac{\pi}{\alpha\epsilon})$ for α-fat wedges in $\mathbb{R}^2$. For the dual problem a linear size ϵ-net is shown in [4], using the combinatorial complexity of the union of objects.

2) Linear size ϵ-nets for quadrants and three-sided axis-parallel rectangles (unbounded axis-parallel rectangles) in $\mathbb{R}^2$.

3) An alternate construction using elementary geometry to get weak ϵ-net of size $\frac{2^d}{\epsilon}$ for d-hypercubes and $O(\frac{1}{\epsilon})$ size weak net for disks in $\mathbb{R}^2$. These results can also be derived from the solution to Hadwiger-Debrunner (p,q) problem for d-hypercubes and balls. However, the proofs are more involved. See [5]. For the case of disks in $\mathbb{R}^2$, $O(\frac{1}{\epsilon})$ size (strong) ϵ-net exist. See [7, 10].

2 ϵ-nets for α-fat wedges in $\mathbb{R}^2$

In this section, we present our main result, ϵ-nets for α-fat wedges in $\mathbb{R}^2$. Without loss of generality, we assume that points are in general position with no two points having the same X or Y coordinate.

Definition 2.1: In $\mathbb{R}^2$, a wedge is defined as the region of intersection of two non-parallel halfspaces. An α-fat wedge is a wedge having an angle of intersection of at least α-radians between the two lines that define the wedge.

Definition 2.2: An axis-aligned wedge is a wedge with angle less than $\frac{\pi}{2}$, formed by the intersection of two halfspaces one of which is either parallel to horizontal axis or vertical axis.

The intersection of a horizontal halfspace with any other halfspace creates four different types of axis-aligned wedges depending upon the direction the open face extends. Similarly, the intersection of a vertical halfspace with any other halfspace creates four different types axis-aligned wedges. Hence we distinguish eight different types of axis-aligned wedges and call them Type 1, Type 2 etc.

Definition 2.3: A Type 1 wedge is an axis-aligned wedge formed by the intersection of a horizontal halfs-

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pace ($y \geq y_0$) with another halfspace whose defining line has positive slope (The wedge W in Figure 2 is a Type 1 wedge). We show a simple construction to obtain small size ϵ-nets for Type 1 wedges.

Lemma 1 *ϵ-nets of size $O(\frac{1}{\epsilon})$ exist for Type 1 wedges.*

Proof. Divide the input point set horizontally into $\frac{2}{\epsilon}$ partitions, each containing $\frac{\epsilon n}{2}$ points. Let M denote the set of points we choose as an ϵ-net. For every partition $i, 1 \leq i \leq \frac{2}{\epsilon}$, let P_i denote the set of points lying on or above the partition i. Let H_i denote the convex hull of P_i. Let H_i' denote the ordered set of points lying on the boundary of H_i, ordered in anti-clockwise direction starting with the topmost point of P_i. For every point $p \in H_i'$, let $N(p)$ denote the point following p in the ordered list H_i'. For the last point of H_i', $N(p)$ is defined as the first element of H_i'. Let H_i'' be the subsequence of H_i' consisting of points belonging to the ith partition (the points in H_i'' appear in the same order as they appear in H_i'). Since the point with lowest Y-coordinate of any point set will be on the convex hull, H_i'' is not empty. For every partition $i, 1 \leq i \leq \frac{2}{\epsilon}$, let p_i denote the last point in the ordered list H_i''. For every partition $i, 1 \leq i \leq \frac{2}{\epsilon}$, include in M, the point p_i and $N(p_i)$, i.e., $M = \bigcup_{i=1}^{\frac{2}{\epsilon}} \{p_i, N(p_i)\}$ (Refer Figure 1). Since we are picking two points for every partition, $|M| \leq \frac{4}{\epsilon}$. We now show that, M indeed forms a valid ϵ-net.

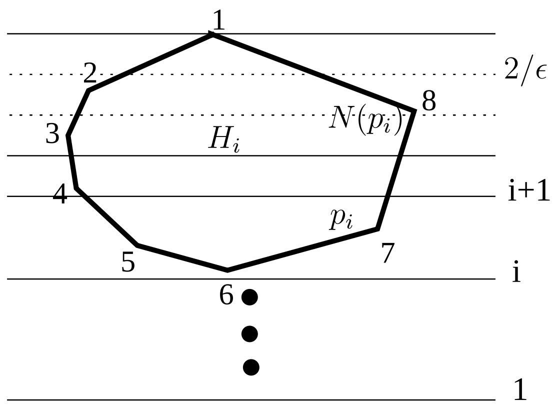

Figure 1: $H_i' = \{1, 2, 3, 4, 5, 6, 7, 8\}, H_i'' = \{5, 6, 7\}, p_i = 7$ and $N(p_i) = 8$.

Let W be a Type 1 wedge containing more than ϵn points, which means, W has to take points from at least three partitions. Let i be the partition containing the horizontal line of W. Let $j, k, i < j < k \leq \frac{2}{\epsilon}$ be the indices such that W takes at least one point from the jth and kth partition. We claim that W contains at least one of p_j or $N(p_j)$.

W intersects the convex hull H_j as it takes points from the jth partition (see Figure 2). Since W also takes points from the kth partition, it has to either contain or intersect the edge $(p_j, N(p_j))$ of H_j. In both the cases, W contains at least one of p_j or $N(p_j)$. $\square$

The ϵ-net construction for the Type 1 wedges can be suitably modified to get an ϵ-net of size at most $\frac{4}{\epsilon}$ for all the other types of axis-aligned wedges. This proves that, ϵ-nets of size at most $\frac{32}{\epsilon}$ exist for the axis-aligned wedges. Now we are ready to prove the main result.

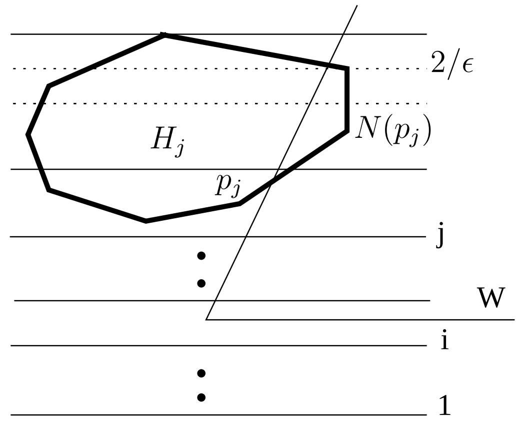

Figure 2: A Type 1 wedge anchored at the partition i and intersecting the edge $(p_j, N(p_j))$ of H_j

Theorem 2 *ϵ-nets of size $O(\frac{\pi}{\alpha\epsilon})$ exist for α-fat wedges.*

Proof. The main idea behind the construction of ϵ-net M for α-fat wedges is to find an axis-aligned wedge contained fully in the α-fat wedge and having a good fraction of ϵn points of the wedge. Then we can use the construction given in Lemma 1 to stab such a wedge. To do this, we construct a sequence of ϵ-nets and include them in M.

1. Construct an $\frac{\epsilon}{3}$-net M_h for halfspaces in R^2.
2. Construct an $\frac{\epsilon}{3}$-net M' for axis-aligned wedges as described in Lemma 1.
3. If α is less than $\frac{\pi}{2}$, do the following. For $\forall i, 1 \leq i \leq \lceil \frac{\pi}{2\alpha} \rceil$ rotate the coordinate axes by $i\alpha$ radians in clockwise direction and construct an $\frac{\epsilon}{2}$-net M_i for axis-aligned wedges.
4. Take $M = M_h \cup M' \cup \{\bigcup_i M_i\}$

We show that M is a valid ϵ-net for α-fat wedges. Consider any wedge W forming an angle $\theta, \theta \geq \alpha$, and containing ϵn points. If $\theta \geq \frac{\pi}{2}$, then W contains either an axis-aligned wedge having at least $\frac{\epsilon n}{3}$ points or contains a halfspace having at least $\frac{\epsilon n}{3}$ points. In either case, W contains one of the points of M. If $\theta < \frac{\pi}{2}$ then at one of the orientations of the coordinate axes as described in step 3, W contains an axis-aligned wedge having at least $\frac{\epsilon n}{2}$ points. Therefore M forms a valid ϵ-net.

There are many constructions known to get ϵ-net of size at most $\frac{2}{\epsilon}$ for halfspaces in $\mathbb{R}^2$. Hence, $|M| = O(\frac{\pi}{\alpha\epsilon})$. $\square$

Corollary 1: ϵ-nets of size at most $\frac{64}{\epsilon}$ exist for translations and rotations of a quadrant.

Proof. This follows from the observation that every orientation of a quadrant contains an axis-aligned wedge containing at least $\frac{\epsilon n}{2}$ points. $\square$

3 ϵ-nets for axis-parallel three-sided rectangles

In this section, we consider three-sided axis-parallel rectangles (rectangles with one of the sides open) in $\mathbb{R}^2$ and show by elementary construction that linear size ϵ-nets exist for them. However, for arbitrary orientations of three-sided rectangles, a non-linear lower bound is shown in [1].

Theorem 3 *ϵ-nets of size $O(\frac{1}{\epsilon})$ exist for axis-parallel three sided rectangles in $\mathbb{R}^2$.*

Proof. We assume for simplicity that no two points have the same X or Y coordinate. This assumption can be removed by a trivial modification to our proof. Partition the input point set horizontally and vertically into $\frac{2}{\epsilon}$ blocks such that, each horizontal and each vertical block contains $\frac{\epsilon n}{2}$ points. Let M denote the set of points we chose as ϵ-net. From every horizontal block, include in M, points with the highest and the lowest value of X coordinate. Similarly, from every vertical block, include in M, points with the highest and the lowest value of Y coordinate. Clearly, $|M| \leq \frac{8}{\epsilon}$. We show that M forms an ϵ-net for three sided axis-parallel rectangles. To see this, without loss of generality, consider any axis-parallel three-sided rectangle R with the open region extending towards top. Let l,r,b denote the left, right and bottom sides of R. Assume for contradiction that R does not contain any points from M. To contain more than ϵn points, R has to include points from at least three horizontal and three vertical blocks. Consider the vertical blocks which do not contain the sides l and r. Since from every vertical block, M contains the point with highest Y coordinate, R cannot include points from these blocks without containing the point with highest Y coordinate. Therefore, R is effectively including points from at most two blocks. A contradiction. $\square$

Note: The above technique also gives us an ϵ-net of size at most $\frac{4}{\epsilon}$ for axis-parallel quadrants, by considering horizontal (or vertical) partitions only, and taking points as described above.

4 Weak ϵ-nets

In this section we give simple constructions to get linear size weak ϵ-nets for axis-parallel d dimensional hypercubes (d-hypercubes) and disk in $\mathbb{R}^2$.

4.1 Weak ϵ-nets for axis-parallel d-hypercubes

Theorem 4 *Weak ϵ-nets of size $\frac{2^d}{\epsilon}$ exist for axis-parallel d-hypercubes.*

Proof. Let P denote the input point set and M denote the set of points we choose as ϵ-net. We consider the smallest d-hypercube containing ϵn points, include all its 2^d vertices in M and recurse on the remaining points. We formally state the construction as follows: For any d-hypercube C, let $P(C)$ denote the set of points enclosed by C. Let C_i be the smallest d-hypercube containing ϵn points on the point set $P \setminus \bigcup_{j=1}^{i-1} P(C_j)$. For all $i, 1 \leq i \leq \frac{1}{\epsilon}$, include all the vertices of C_i in M. Since at each iteration we pick 2^d points, $|M| = \frac{2^d}{\epsilon}$.

We show that, M is a weak ϵ-net for axis-parallel d-hypercubes. Consider any axis-parallel d-hypercube C which contains more than ϵn points. Let $S \subseteq \{C_i | 1 \leq i \leq \frac{1}{\epsilon}\}$ be the set of d-hypercubes that C intersects. Let C_j be the d-hypercube with the smallest index in S. Since at each iteration we pick the smallest d-hypercube containing ϵn points, C_j cannot be larger than C. Therefore, C contains one of the vertices of C_j. Hence, M is a weak ϵ-net for d-hypercubes. $\square$

4.2 Weak ϵ-nets for disks

Theorem 5 *Weak ϵ-nets of size $\frac{13}{\epsilon}$ exist for disks.*

Proof. We use a similar technique as described in Theorem 4. Let P denote the input point set and M denote the set of points we choose as ϵ-net. For any disk C, let $P(C)$ denote the set of points enclosed by C. Let C_i be the smallest disk containing ϵn points on the point set $P \setminus \bigcup_{j=1}^{i-1} P(C_j)$. For all $i, 1 \leq i \leq \frac{1}{\epsilon}$, let C_i' denote the concentric circle with radius $\frac{3}{2}$ times the radius of C_i. From the circumference of C_i, $1 \leq i \leq \frac{1}{\epsilon}$, include in M, five equally spaced points. Similary, from the circumference of C_i', $1 \leq i \leq \frac{1}{\epsilon}$, include in M, eight equally spaced points. Since, at each iteration we pick exactly thirteen points, $|M| = \frac{13}{\epsilon}$. We shall show that M is a valid weak ϵ-net for disks. Towards this end, we shall make an elementary observation.

Claim: Let C_1, C_2 be concentric circles of radius r and $\frac{3r}{2}$. Let C' be circle of radius r which intersects C_1. Then, C' will either enclose an arc of length at least $\frac{1}{5}$th fraction of circumference of C_1 or enclose an arc of length at least $\frac{1}{8}$th fraction of circumference of C_2 .

Refer figure 3. Consider the case when C' touches the circle C_1. Using the cosine rule, it follows that $\angle QPA$ is at least $25°$ and $\angle BPA$ is at least $50°$. Therefore C' encloses an arc of length at least $\frac{1}{8}$th fraction of circumference of C_2.

Now consider the case when center of C' lies on the circumference of C_2. Refer figure 4. In this case, the

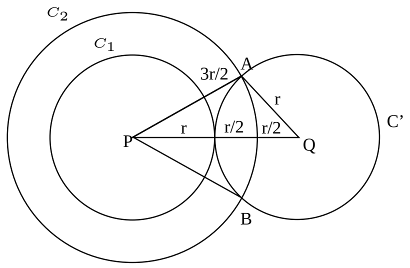

Figure 3: Circles C_1 and C_2 are concentric circles with radius r and $\frac{3r}{2}$. Circle C' touches C_1.

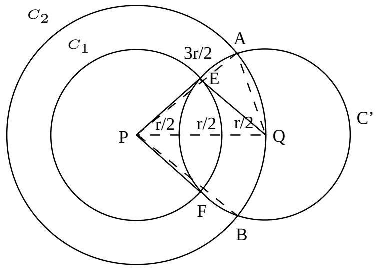

Figure 4: Circles C_1 and C_2 are concentric circles with radius r and $\frac{3r}{2}$. Center of C' lies on circumference of C_2.

$\angle QPA$ is at least $35°$ and $\angle BPA$ is at least $70°$. So, C' still encloses an arc of length at least $\frac{1}{8}$th fraction of circumference of C_2. It is easy to see that if the center of C' lies in between these two configurations, length of the arc enclosed by C' increases monotonically.

It also follows from the cosine rule that, the $\angle QPE$ is at least $40°$ and $\angle EPF$ is at least $80°$. Hence at this configuration, C' will enclose an arc of length at least $\frac{1}{5}$th fraction of circumference of C_1. If C' intersects the circle C_1 more deeply, it will enclose a larger fraction of circumference of C_1. This proves the claim. It is clear that the above claim holds when the radius of C' is greater than r.

Now consider any disk C containing more than ϵn points. Let $S \subseteq \{C_i | 1 \leq i \leq \frac{1}{\epsilon}\}$ be the set of disks that C intersects and let C_j be the disk with the smallest index in S. Let C'_j denote the concentric disk of radius $\frac{3}{2}$ times radius of C_j. Since at each iteration we pick the smallest disk containing ϵn points, C_j cannot be larger than C. Therefore, from the observation mentioned above, C will either enclose an arc of length at least $\frac{1}{5}$th fraction of circumference of C_j or enclose an arc of length at least $\frac{1}{8}$th fraction of the circumference C'_j. Since M contains five equally spaced points from the circumference of C_j and eight equally spaced points from the circumference of C'_j, C has to contain at least one of these points. Hence M is a valid ϵ-net. $\square$

Conclusion

In this paper, we have shown a simple construction to get small size ϵ-nets for α-fat wedges. Since arbitrary wedges do not admit linear size ϵ-nets (they do not admit linear size weak ϵ-nets as well), it is an interesting open question to get tight bounds on the size of ϵ-nets. Another interesting open question is to find tight bounds on the size of weak ϵ-nets for axis-parallel rectangles.

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