On a Dispersion Problem in Grid Labeling

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Abstract

Given k labelings of a finite d-dimensional grid, define the *combined distance* between two labels to be the sum of the ℓ_1-distance between the two labels in each labeling. We present asymptotically optimal constructions of k labelings of cubical d-dimensional grids which maximize the minimum combined distance.

1 Introduction

Let L_1 and L_2 be two bijections from the cells of an $n \times n$ grid to a label set S of n^2 symbols. Then each symbol in S labels two cells, one in L_1 and one in L_2. Define the *combined distance* between two symbols x and y in S as the distance between the two cells in L_1 plus the distance between the two cells in L_2 that are labeled by x and y. How to arrange the symbols of the two labelings such that the minimum combined distance between any two symbols is maximized? We refer to Figure 1 for an example.

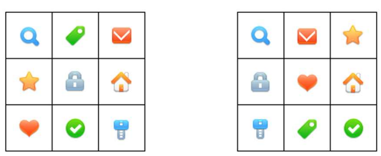

Figure 1: Two labelings of a 3×3 grid. With the first labeling fixed, the second labeling is one of 840 solutions for which the minimum combined distance is 3.

This problem was posed at the open problems session of CCCG 2009 [4] by Belén Palop, who formulated the problem from her research with Zhenghao Zhang in wireless communication. This problem has many applications to wireless communication, in particular, permutation code generation [7, Chapter 9]. A permutation code uses a grid of symbols for each channel when transmitting data over multiple channels; transmission errors are more easily detected if the combined distance between any pair of symbols in the grids is large.

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The problem is also related to Latin hypercube designs [2, 3]. A *Latin hypercube design* (LHD) is an arrangement of n points in a k-dimensional grid with n distinct coordinates in each dimension, such that no two points share a coordinate in any dimension. In other words, it is a set of n non-attacking rooks in a k-dimensional chessboard; for the sake of understanding, we will prefer the term *rook placement* rather than LHD in this article. LHDs are useful in obtaining approximation models for black-box functions that may have too many combinations of input parameters and need to be tested on only a reduced subset of the combinations.

See [5] for a survey on related topics in graph labeling.

The grid labeling problem illustrated above was defined for two labelings of a square grid, and can be naturally generalized. We now introduce some formal definitions. Throughout the article, we denote by $\langle\, n \,\rangle$ the set $\{0, 1, 2, \ldots, n-1\}$. We consider the d-dimensional grid $\langle\, n \,\rangle^d$, with n distinct coordinates in each dimension. A *labeling* of $\langle\, n \,\rangle^d$ is a bijection $L : \langle\, n \,\rangle^d \to \langle\, n^d \,\rangle$ which assigns a *label* of $\langle\, n^d \,\rangle$ to each *grid cell* of $\langle\, n \,\rangle^d$. For any two labels $x, y \in \langle\, n^d \,\rangle$, we denote by $\mathrm{dist}(L, x, y)$ the ℓ_1-distance $\left\| L^{-1}(x) - L^{-1}(y) \right\|_1$ between the grid cells of $\langle\, n \,\rangle^d$ respectively labeled by x and y in the labeling L. Given k labelings $L_1, \ldots, L_k$ of $\langle\, n \,\rangle^d$, we define the *combined distance* between the labels $x, y \in \langle\, n^d \,\rangle$ as

$$\textsc{cd}(L_1, \ldots, L_k, x, y) := \sum_{i=1}^{k} \mathrm{dist}(L_i, x, y),$$

and the *minimum combined distance* of $L_1, \ldots, L_k$ as

$$\textsc{mcd}(L_1, \ldots, L_k) := \min_{x,y \in \langle\, n^d \,\rangle} \textsc{cd}(L_1, \ldots, L_k, x, y).$$

We study the maximal value of this minimum:

$$\gamma(k, n, d) := \max_{L_1,\ldots,L_k} \textsc{mcd}(L_1, \ldots, L_k),$$

where $L_1, \ldots, L_k$ range over all combinations of k labelings of $\langle\, n \,\rangle^d$.

The number $\gamma(k, n, 1)$ has been studied in the context of Latin hypercube designs [2, 3]. The following bounds were previously known:

Theorem 1 (van Dam et al. [2, 3]) *For $k, n \geq 2$,*

$$\gamma(k, n, 1) \leq \left\lfloor \frac{k}{3}(n+1) \right\rfloor.$$

Moreover, $\gamma(2, n, 1) = \left\lfloor \sqrt{2n+2} \right\rfloor$ for any $n \geq 2$.

We obtain asymptotically tight bounds on the number $\gamma(k,n,1)$ in the following theorem:

Theorem 2 *For any integers $k \geq 2$ and $n \geq 2$,*

$$k\left\lfloor \left(\frac{n}{k}\right)^{1/k} \right\rfloor^{k-1} \leq \gamma(k,n,1) \leq \frac{n-1}{(n/k!)^{1/k}-1}.$$

Our next theorem generalizes Theorem 2:

Theorem 3 *For any integers $k \geq 2$, $n \geq 2$, and $d \geq 1$,*

$$k\left\lfloor \left(\frac{n}{k}\right)^{1/k} \right\rfloor^{k-1} \leq \gamma(k,n,d) \leq \frac{n-1}{(n^d/(dk)!)^{1/(dk)}-1}.$$

The following corollary is immediate:

Corollary 4 $\gamma(k,n,d) = \Theta(n^{1-1/k})$ *for fixed k and d.*

Let us briefly comment on the method we use to prove the lower bound of Theorem 2. Instead of providing explicit but complicated formulas for the k labelings maximizing the combined distances, we use a more geometric approach. We first provide simple and explicit formulas for the k labelings only for certain values of n, and we then use the geometric interpretation in terms of rook placements to generate good labelings for arbitrary values of n. This approach enables us to restrict the proof to friendly values of n, and thus to avoid unnecessary technical calculations for general values of n. Let us underline that even if we do not provide explicit formulas, the proof is completely constructive: it provides a simple way to construct k-tuples of labelings of $\langle n \rangle^k$ whose minimum combined distance is at least the lower bound of Theorem 2.

Observe that our lower bounds, in conjunction with the upper bounds, yield a very simple $O(kn^d)$-time constant-factor approximation algorithm for the optimization problem of maximizing the combined distance of k labelings of a d-dimensional grid, for fixed k and d.

2 Labelings with large minimum combined distance

We first construct k labelings of a 1-dimensional array of length n with large minimum combined distance for certain specific values of n: namely, we present this construction only for $n = km^k$ and $m \geq 2$. For a fixed integer m we construct k labelings $B_0, \ldots, B_{k-1}$ of the array $\langle km^k \rangle$. To construct the labeling B_i, we first assign a color $\alpha_i(x)$ to each cell x of $\langle km^k \rangle$ such that

$$\alpha_i(x) := \left\lfloor \frac{x}{m^{i-1}} \right\rfloor \bmod m.$$

Intuitively, for $1 \leq i \leq k-1$, the cell x is colored by $\alpha_i(x)$ according to its ith least significant digit in its m-ary decomposition. Observe that the color $\alpha_0(x)$ is always equal to 0. The labeling B_i is then defined for all cells $x \in \langle km^k \rangle$ by

$$B_i(x) := \left(x - km^{k-1}\alpha_i(x)\right) \bmod km^k.$$

Note that B_0 is the identity permutation.

In other words, for all $0 \leq p \leq m-1$, the labeling B_i cyclically permutes the set of all cells x with color $\alpha_i(x) = p$, and the amplitude of this permutation is proportional to p. In particular, we have $\alpha_i(x) = \alpha_i(B_i(x))$ and it is easy to describe the inverse permutation of B_i for all labels $x \in \langle km^k \rangle$ as

$$B_i^{-1}(x) = \left(x + km^{k-1}\alpha_i(x)\right) \bmod km^k.$$

Proposition 5 *The minimum combined distance of the k labelings $B_0, \ldots, B_{k-1}$ of $\langle km^k \rangle$ is at least km^{k-1}.*

Proof. Let x and y be two distinct labels of $\langle km^k \rangle$, and for $0 \leq i \leq k-1$, write

$$B_i^{-1}(x) = x + km^{k-1}\alpha_i(x) + r_i km^k$$
$$\text{and} \quad B_i^{-1}(y) = y + km^{k-1}\alpha_i(y) + s_i km^k$$

for some integers r_i and s_i. We consider two cases:

(1) If $\alpha_i(x) = \alpha_i(y)$ for all i, then $x - y$ is a nonzero multiple of m^{k-1}. Thus, for all i, the difference $B_i^{-1}(x) - B_i^{-1}(y) = x - y + (r_i - s_i)km^k$ is also a nonzero multiple of m^{k-1}, and $\mathrm{CD}(B_0, \ldots, B_{k-1}, x, y) = \sum_{i=0}^{k-1} \left|B_i^{-1}(x) - B_i^{-1}(y)\right| \geq km^{k-1}$.

(2) Otherwise, $\alpha_j(x) \neq \alpha_j(y)$ for some j. Then $\mathrm{CD}(B_0, \ldots, B_{k-1}, x, y) \geq |B_j^{-1}(x) - B_j^{-1}(y)| + |x - y| \geq |B_j^{-1}(x) - B_j^{-1}(y) - x + y| = km^{k-1}|\alpha_j(x) - \alpha_j(y) + (r_j - s_j)m| \geq km^{k-1}$. The last inequality holds since $1 \leq |\alpha_j(x) - \alpha_j(y)| \leq m-1$. $\square$

Example 6 For $k = 2$ and $m = 3$, this construction yields the two labelings of $\langle 18 \rangle$ in Figure 2, with minimum combined distance 6. The numbers on top are the ternary decompositions of the array cell indices.

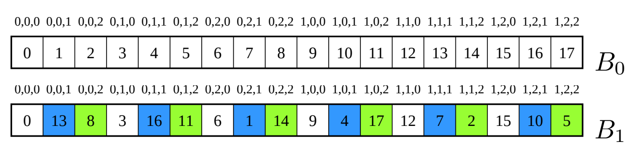

Figure 2: The labelings B_0 and B_1 for $k = 2$ and $m = 3$.

3 Rook placements

We now interpret the minimum combined distance of k labelings of a 1-dimensional array $\langle n \rangle$ as the minimum distance in a rook placement in the k-dimensional hypercube $\langle n \rangle^k$. Let us first state a precise definition:

Definition 7 *A (k,n)-rook placement is a subset R of the k-dimensional hypercube $\langle n \rangle^k$ with precisely one element in the subspace $\langle n \rangle^{p-1} \times \{q\} \times \langle n \rangle^{k-p}$ for each $1 \leq p \leq k$ and $0 \leq q \leq n-1$.*

In other words, a (k,n)-rook placement is a maximal set of non-attacking rooks in $\langle n \rangle^k$, where a rook positioned in $(x_1, \ldots, x_k)$ can attack the subspaces $\langle n \rangle^{p-1} \times \{x_p\} \times \langle n \rangle^{k-p}$ for $1 \leq p \leq k$ (see Figure 3).

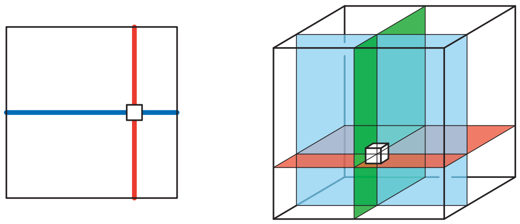

Figure 3: The affine spaces a rook can attack.

There is a correspondence between k-tuples of labelings of the 1-dimensional array $\langle\, n\, \rangle$ and (k, n)-rook placements:

- given k labelings $L_1, \ldots, L_k$ of $\langle\, n\, \rangle$, the subset $R(L_1, \ldots, L_k) := \{(L_1^{-1}(x), \ldots, L_k^{-1}(x)) \mid x \in \langle\, n\, \rangle\}$ of $\langle\, n\, \rangle^k$ is a (k, n)-rook placement;
- reciprocally, a (k, n)-rook placement R has n rooks, whose pth coordinates are all distinct (for each $1 \leq p \leq k$). If we arbitrarily label the rooks from 0 to $n - 1$, the order of the rooks according to their pth coordinate defines a labeling $L_p(R)$ of $\langle\, n\, \rangle$.

This correspondence preserves metric properties: the combined distance between two labels x and y in k labelings $L_1, \ldots, L_k$ of $\langle\, n\, \rangle$ is the ℓ_1-distance between the two rooks $(L_1^{-1}(x), \ldots, L_k^{-1}(x))$ and $(L_1^{-1}(y), \ldots, L_k^{-1}(y))$ in the (k, n)-rook placement $R(L_1, \ldots, L_k)$. We call *minimum distance* of a finite point set S the minimum pairwise ℓ_1-distance between two points of S.

To illustrate the interest of this geometric point of view, let us first prove the upper bound of Theorem 2:

Lemma 8 *For any integers $k \geq 2$ and $n \geq 2$,*

$$\gamma(k, n, 1) \leq \frac{n - 1}{(n/k!)^{1/k} - 1}.$$

Proof. We prove the result in the setting of rook placements by a simple volume argument. Consider a (k, n)-rook placement R with minimum distance δ. Then the ℓ_1-balls of radius $\delta/2$ centered at the rooks of R are disjoint and contained in $[-\delta/2, n - 1 + \delta/2]^k$. Since each ball has volume $\delta^k/k!$, this yields the inequality $n\delta^k/k! \leq (n - 1 + \delta)^k$, and thus the upper bound of the lemma. □

To prove the lower bound of Theorem 2, we will use more general configurations of integer points in $\mathbb{R}^k$ to obtain (k, n)-rook placements with large minimum distance, for all values of n. The principal ingredient of our constructions is the following proposition:

Proposition 9 *If there exists a set of n integer points in $\mathbb{Z}^k$ with minimum distance δ such that the projection of these points on each axis is an interval of consecutive integers (with possible repetitions), then there exists a (k, n)-rook placement with minimum distance δ.*

Proof. Let S be such a set of n integers. We label the points of S arbitrarily from 0 to $n - 1$. For each direction i, we then construct a labeling L_i of $\langle\, n\, \rangle$ which respects the order of the ith coordinate of the points of S, and where points with equal ith coordinate are ordered arbitrarily. Since the projection of S in each direction covered an interval of integers, the distance between two points in each direction can only increase during this construction, and the minimum distance of the (k, n)-rook placement $R(L_1, \ldots, L_k)$ is at least that of S. □

A simple way to obtain such point sets S on which we can easily control the minimum distance is to use lattices of $\mathbb{R}^k$. Remember that a *lattice* of $\mathbb{R}^k$ is the set of integer linear combinations of k linearly independent vectors of $\mathbb{R}^k$; see [6, Chapter 1]. We call a *(k, n)-rook lattice* any sublattice L of the integer lattice $\mathbb{Z}^k$ whose trace $L \cap \langle\, n\, \rangle^k$ on the hypercube $\langle\, n\, \rangle^k$ is a (k, n)-rook placement, and which contains ne_1 (e_1 is the first vector of the canonical basis of $\mathbb{R}^k$). Applying Proposition 9, a good (k, ν)-rook lattice provides good (k, n)-rook placements not only for $n = \nu$, but for any larger value of n:

Proposition 10 *If there exists a (k, ν)-rook lattice with minimum distance δ, then there exists a (k, n)-rook placement with minimum distance δ for all $n \geq \nu - 1$.*

Proof. Let L be a (k, ν)-rook lattice of minimum distance δ. For $n = \nu - 1$, consider the point configuration $L \cap \{1, \ldots, \nu - 1\}^k$: it has minimum distance δ and projects bijectively on $\{1, \ldots, \nu - 1\}$ in each direction. For $n \geq \nu$, consider the trace of L on $\langle\, n\, \rangle \times \langle\, \nu\, \rangle^{k-1}$. It projects bijectively on $\langle\, n\, \rangle$ in the first direction and surjectively on $\langle\, \nu\, \rangle$ in all the other directions. The result thus follows from Proposition 9. □

Example 11 (Rook placements in the square) We consider two families of lattices of $\mathbb{R}^2$ (see Figure 4):

(a) The lattice generated by (m, m) and $(1, 2m + 1)$ is a $(2, 2m^2)$-rook lattice with minimum distance $2m$.

(b) The lattice generated by $(m+1, m)$ and $(1, 2m+1)$ is a $(2, 2m^2 + 2m + 1)$-rook lattice with minimum distance $2m + 1$.

From these two families and using Proposition 10, we derive the following lower bound in Theorem 1:

Proposition 12 *For any n, $\gamma(2, n, 1) \geq \lfloor \sqrt{2n + 2} \rfloor$.*

Proof. Let m be an integer. Since there exists a $(2, 2m^2)$-rook lattice with minimum distance $2m$, Proposition 10 implies $\lfloor \sqrt{2n + 2} \rfloor = 2m \leq \gamma(2, n, 1)$ for any integer n with $2m^2 - 1 \leq n \leq 2m^2 + 2m - 1$. Similarly, since there exists a $(2, 2m^2 + 2m + 1)$-rook lattice with minimum distance $2m + 1$, Proposition 10 implies $\lfloor \sqrt{2n + 2} \rfloor = 2m + 1 \leq \gamma(2, n, 1)$ for any integer n with $2m^2 + 2m \leq n \leq 2m^2 + 4m$. □

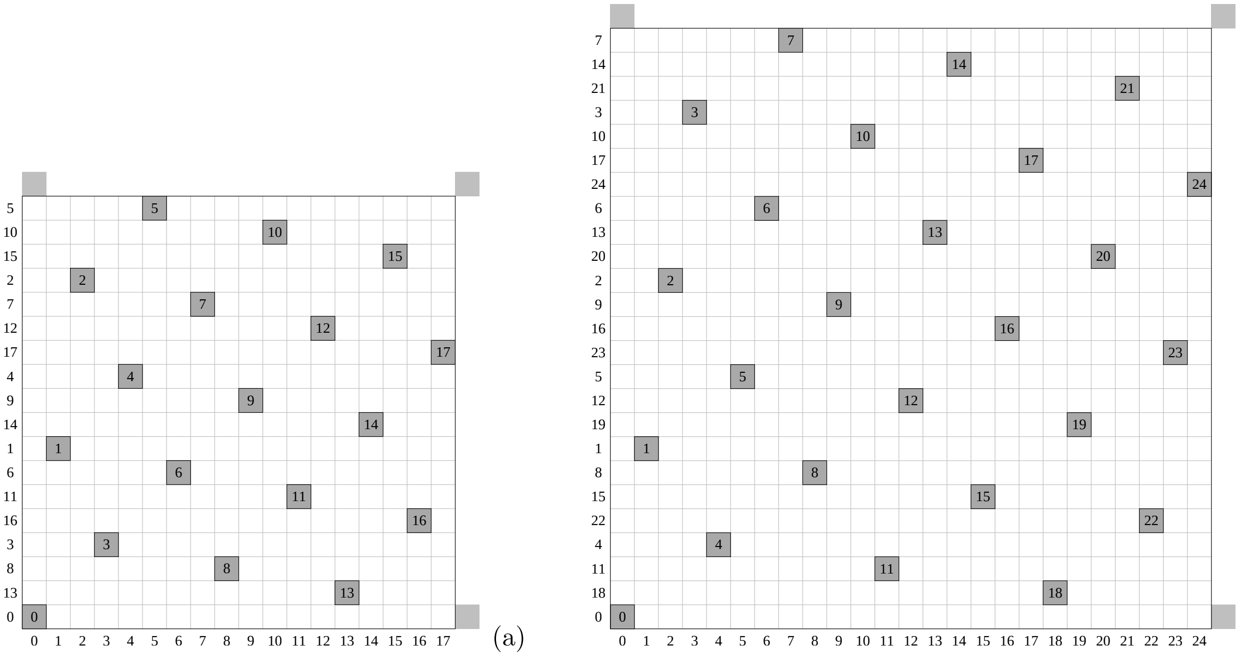

Figure 4: Examples of two optimal families of rook lattices in the square. (a) Lattice generated by the vectors (m, m) and $(1, 2m+1)$, for $m = 3$. (b) Lattice generated by the vectors $(m+1, m)$ and $(1, 2m+1)$, for $m = 3$.

We have seen in Lemma 8 that $\gamma(2, n, 1)$ is bounded by $(n-1)/(\sqrt{n/2}-1)$. Together with Proposition 12, this implies that $\gamma(2, n, 1) \sim \sqrt{2n}$. In fact, using a similar but slightly refined packing argument as in our proof of Lemma 8, van Dam et al. [2] proved that the bound in Proposition 12 is in fact the exact value of $\gamma(2, n, 1)$:

$$\gamma(2, n, 1) = \left\lfloor \sqrt{2n+2} \right\rfloor .$$

The (k, km^k)-rook placement $R(B_0, \ldots, B_{k-1})$ is not the trace of a lattice on $\langle\, km^k \,\rangle$ when $k \geq 3$. However, it is still sufficiently regular to apply Proposition 9:

Lemma 13 *For any integers $k \geq 2$ and $n \geq 2$,*

$$\gamma(k, n, 1) \geq k \left\lfloor \left(\frac{n}{k}\right)^{1/k} \right\rfloor^{k-1} .$$

Proof. Let $m := \left\lfloor \left(\frac{n}{k}\right)^{1/k} \right\rfloor$. Let S denote the set obtained by translations of the (k, km^k)-rook placement $R(B_0, \ldots, B_{k-1})$ by any integer multiple of $km^k e_1$. In other words, $S = \left\{(x, B_1^{-1}(x), \ldots, B_{k-1}^{-1}(x)) \mid x \in \mathbb{Z}\right\}$. The trace of S on $\langle\, n \,\rangle \times \langle\, km^k \,\rangle^{k-1}$ projects bijectively on $\langle\, n \,\rangle$ on the first coordinate and surjectively on $\langle\, km^k \,\rangle$ on all other coordinates. A similar analysis as in the proof of Proposition 5 ensures that the minimum distance of S, like the minimum distance of $R(B_0, \ldots, B_{k-1})$, is at least km^{k-1} too. Propositions 5 and 9 thus provide a (k, n)-rook placement whose minimum distance is at least km^{k-1}. $\square$

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In this Appendix, we present three further results on combined distances in labelings which we had to skip in our extended abstract due to space limitations:

(a) We first provide another construction of labelings of the 1-dimensional array. Although its minimum combined distance is smaller than that of the construction presented in Section 2, this new construction is particularly simple and the corresponding rook placement can be interpreted as the trace of a rook lattice in any dimension.

(b) We then extend our constructions to labelings of d-dimensional grids to obtain a proof of Theorem 3.

(c) We finally discuss an extension of the problem to labelings of arbitrary graphs (not only grid graphs). We show that the problem of deciding whether a graph has two labelings with combined distance at least 3 is already as hard as graph isomorphism.

A An alternative simple construction

As the construction in Section 2, we present this new construction only for special values of n, namely for $n = m^k$ and $m \geq 2$. Let $\phi : \langle\, m \,\rangle^k \to \langle\, m^k \,\rangle$ be the bijection defined as $\phi(x_{k-1}, \ldots, x_0) := \sum_{j=0}^{k-1} x_j m^j$. Its reciprocal bijection ϕ^{-1} is the decomposition of an integer in the m-ary number system, using k digits. Observe that we write the least significant digit to the right to be consistent with the usual conventions. Let $\sigma : \langle\, m \,\rangle^k \to \langle\, m \,\rangle^k$ be the cyclic permutation defined as

$$\sigma(x_{k-1}, \ldots, x_1, x_0) := (x_0, x_{k-1}, \ldots, x_1).$$

For $0 \leq i \leq k-1$, we define a labeling A_i of $\langle\, m^k \,\rangle$ as

$$A_i := \phi \circ \sigma^i \circ \phi^{-1}.$$

In other words, the m-ary decompositions of a label and of its position in the labeling A_i are just cyclically permuted by σ^i. Observe that the inverse permutation of A_i is

$$A_i^{-1} = \phi \circ \sigma^{k-i} \circ \phi^{-1}.$$

Proposition 14 *The minimum combined distance of the k labelings $A_0, \ldots, A_{k-1}$ of $\langle\, m^k \,\rangle$ is bounded by*

$$\textsc{mcd}(A_0, \ldots, A_{k-1}) \geq m^{k-1} - \frac{m^{k-1}-1}{m-1}.$$

Proof. Observe first that for any two elements $(x_{k-1}, \ldots, x_0)$ and $(y_{k-1}, \ldots, y_0)$ of $\langle\, m \,\rangle^k$, the distance between the cells $\phi(x_{k-1}, \ldots, x_0)$ and $\phi(y_{k-1}, \ldots, y_0)$ in the array $\langle\, m^k \,\rangle$ is at least

$$\begin{aligned} &|\phi(x_{k-1}, \ldots, x_0) - \phi(y_{k-1}, \ldots, y_0)| \\ &\qquad \geq m^{k-1}|x_{k-1} - y_{k-1}| - \sum_{j=0}^{k-2} m^j |x_j - y_j|. \end{aligned}$$

Consequently, for any two distinct elements $(x_{k-1}, \ldots, x_0)$ and $(y_{k-1}, \ldots, y_0)$ of $\langle\, m \,\rangle^k$, the combined distance $\textsc{cd}(A_0, \ldots, A_{k-1}, x, y)$ between the labels $x := \phi(x_{k-1}, \ldots, x_0)$ and $y := \phi(y_{k-1}, \ldots, y_0)$ in the k labelings $A_0, \ldots, A_{k-1}$ is at least

$$\begin{aligned} &\textsc{cd}(A_0, \ldots, A_{k-1}, x, y) \\ &\quad = \sum_{i=0}^{k-1} \left| A_i^{-1}(x) - A_i^{-1}(y) \right| \\ &\quad = \sum_{i=0}^{k-1} |\phi(x_{k-i-1}, \ldots, x_0, x_{k-1}, \ldots, x_{k-i}) \\ &\qquad\qquad - \phi(y_{k-i-1}, \ldots, y_0, y_{k-1}, \ldots, y_{k-i})| \\ &\quad \geq \sum_{i=0}^{k-1} \Big(m^{k-1}|x_{k-i-1} - y_{k-i-1}| \\ &\qquad\qquad - \sum_{j=0}^{k-2} m^j |x_{(j-i) \bmod k} - y_{(j-i) \bmod k}| \Big) \\ &\quad = \left(\sum_{i=0}^{k-1} |x_i - y_i| \right) \left(m^{k-1} - \sum_{j=0}^{k-2} m^j \right) \\ &\quad \geq m^{k-1} - \frac{m^{k-1}-1}{m-1}. \end{aligned}$$

□

Example 15 For $k = 2$ and $m = 4$, this construction yields the two labelings of $\langle\, 16 \,\rangle$ with minimum combined distance 5 shown in Figure 5. The numbers on top are the m-ary decompositions of the numbers in the array cells.

0,0	0,1	0,2	0,3	1,0	1,1	1,2	1,3	2,0	2,1	2,2	2,3	3,0	3,1	3,2	3,3	
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	A_0

0,0	1,0	2,0	3,0	0,1	1,1	2,1	3,1	0,2	1,2	2,2	3,2	0,3	1,3	2,3	3,3	
0	4	8	12	1	5	9	13	2	6	10	14	3	7	11	15	A_1

Figure 5: The two labelings A_0 and A_1 when $n = 16$, $k = 2$ and $m = 4$.

We can now revisit this new construction in terms of rook lattices. Denote by $(e_0, \ldots, e_{k-1})$ the canonical basis of $\mathbb{R}^k$. Consider the lattice $U(k, m)$ of $\mathbb{R}^k$ generated by the vectors $u_j := \sum_{i=0}^{k-1} m^{(j+i) \bmod k} e_i$, for $0 \leq j \leq k-1$. In other words, the matrix whose column vectors are $u_0, \ldots, u_{k-1}$ is a circulant matrix $M(k, n)$ whose first row is $(1, m, \ldots, m^{k-1})$. See Figure 6 for an example.

Lemma 16 *The (k, m^k)-rook placement $R(A_0, \ldots, A_{k-1})$ is formed by the points of $U(k, m)$ located in the hypercube $\langle\, m^k - 1 \,\rangle^k$ together with the point $(m^k - 1) \sum_{i=0}^{k-1} e_i$.*

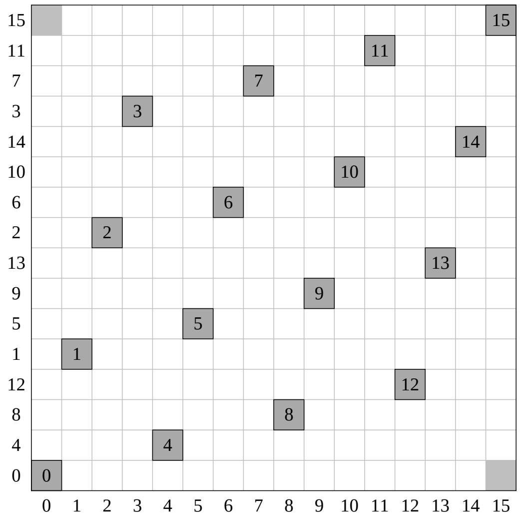

Figure 6: The lattice corresponding to the example in Figure 5 of construction A.

Proof. For any $x := \phi(x_{k-1}, \ldots, x_0) \in \langle\, m^k \,\rangle$, the rook labeled by x in $R(A_0, \ldots, A_{k-1})$ is positioned at

$$\sum_{i=0}^{k-1} A_i^{-1}(x) e_i = \sum_{i=0}^{k-1} \left(\sum_{\ell=0}^{k-1} x_{(\ell-i) \bmod k}\, m^{\ell} \right) e_i$$
$$= \sum_{i=0}^{k-1} \left(\sum_{j=0}^{k-1} x_j\, m^{(j+i) \bmod k} \right) e_i$$
$$= \sum_{j=0}^{k-1} x_j \left(\sum_{i=0}^{k-1} m^{(j+i) \bmod k} e_i \right) = \sum_{j=0}^{k-1} x_j u_j,$$

and thus is an element of the lattice $U(k,m)$. In particular, since $m^k - 1 = \sum_{j=0}^{k-1} m^j = \phi(1, \ldots, 1)$, we have $A_i^{-1}(m^k - 1) = m^k - 1$ for all $i = 0, \ldots, k-1$. So the last rook is positioned at $(m^k - 1) \sum_{i=0}^{k-1} e_i$. Thus, $R(A_0, \ldots, A_{k-1})$ is a subset of

$$\left(U(k,m) \cap \langle\, m^k - 1 \,\rangle^k \right) \cup \left\{ (m^k - 1) \sum_{i=0}^{k-1} e_i \right\}.$$

To prove the reciprocal inclusion, we show that $U(k,m) \cap \langle\, m^k - 1 \,\rangle^k$ is a set of non-attacking rooks of $\langle\, m^k - 1 \,\rangle^k$. Since $U(k,m)$ is a lattice, it is sufficient to prove that the rook at the origin is not attacked by any other rook of $U(k,m) \cap \langle\, m^k - 1 \,\rangle^k$. Assume for contradiction that the rook at the origin is attacked by another rook of $U(k,m) \cap \langle\, m^k - 1 \,\rangle^k$. Let x be the label of this attacking rook. Then we can write $x := \sum_{i=0}^{k-1} x_i e_i$ with $0 \le x_i \le m^k - 2$. By inversion of the circulant matrix $M(k,n)$ whose coefficients are the coordinates of the vectors of the base $(u_0, \ldots, u_{k-1})$ in the base $(e_0, \ldots, e_{k-1})$, we obtain that for all $0 \le i \le k-1$,

$$m u_{(i-1) \bmod k} - u_i$$
$$= m \sum_{j=0}^{k-1} m^{((i-1)+j) \bmod k} e_j - \sum_{j=0}^{k-1} m^{(i+j) \bmod k} e_j$$
$$= \sum_{j=0}^{k-1} \left(m^{((i+j-1) \bmod k)+1} - m^{(i+j) \bmod k} \right) e_j$$
$$= (m^k - 1) e_i.$$

Thus

$$x = \sum_{i=0}^{k-1} x_i e_i = \sum_{i=0}^{k-1} x_i \frac{m u_{(i-1) \bmod k} - u_i}{m^k - 1}$$
$$= \sum_{j=0}^{k-1} \frac{m x_{(j+1) \bmod k} - x_j}{m^k - 1} u_j.$$

For the rook labeled by x to attack the rook at the origin, at least one of x_i must be zero. On the other hand, for the rook labeled by x to be different from the rook at the origin, at least one of x_i must be non-zero. Assume without loss of generality that $x_{(j+1) \bmod k} = 0$ and $1 \le x_j \le m^k - 2$. Then the jth coordinate of x in the basis $u_0, \ldots, u_{k-1}$ is between $-\frac{1}{m^k-1}$ and $-\frac{m^k-2}{m^k-1}$, which is strictly contained between 0 and -1. Thus the rook labeled by x is not in $U(k,m)$, which is a contradiction. $\square$

In other words, $U(k,m)$ is a (k, m^k)-rook lattice whose minimum distance is at least $m^{k-1} - \frac{m^{k-1}-1}{m-1}$. Applying Proposition 10, we obtain that for any $n \in \mathbb{N}$,

$$\gamma(k, n, 1) \ge \left\lfloor n^{1/k} \right\rfloor^{k-1} - \frac{\left\lfloor n^{1/k} \right\rfloor^{k-1} - 1}{\left\lfloor n^{1/k} \right\rfloor - 1}.$$

B Labelings of d-dimensional Grids

In this section we prove Theorem 3. To generalize the lower bound from a one-dimensional array to a d-dimensional grid, we simply treat the d dimensions independently. The movement of a symbol in the $k-1$ labelings $L_1, \ldots, L_{k-1}$ in each direction depends only on the location of the symbol in the labeling L_0 in that particular direction, as described in the previous Sections. Thus we obtain a lower bound for the d-dimensional grid that is exactly the same as the lower bound for the one-dimensional array.

Example 17 For $k = 2$, $n = 8$, and $d = 2$, construction B yields the two labelings with minimum combined distance 4 shown in Figure 7.

In turn, the upper bound for general d is obtained by an adapted packing argument. As in the case when

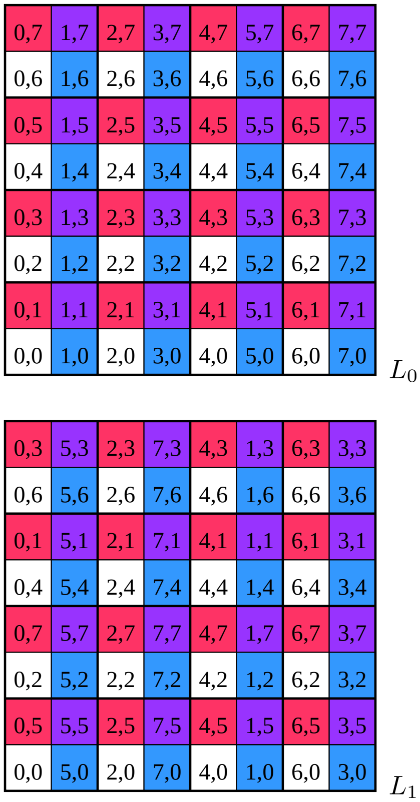

Figure 7: Two labelings L_0 and L_1 of a square grid, obtained by construction B. For convenience, in this example we label each direction independently by using $\langle\, n\,\rangle^d$ labels, instead of $\langle\, n^d\,\rangle$ labels.

$d = 1$, we can represent k labelings $L_1, \ldots, L_k$ of a d-dimensional grid $\langle\, n\,\rangle^d$ by the point configuration $R(L_1, \ldots, L_k) := \{(L_1^{-1}(x), \ldots, L_k^{-1}(x)) \mid x \in \langle\, n\,\rangle^d\}$ of $(\langle\, n\,\rangle^d)^k \simeq \langle\, n\,\rangle^{dk}$. The combined distance between two labels $x, y \in \langle\, n^d\,\rangle$ is given by the ℓ_1-distance of the corresponding rooks $(L_1^{-1}(x), \ldots, L_k^{-1}(x))$ and $(L_1^{-1}(y), \ldots, L_k^{-1}(y))$ of $R(L_1, \ldots, L_k)$. Consequently, if $L_1, \ldots, L_k$ are k labelings of $\langle\, n\,\rangle^d$ with minimum combined distance δ, then the ℓ_1-balls of radius $\delta/2$ centered at the rooks of $R(L_1, \ldots, L_k)$ are disjoint and contained in the hypercube $[-\delta/2, n-1+\delta/2]^{dk}$. Since each of these balls has volume $\delta^{dk}/(dk)!$, this yields the inequality $n^d\delta^{dk}/(dk)! \le (n-1+\delta)^{dk}$, and thus the upper bound of Theorem 3.

C Connection to Graph Isomorphism

In this section, we discuss the extension of our problem to labelings of general graphs. Let G be a graph of n vertices, and let S be a set of n symbols. Define a *labeling* of the graph G as a bijection that assigns a distinct symbol in S to each vertex in G, and define the *distance* between two vertices u and v in G as the number of edges in a shortest path between them. Then define the combined distance of multiple labelings of a graph in a similar way as that for a grid. We have the following theorem:

Theorem 18 *Deciding whether a graph has two labelings with combined distance at least* 3 *is at least as hard as graph isomorphism.*

To show this theorem, we first prove the following lemma:

Lemma 19 *A graph has two labelings with combined distance at least* 3 *if and only if the graph is a subgraph of its complement.*

Proof. We first prove the direct implication. Suppose that a graph G has two labelings L_1 and L_2 with combined distance at least 3. Then any two symbols assigned by one labeling to two adjacent vertices in G must be assigned by the other labeling to two non-adjacent vertices in G. That is, any two symbols assigned by one labeling to two adjacent vertices in G must be assigned by the other labeling to two adjacent vertices in the complement G' of G. Thus the two labelings L_1 and L_2 specify a bijection f from the vertices of G to the vertices of G' such that two vertices u and v are adjacent in G only if the corresponding two vertices $f(u)$ and $f(v)$ are adjacent in G'. Therefore G is a subgraph of its complement G'.

We next prove the reverse implication. Suppose G is a subgraph of its complement G'. Let f be a bijection from the vertices of G to the vertices of G' such that two vertices u and v are adjacent in G only if the corresponding two vertices $f(u)$ and $f(v)$ are adjacent in G'. Then in the graph G, two vertices u and v are adjacent only if the two vertices $f(u)$ and $f(v)$ are non-adjacent. Let L_1 and L_2 be two labelings of G such that the symbol assigned to a vertex v by L_1 is the same as the symbol assigned to the corresponding vertex $f(v)$ by L_2. Then the combined distance of the two labelings L_1 and L_2 is at least 3. □

The *graph isomorphism* problem is that of deciding whether two graphs are isomorphic. Remember that two graphs $G_1 := (V_1, E_1)$ and $G_2 := (V_2, E_2)$ are *isomorphic* if there is a bijection f from V_1 to V_2 such that any two vertices u and v are adjacent in G_1 if and only if the corresponding two vertices $f(u)$ and $f(v)$ are adjacent in G_2. A graph is *self-complementary* if it is isomorphic to its complement. It is known that self-complementary graph recognition is polynomial-time equivalent to graph isomorphism [1]. Observe that a graph is isomorphic to its complement if and only if

(1) the graph is a subgraph of its complement, and

(2) the graph and its complement have the same number of edges.

Condition (2) can be easily checked in linear time. It then follows from Lemma 19 that deciding whether a graph has two labelings of combined distance at least 3 is at least as hard as graph isomorphism. This completes the proof of Theorem 18.