# Some Properties of Higher Order Delaunay and Gabriel Graphs

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## Abstract

We consider two classes of higher order proximity graphs defined on a set of points in the plane, namely, the $k$-Delaunay graph and the $k$-Gabriel graph. We give bounds on the following combinatorial and geometric properties of these graphs: spanning ratio, diameter, chromatic number, and minimum number of layers necessary to partition the edges of the graphs so that no two edges of the same layer cross.

## 1 Introduction and basic notation

Let $S$ be a set of $n$ points in the plane in general position (no three are collinear and no four are concyclic). A proximity graph on $S$ is a geometric graph where two points are adjacent if they satisfy some specific proximity criterion. Proximity graphs have been widely studied due to their theoretical interest and to their applications in situations where it is necessary to extract the "shape" of a set of points (see [10] for a survey).

Adjacency in many proximity graphs is defined in terms of an empty region associated to any pair of points. To provide more flexibility the definition of the graphs can be relaxed to allow up to $k$ points to lie in the neighborhood region. This gives rise to higher order proximity graphs. In this paper we deal with two such graphs.

We consider the *$k$-Delaunay graph* of $S$ (denoted $k$-$\mathsf{DG}(S)$), where a straight-line segment connects points $p_i, p_j \in S$ if there exists a circle $C(p_i, p_j)$ through $p_i$ and $p_j$ with at most $k$ points of $S$ in its interior. The standard Delaunay triangulation corresponds to 0-$\mathsf{DG}(S)$ and will be denoted by $\mathsf{DT}(S)$.

We also study the *$k$-Gabriel graph* of $S$ (denoted by $k$-$\mathsf{GG}(S)$), where a straight-line segment connects points $p_i, p_j \in S$ if the closed disk centered at the midpoint of the segment $\overline{p_i p_j}$ with both $p_i$ and $p_j$ on its boundary contains at most $k$ points from $S$ different from $p_i, p_j$. The standard Gabriel graph corresponds to 0-$\mathsf{GG}(S)$ and will be denoted by $\mathsf{GG}(S)$.

The combinatorial and geometric properties of these graphs have been widely studied for the case $k = 0$ (see [10]). However, not so much is known for higher values of $k$. Some results are given in [1, 16], but the topic has still not been explored in full depth; a systematic study is being developed in [15].

The first property considered in this paper is the spanning ratio, a parameter capturing to what extent traveling along a graph is much longer than traveling along the plane (the formal definition is given below). For $k = 0$, the spanning ratio of several proximity graphs has been studied in the literature [5, 6, 9, 11], and determining the exact value of the spanning ratio of the Delaunay triangulation remains a challenging open problem. Our main goal here is to study the relationship between $k$ and the spanning ratio.

We also study the diameter of $k$-$\mathsf{DG}(S)$ and $k$-$\mathsf{GG}(S)$, which can be seen as a combinatorial counterpart to the spanning ratio.

Finally, we give bounds on the minimum number of layers necessary to partition the edges of $k$-$\mathsf{DG}(S)$ or $k$-$\mathsf{GG}(S)$ so that no two edges of the same layer cross. From a theoretical point of view, this is related to a more general problem that remains unsolved (see, for example, [4, 12]): for every geometric graph $G$ with at most $\lambda$ pairwise crossing edges, can the edges of $G$ can be colored with $f(\lambda)$ colors such that crossing edges receive distinct colors? In our particular case, the answer is affirmative, as it can be shown that the graphs $k$-$\mathsf{GG}(S)$ and $k$-$\mathsf{DG}(S)$ contain at most $2k+1$ pairwise crossing edges. In Section 6 we give a quadratic upper bound on the number of colors required.

From a more practical point of view, $\mathsf{DT}(S)$ and $\mathsf{GG}(S)$ satisfy some properties that make them interesting in the context of routing in wireless networks [7, 13]. Finding ways to extract plane layers from $k$-$\mathsf{DG}(S)$ or $k$-$\mathsf{GG}(S)$ may have applications in this setting.

For all $k \geq 0$, the following relations hold: (i) $k$-$\mathsf{DG}(S) \subseteq (k+1)$-$\mathsf{DG}(S)$, (ii) $k$-$\mathsf{GG}(S) \subseteq (k+1)$-$\mathsf{GG}(S)$, (iii) $k$-$\mathsf{GG}(S) \subseteq k$-$\mathsf{DG}(S)$.

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## 2 Spanning ratio

Let $G$ be a geometric graph on $S$ and $P = \{p_1 p_2 \cdots p_l\}$ be a path in $G$. We define the *geometric length* of $P$ as $\sum_{i=1}^{l-1} |p_i p_{i+1}|$, where $|p_i p_j|$ is the Euclidean distance between $p_i$ and $p_j$. The *geometric distance* between points $p_i, p_j \in S$, denoted by $d_g(p_i, p_j)$, is the minimum over the geometric length of all paths in $G$ connecting $p_i$ and $p_j$. The *spanning ratio* of $G$ is defined as

$$SR(G) = \max_{p_i \neq p_j \in S} \frac{d_g(p_i, p_j)}{|p_i p_j|}.$$

The number of edges of $k$-Delaunay graphs grows with $k$. Consequently, it would be reasonable to believe that the spanning ratio of these graphs decreases as $k$ increases. Surprisingly, the next theorem shows that in the worst case the spanning ratio of $k$-DG is not smaller than the spanning ratio of the Delaunay triangulation.

**Theorem 1** *For any set $S$ of $n$ points in the plane, any constant value of $k$, and any $\epsilon > 0$, there exists a set of points $S'$ such that $SR(k\text{-}\mathsf{DG}(S')) \geq SR(\mathsf{DT}(S)) - \epsilon$.*

**Proof.** Consider the Delaunay triangulation of $S$. Since $S$ is in general position, the combinatorial structure of the graph does not change when moving each point in $S$ at most $\epsilon'$, for sufficiently small values of $\epsilon' > 0$. The supremum of the values of $\epsilon'$ satisfying this property is called the *tolerance* of $\mathsf{DT}(S)$ and is denoted by $\mathrm{tol}\,(\mathsf{DT}(S))$ [2].

Let us suppose that

$$SR(\mathsf{DT}(S)) = \frac{d_g(p_i, p_j)}{|p_i p_j|}.$$

Given $\epsilon > 0$, for each $p_l \in S$, define $p_{l,0} = p_l$ and place $k$ new points $p_{l,1}, p_{l,2}, \ldots, p_{l,k}$ at distance from $p_{l,0}$ less than $\min\{\mathrm{tol}\,(\mathsf{DT}(S)), \frac{|p_i p_j|\epsilon}{2n}\}$. Let $S'$ be the resulting set of points. By construction, if $p_l$ and $p_m$ are not adjacent in $\mathsf{DT}(S)$, then $p_{l,\nu}$ and $p_{m,\iota}$ are not adjacent in $k$-$\mathsf{DG}(S')$ for any $\nu, \iota \in \{0, 1, \ldots, k\}$. Therefore, in $k$-$\mathsf{DG}(S')$,

$$\frac{d_g(p_{i,0}, p_{j,0})}{|p_{i,0} p_{j,0}|} \geq SR(\mathsf{DT}(S)) - \epsilon.$$

□

For $k$-Gabriel graphs we provide the following bounds:

**Theorem 2** *For any set $S$ of $n$ points in the plane and $k \leq n-2$, the spanning ratio of $k$-$\mathsf{GG}(S)$ is $O(\sqrt{n})$. There exist sets of $n$ points in the plane whose $k$-Gabriel graphs have spanning ratio $\Theta(\sqrt{\frac{n}{k}})$.*

**Proof.** The first part follows from a result in [5] stating that the spanning ratio of the 0-Gabriel graph of any $n$-point set is at most $\frac{4\pi}{3}\sqrt{2n-4}$.

As for the second part, consider the Gabriel graph tower construction in [5] with $\lceil \frac{n}{k+1} \rceil$ points, which has spanning ratio $\Theta(\sqrt{\frac{n}{k}})$. For sufficiently small values of $\epsilon' > 0$, each point can be moved at most $\epsilon'$ without changing the combinatorial structure of the graph. Now, proceeding as in the proof of Theorem 1, we obtain a point set whose $k$-Gabriel graph has spanning ratio $\Theta(\sqrt{\frac{n}{k}})$. □

## 3 Diameter

We define the *combinatorial length* of a path $P$ on a geometric graph $G$ as the number of its edges. The *combinatorial distance* between points $p_i, p_j \in S$, denoted by $d_c(p_i, p_j)$, is the minimum over the combinatorial length of all paths in $G$ connecting $p_i$ and $p_j$. The *diameter* of $G$, denoted by $D(G)$, is defined as the maximum over the combinatorial distance of all pairs of points in $S$.

**Theorem 3** *Let $S$ be a set of $n$ points in the plane and $k \leq \lceil n/2 \rceil - 1$. Let $i$ be the integer such that $\lceil n/2^{i+1} \rceil - 1 \leq k < \lceil n/2^i \rceil - 1$. Then $D(k\text{-}\mathsf{DG}(S)) \leq 2^i$. There exist sets of $n$ points in the plane whose $k$-Delaunay graphs have diameter $\lfloor \frac{\lceil n/(k+1) \rceil}{2} \rfloor$.*

**Proof.** It suffices to prove the upper bound for values of $k$ of the form $k = \lceil n/2^{i+1} \rceil - 1$. We use induction on $i$.

For $i = 0$, we want to prove that $(\lceil \frac{n}{2} \rceil - 1)$-$\mathsf{DG}(S)$ has diameter 1, i.e., is the complete graph. Let $p_j$ and $p_l$ be two points in $S$. The line passing through $p_j$ and $p_l$ divides the plane into two open half-planes, one of which contains at most $\lceil n/2 \rceil - 1$ points in $S$. It is easy to see that there exists a circle through $p_j$ and $p_l$ that does not contain any point in $S$ lying on the opposite half-plane. This circle contains no more than $\lceil n/2 \rceil - 1$ points of $S$ in its interior, hence $p_j$ and $p_l$ are adjacent in $(\lceil \frac{n}{2} \rceil - 1)$-$\mathsf{DG}(S)$.

Now assume that the result holds for some fixed $i$. Then $(\lceil \frac{n}{2^{i+1}} \rceil - 1)$-$\mathsf{DG}(S)$ contains a path connecting $p_j$ and $p_l$ with combinatorial length at most $2^i$. Let $(p_a, p_b)$ be an edge of such path. Since $(p_a, p_b)$ belongs to $(\lceil \frac{n}{2^{i+1}} \rceil - 1)$-$\mathsf{DG}(S)$, there exists a circle $C$ through $p_a$ and $p_b$ whose interior contains no more than $\lceil n/2^{i+1} \rceil - 1$ points of $S$. If $C$ contains at most $\lceil n/2^{i+2} \rceil - 1$ points of $S$ in its interior, then $(p_a, p_b)$ is an edge of $(\lceil \frac{n}{2^{i+2}} \rceil - 1)$-$\mathsf{DG}(S)$. Otherwise, for each $p_m$ in the interior of $C$ and $\nu \in \{a, b\}$, define $C_{\nu,m}$ as the circle tangent to $C$ at point $p_\nu$ containing $p_m$ on its boundary. Either there exists a point $p_m$ in the interior of $C$ such that $C_{a,m}$ contains $\lceil n/2^{i+2} \rceil - 1$ points of $S$ in its interior, or there exist two points $p_m, p_{m'}$ in the interior of $C$ such that $C_{a,m} = C_{a,m'}$ contain $\lceil n/2^{i+2} \rceil - 2$ points of $S$ in their interior. Here we deal with the first case; the second case is analogous. Let $C_1 = C_{a,m}$, where $p_m$ is such that $C_{a,m}$ contains $\lceil n/2^{i+2}\rceil - 1$ points of $S$ in its interior. Let $C_2 = C_{b,m}$. See Figure 1. If $p_m$ is the only point of $S$ in the intersection of $C_1$ and $C_2$, then $C_2$ contains at most $\lceil n/2^{i+2}\rceil - 1$ points of $S$ in its interior and both edges $(p_a, p_m), (p_m, p_b)$ belong to $(\lceil \frac{n}{2^{i+2}}\rceil - 1)$-$\mathsf{DG}(S)$. Otherwise let $p_{m''}$ be a point in the intersection of $C_1$ and $C_2$ such that the radius of $C_{b,m''}$ is minimum. Then the only points of $S$ in the intersection of $C_1$ and $C_{b,m''}$ are $p_{m''}$ and possibly another point in the boundary of $C_{b,m''}$. Thus $(p_a, p_{m''})$ and $(p_{m''}, p_b)$ are edges in $(\lceil \frac{n}{2^{i+2}}\rceil - 1)$-$\mathsf{DG}(S)$.

In conclusion, each edge of the path in $(\lceil \frac{n}{2^{i+1}}\rceil - 1)$-$\mathsf{DG}(S)$ connecting $p_j$ and $p_l$ can be replaced by at most two edges in $(\lceil \frac{n}{2^{i+2}}\rceil - 1)$-$\mathsf{DG}(S)$. Therefore, in the last graph, there exists a path from $p_j$ to $p_l$ of combinatorial length less than or equal to $2^{i+1}$.

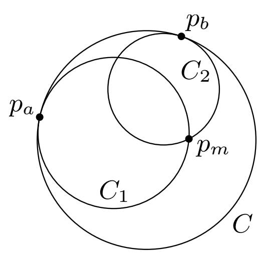


Figure 1: Points $p_a, p_b$, and $p_m$, and circles $C, C_1$, and $C_2$ in the proof of Theorem 3.

The lower bound is attained by any set of $\lceil \frac{n}{k+1}\rceil$ points such that its Delaunay triangulation is a sequential triangulation. As in Theorem 1, each point (except possibly one) can be replaced by $k+1$ points so that any two points are adjacent if and only if they belong to the same cluster or their original points were adjacent. The $k$-Delaunay graph of this point set has diameter $\lfloor \frac{\lceil n/(k+1)\rceil}{2} \rfloor$. $\square$

In general, the $k$-Gabriel graph has fewer edges than the $k$-Delaunay graph, so its diameter is usually greater:

**Theorem 4** *For any set $S$ of $n$ points in the plane and $k \le n-2$, $D(k\text{-}\mathsf{GG}(S)) \le \lceil 3n/k \rceil$. There exist sets of $n$ points in the plane whose $k$-Gabriel graphs have diameter $\lceil \frac{n-1}{k+1} \rceil$.*

**Proof.** The upper bound follows from a general result on the diameter of a graph with given minimum degree (see [14]) together with the fact that the vertices of any $k$-Gabriel graph have degree at least $k$. As for the lower bound, let $S = \{p_1, \dots, p_n\}$ be a set of $n$ points sorted by $x$ coordinate in an infinitesimally perturbed horizontal line. Then $k$-$\mathsf{GG}(S)$ contains the edge $(p_i, p_j)$ if and only if $|i-j| \le k+1$. Thus $d_c(p_1, p_n) = \lceil \frac{n-1}{k+1} \rceil$. $\square$

## 4 Chromatic number

A *j-coloring* of a graph $G = (V, E)$ is a mapping $f : V \to \{1, 2, \dots, j\}$ such that $f(v) \neq f(w)$ for every edge $(v, w)$ of $G$. The *chromatic number* of $G$, denoted by $\chi(G)$, is the minimum $j$ such that $G$ is $j$-colorable.

Since the main result in Section 6 is given in terms of the chromatic number of $k$-$\mathsf{DG}(S)$, we provide an upper bound on this parameter:

**Theorem 5** *For any set $S$ of $n$ points in the plane and $k \le \lceil n/2 \rceil - 1$, $\chi(k\text{-}\mathsf{GG}(S)) \le \chi(k\text{-}\mathsf{DG}(S)) \le 6(k+1)$.*

**Proof.** The number of edges of $k$-$\mathsf{DG}(S)$ does not exceed $3(k+1)n - 3(k+1)(k+2)$ [1]. Consequently, the graph contains a vertex of degree at most $6k+5$. Observe that, if $(p_i, p_j)$ is an edge of $k$-$\mathsf{DG}(S)$, this edge is also present in $k$-$\mathsf{DG}(S \setminus \{p_l\})$ for any $p_l \in S$ $(p_l \neq p_i, p_j)$. Thus, if $k$-$\mathsf{DG}(S) \setminus S'$ is an induced subgraph of $k$-$\mathsf{DG}(S)$ on $n'$ vertices, then it is a subgraph of $k$-$\mathsf{DG}(S \setminus S')$ and it has no more than $3(k+1)n' - 3(k+1)(k+2)$ edges. Hence we can color $k$-$\mathsf{DG}(S)$ with $6k+6$ colors applying the minimum degree greedy algorithm [8]. $\square$

Next we describe a point set for which these graphs have high chromatic number:

**Proposition 6** *For any $n \ge 3$ and $k \le \frac{n-3}{2}$, there exists a set $S$ of $n$ points in the plane whose $k$-Gabriel and $k$-Delaunay graphs have chromatic number at least $2k+3$.*

**Proof.** Let $S = \{p_1, p_2, \dots, p_{2k+3}\}$ denote the set of vertices of a slightly perturbed regular $(2k+3)$-gon. These points form a $(2k+3)$-clique in $k$-$\mathsf{GG}(S)$. Therefore the chromatic number of the graph is at least $2k+3$. If $n > 2k+3$, it suffices to add to $S$ additional points far from $p_1, \dots, p_{2k+3}$, so that the adjacencies are preserved. $\square$

## 5 Constrained geometric thickness of $1$-$\mathsf{DG}(S)$ and $1$-$\mathsf{GG}(S)$

Suppose that we want to partition the edges of a geometric graph $G$ into layers in such a way that no two edges of the same layer cross. We define the *constrained geometric thickness of* $G$, denoted by $\theta_c(G)$, as the smallest number of necessary layers. Observe that, in contrast to the notion of *geometric thickness* of a combinatorial graph, when it comes to the constrained geometric thickness the embedding of the graph is fixed. In this section we give bounds on the constrained geometric thickness of $1$-$\mathsf{DG}(S)$ and $1$-$\mathsf{GG}(S)$.

Let us first introduce some definitions and recall some properties of $1$-$\mathsf{DG}(S)$.

Edges of $\mathsf{DT}(S)$ are said to have *order 0*. The edges of *order* $k \ge 1$ are those belonging to $k$-$\mathsf{DG}(S)$, but not to $(k-1)$-$\mathsf{DG}(S)$.

Let $(p_i, p_j)$ be an edge of order 1. Then $(p_i, p_j)$ is an edge in $\mathsf{DT}(S \setminus p_l)$ for a certain $p_l \in S$. We will say that $(p_i, p_j)$ *is generated* by $p_l$. Observe that: (i) $(p_i, p_j)$ is generated by $p_l$ if and only if there exists a circle through $p_i$ and $p_j$ whose interior contains $p_l$ and no other point in $S$; (ii) every edge of order 1 is generated by at most one point on each side of the line determined by the edge; (iii) if $(p_i, p_j)$ is generated by $p_l$, then $(p_l, p_i)$ and $(p_l, p_j)$ are edges in $\mathsf{DT}(S)$. (See [1].)

**Lemma 7 [3]** *Let $(p_i, p_j)$, $(p_l, p_m)$ be two crossing edges in 1-$\mathsf{DG}(S)$. If both edges have order 1, then one of them can only be generated by the endpoints of the other. If $(p_l, p_m)$ has order 0 and $(p_i, p_j)$ has order 1, then $(p_i, p_j)$ can only be generated by $p_l$ and $p_m$.*

We now prove the main result of this section:

**Theorem 8** *For any set $S$ of $n$ points in the plane,* $2 \leq \theta_c(1\text{-}\mathsf{DG}(S)) \leq \chi(\mathsf{DT}(S)) \leq 4$.

**Proof.** The graph $\mathsf{DT}(S)$ is maximal planar, hence each edge of order 1 crosses at least one edge in $\mathsf{DT}(S)$. Since the number of edges of order 1 is strictly greater than zero [1], at least two layers are needed.

We now prove the upper bound. Let $f$ be a $\chi(\mathsf{DT}(S))$-coloring of the vertices of $\mathsf{DT}(S)$. We define a $\chi(\mathsf{DT}(S))$-coloring of the edges of 1-$\mathsf{DG}(S)$ as follows. Let $(p_i, p_j)$ be an edge of 1-$\mathsf{DG}(S)$. If $(p_i, p_j)$ has order 1 and is generated by $p_l$, we assign it the color $f(p_l)$ (if $(p_i, p_j)$ is generated by two points, we arbitrarily assign one of the two colors). If $(p_i, p_j)$ belongs to $\mathsf{DT}(S)$, we assign it an arbitrary color different from $f(p_i)$ and $f(p_j)$.

Next we prove that each color class is plane.

Suppose that $(p_i, p_j)$ and $(p_l, p_m)$ are two crossing edges of order 1. By Lemma 7, one of them can only be generated by the endpoints of the other. Let us assume that this is the case of edge $(p_i, p_j)$. Then $(p_i, p_j)$ has color $f(p_l)$ or $f(p_m)$. Since the points generating $(p_l, p_m)$ are connected to both $p_l$ and $p_m$ in $\mathsf{DT}(S)$, their color is different from $f(p_l)$ and $f(p_m)$. Consequently, $(p_l, p_m)$ is assigned a color different from $f(p_l)$ and $f(p_m)$.

Suppose that $(p_i, p_j)$ and $(p_l, p_m)$ are two crossing edges, where $(p_i, p_j)$ has order 1 and $(p_l, p_m)$ has order 0. The color of $(p_l, p_m)$ is different from $f(p_l)$ and $f(p_m)$. By Lemma 7, $(p_i, p_j)$ can only be generated by $p_l$ and $p_m$. Hence its color is $f(p_l)$ or $f(p_m)$. □

**Corollary 9** *For any set $S$ of $n$ points in the plane,* $\theta_c(1\text{-}\mathsf{GG}(S)) \leq \chi(\mathsf{DT}(S))$.

We now give a worst-case lower bound on the constrained geometric thickness of 1-$\mathsf{DG}(S)$ and 1-$\mathsf{GG}(S)$:

**Proposition 10** *For any $n \geq 6$, there exists a set $S$ of $n$ points in the plane such that $\theta_c(1\text{-}\mathsf{DG}(S)) \geq \theta_c(1\text{-}\mathsf{GG}(S)) \geq 3$.*

**Proof.** Figure 2 shows a set of 6 points whose 1-Gabriel graph contains three pairwise intersecting edges. Thus its constrained geometric thickness is at least three. For larger values of $n$ it suffices to add $n-6$ points outside the disks. □

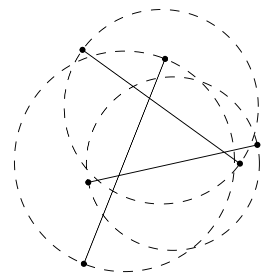

Figure 2: Example of a set of 6 points whose 1-Gabriel graph has constrained geometric thickness at least 3.

## 6 Constrained geometric thickness of $k$-$\mathsf{DG}(S)$ and $k$-$\mathsf{GG}(S)$

The arguments in the preceding section are generalized in Theorem 11. First we make some observations on the structure of $k$-$\mathsf{DG}(S)$.

Let $(p_i, p_j)$ be an edge of order $k$. Then $(p_i, p_j)$ is an edge in $\mathsf{DT}(S \setminus \{p_l^1, p_l^2, \ldots, p_l^k\})$ for some $\{p_l^1, \ldots, p_l^k\} \in S$. We will say that $(p_i, p_j)$ *is generated* by $\{p_l^1, \ldots, p_l^k\}$. It holds that: (i) $(p_i, p_j)$ is generated by $\{p_l^1, \ldots, p_l^k\}$ if and only if there exists a circle through $p_i$ and $p_j$ whose interior contains $p_l^1, \ldots, p_l^k$ and no other point in $S$; (ii) if $(p_i, p_j)$ is generated by $\{p_l^1, \ldots, p_l^k\}$, then $(p_l^\nu, p_i)$ and $(p_l^\nu, p_j)$ are edges in $(k-1)$-$\mathsf{DG}(S)$ for all $\nu \in \{1, \ldots, k\}$.

**Theorem 11** *For any set $S$ of $n$ points in the plane and $k \leq \lceil n/2 \rceil - 1$, $\theta_c(k\text{-}\mathsf{DG}(S)) \leq \frac{\chi^2((k-1)\text{-}\mathsf{DG}(S))}{2}$.*

**Proof.** We define a $\frac{\chi^2((k-1)\text{-}\mathsf{DG}(S))}{2}$-coloring of the edges of $k$-$\mathsf{DG}(S)$ such that within each color class no two edges cross.

Consider a $\chi((k-1)\text{-}\mathsf{DG}(S))$-vertex coloring $f$ of $(k-1)$-$\mathsf{DG}(S)$. If $(p_i, p_j)$ is an edge of $k$-$\mathsf{DG}(S)$, the color assigned to $(p_i, p_j)$ is the tuple $\{f(p_i), f(p_j)\}$.

Let us prove that no two edges of the same color cross. Suppose that $(p_i, p_j)$ and $(p_l, p_m)$ are two crossing edges in $k$-$\mathsf{DG}(S)$, where $(p_i, p_j)$ has order $s$ and $(p_l, p_m)$ has order $t$, with $0 \leq s, t \leq k$. Without loss of generality, let us assume that $s \geq 1$ and that the circle $C(p_i, p_j)$ contains $p_l$ in its interior. Then $p_l$ is connected to $p_i$ and $p_j$ in the graph $(s-1)$-$\mathsf{DG}(S) \subseteq (k-1)$-$\mathsf{DG}(S)$. Therefore $f(p_l) \neq f(p_i), f(p_j)$. □

**Corollary 12** *For any set $S$ of $n$ points in the plane and $k \leq \lceil n/2 \rceil - 1$, $\theta_c(k\text{-}\mathsf{GG}(S)) \leq \theta_c(k\text{-}\mathsf{DG}(S)) \leq 18k^2$.*

Unfortunately, in this case our worst-case upper and lower bounds do not have the same order of magnitude:

**Proposition 13** *For any $n \geq 3$ and $k \leq \frac{n-3}{2}$, there exists a set $S$ of $n$ points in the plane whose $k$-Gabriel and $k$-Delaunay graphs have thickness at least $k+1$.*

**Proof.** Consider the point set in the proof of Proposition 6, with the points labelled in clockwise order. The edges $(p_1, p_{k+2}), (p_2, p_{k+3}), \ldots, (p_{k+1}, p_{2k+2})$ belong to the $k$-Gabriel graph and are pairwise crossing. Therefore the thickness of the graph is at least $k+1$. □

## 7 Final remarks

We have studied several properties of two fundamental higher order proximity graphs.

As for open problems, a natural one is to close the gaps between the lower and upper bounds on the spanning ratio of $k$-Gabriel graphs and on the constrained geometric thickness of $k$-Gabriel and $k$-Delaunay graphs. In both cases we are inclined to think that the lower bounds are closer to the true values.

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