Hamiltonian Paths and Cycles in Planar Graphs

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Abstract. We examine the problem of counting the number of Hamiltonian paths and Hamiltonian cycles in outerplanar graphs and planar graphs, respectively. We give an $O(n\alpha^n)$ upper bound and an $\Omega(\alpha^n)$ lower bound on the maximum number of Hamiltonian paths in an outerplanar graph with n vertices, where $\alpha \approx 1.46557$ is the unique real root of $\alpha^3 = \alpha^2 + 1$. For any positive integer $n \geq 6$, we define an outerplanar graph G, called a ZigZag outerplanar graph, such that the number of Hamiltonian paths starting at a single vertex in G is the maximum over all possible outerplanar graphs with n vertices. Finally, we prove a 2.2134^n upper bound on the number of Hamiltonian cycles in planar graphs, which improves the previously best known upper bound 2.3404^n.

1 Introduction

Counting of combinatorial objects is a fundamental problem in combinatorics. Given a graph G with n vertices, a straightforward approach to count the number of Hamiltonian paths in G is to use a naive backtracking algorithm that enumerates all possible paths in G. Since the problem of determining whether any Hamiltonian path exists in a given graph is NP-hard [7], determining their exact number is also NP-hard.

Much research effort has been devoted to counting as well as bounding the number of Hamiltonian paths and Hamiltonian cycles in graphs [1, 3, 4] and various classes of graphs, such as cubic graphs [8, 6], grid graphs [3] and planar graphs [2]. The currently best known upper and lower bounds on the number of Hamiltonian cycles in planar graphs are established by Buchin et al. [2], which are 2.3404^n and 2.0845^n, respectively. They also gave a 2.8927^n upper bound and a 2.4262^n lower bound on the number of simple cycles in planar graphs. Recently, de Mier and Noy [5] proved that the number of simple cycles in an outerplanar graph is $\Theta(1.502837^n)$.

Although there exists a polynomial-time algorithm to determine the number of Hamiltonian paths in the graphs with bounded treewidth [9], finding a tight

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upper bound on that number is a non-trivial task (e.g., counting Hamiltonian paths in a rectangular grid of small width [3]). On the other hand, we can find a fairly tight upper bound on the number of Hamiltonian paths in an outerplanar graph G with n vertices by simply solving a recurrence formula as follows. Let a be a vertex of G (without loss of generality assume that G is maximal). We can define the number of Hamiltonian paths of G starting at a recursively, as shown in Figures 1(a)–(c). The numbers k_1 and k_2 represent the number of vertices (that are not shown explicitly) in the corresponding shaded regions. A partial Hamiltonian path starting at a can be extended along the path shown in bold, where the vertices already visited are shown in gray, the current vertex is shown in white. The vertices still to be visited either lie in the dark gray region or are shown in black. Consequently, the number of Hamiltonian paths starting at a is

$$\begin{aligned} T(n) = \quad & \max\{T(n-k_2-2)+T(n-k_2-3)+T(n-k_1-3)+T(n-k_1-2), \\ & T(n-1)+T(n-3), T(n-k_1-2)+T(n-k_2-2)\}, \end{aligned}$$

which is dominated by $T(n-1)+T(n-3)$ and hence bounded by $O(1.46557^n)$. This also suggests that the number of Hamiltonian paths of an outerplanar graph is maximized when the graph has low maximum degree.

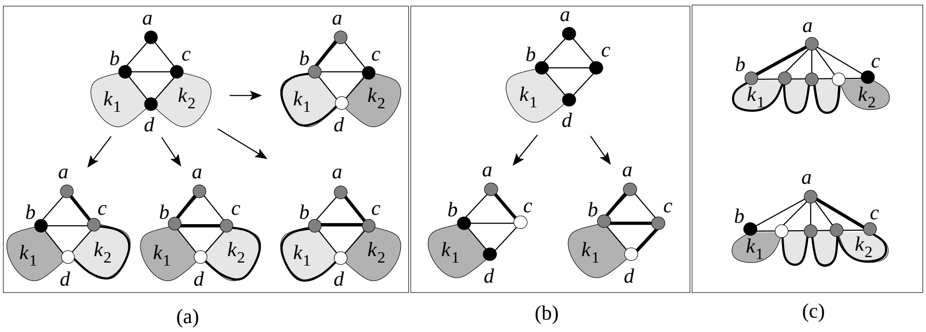

Fig. 1. Counting Hamiltonian Paths. Illustration for the cases when (a) degree(a)=2 and $k_1, k_2 > 0$, (b) degree(a)=2 and $k_2 = 0$, and (c) degree(a)> 2 and $k_1, k_2 \geq 0$.

We give a combinatorial proof for an $O(n\alpha^n)$ upper bound and an $\Omega(\alpha^n)$ lower bound on the maximum number of Hamiltonian paths in an outerplanar graph with n vertices, where $\alpha \approx 1.46557$ is the unique real root of $\alpha^3 = \alpha^2 + 1$. Our proof relies on graph transformation. We show that given a maximal outerplanar graph G with n vertices and a vertex x in G, one can insert/delete constant number of vertices and edges to obtain another combinatorially different maximal outerplanar graph G' with n vertices such that the maximum number of Hamiltonian paths starting at some vertex y in G' is at least as large as the maximum number of Hamiltonian paths in G that start at x. If we apply such a transformation repeatedly, then within $3n/2$ steps we can find an outerplanar graph G'' such that the number of Hamiltonian paths starting at a vertex in G'' is maximum over all the outerplanar graphs with n vertices. Contrary to proofs using recurrence relations, this proof helps characterize some the structural properties of outerplanar graphs. Furthermore, we prove a 2.2134^n upper bound on the number of Hamiltonian cycles in planar graphs, which improves the previously best known upper bound 2.3404^n and reduces the previous gap between the upper and lower bound for the exponential growth from 0.46 to 0.13.

2 Preliminaries

Let G be a graph with n vertices. By $V(G)$ and $E(G)$ we denote the set of vertices and the set of edges in G, respectively. By $|V(G)|$ we denote the number of vertices of G, i.e., $|V(G)| = n$. By (u, v) we denote an edge between the vertices u and v. Let G be a graph and let G' be a subgraph of G. By $G - G'$ we denote the graph obtained by deleting all the vertices of G' from G. A *separating pair* of G is a pair of vertices $\{x, y\}$ whose deletion disconnects G. If x and y are neighbors, then the pair is called a *separating edge*.

A graph is *outerplanar* if it has a planar embedding with all its vertices on the outer face. An outerplanar graph is *maximal* if the addition of any edge violates outerplanarity. Let G be a maximal outerplanar graph with $n > 3$ vertices and let $\{x, y\}$ be a separating edge of G. Then deletion of the vertices x and y from G will give two connected components G' and G''. We call the subgraphs $G - G'$ and $G - G''$ the *split graphs* with respect to $\{x, y\}$. By $\langle u_1, u_2, \ldots, u_k\rangle$ we denote a simple path of k vertices. We now have the following fact.

Fact 1. *Let G be a maximal outerplanar graph with n vertices. For any Hamiltonian path $\langle v_1, v_2, \ldots, v_n\rangle$ in G, the edge (v_1, v_2) must be an outer edge of G. Let (u, v) be an outer edge of G. Then the Hamiltonian path that starts at u and ends at v is unique and lies along the outer face of G.*

It is straightforward to design a backtracking algorithm based on Fact 1 that takes a maximal outerplanar graph G (a fixed combinatorial plane embedding of G) and a vertex x of G as input and then enumerates all the Hamiltonian paths of G that start at vertex x. Starting at x such an algorithm constructs a Hamiltonian path incrementally by visiting the unvisited vertices one after another. At each vertex the algorithm can have at most two choices to move forward to the next vertex and at each forward phase the algorithm is guaranteed to produce a new Hamiltonian path. Once a Hamiltonian path is produced, the algorithm backtracks to find a vertex that can initiate a forward move that has not been taken yet. If there is no such vertex, then the algorithm terminates. We will use this idea of enumerating Hamiltonian paths in our counting technique.

3 Hamiltonian Paths in Outerplanar Graphs

In this section we give an $O(n1.47^n)$ upper bound on the number of Hamiltonian paths in an outerplanar graph with n vertices. Since the addition of an edge in a graph does not decrease the number of Hamiltonian paths, it suffices to consider only maximal outerplanar graphs.

Let G be a maximal outerplanar graph and let v be a vertex of G. By $h(G)_v$ we denote the number of Hamiltonian paths in G starting at vertex v. If the number of Hamiltonian paths starting from v in G is the maximum over all vertices of G, then we say v is an *ace vertex* of G. By $\mathcal{N}(G)$ we denote the number of Hamiltonian paths in G starting at an ace vertex of G. In the following we give an outline of our proof technique.

Step 1: Let $\mathcal{S}_n$ be the set of all maximal outerplanar graphs of n vertices, where $n \geq 6$, whose weak dual is a path. We prove that there exists a graph $G \in \mathcal{S}_n$, such that $\mathcal{N}(G)$ is the maximum over all possible maximal outerplanar graphs of n vertices. See Theorem 1.
Step 2: We then identify such a graph G and refer to that graph as a ZigZag graph. See Theorem 2.
Step 3: Finally, we identify an ace vertex v in G. We give an $O(1.47^n)$ upper bound on $h(G)_v$. Consequently, we obtain an $O(n1.47^n)$ upper bound on the number of Hamiltonian paths in any outerplanar graph of n vertices. See Theorem 3.

Let G be a maximal outerplanar graph with n vertices. Then the *weak dual T of G* is a binary tree which has a vertex for each bounded face of G, and two vertices in T are adjacent if the corresponding faces in G share an edge. Let f be a face in G. Then the node in T that corresponds to the face f is the *dual node* of f. We can define the weak dual T as a rooted ordered binary tree as follows. If $n = 3$, then T contains a single node which is the root r. Otherwise, $n > 3$ and we take any vertex of degree one as the root r of T. Observe that r has only one child v. By convention, we set v to be the left child of r. For any node $u \neq r$ in T, let the parent of u be w and let (a, b) be the common edge of the two faces of G that correspond to the vertices u and w. Let the vertices on the triangular face corresponding to u be a, b, c in clockwise order. Let f and f' be the triangular faces (if any) other than abc that contain the edges (a, c) and (b, c), respectively. Then the dual nodes of f and f' are the left and right children of u, respectively.

We now prove the correctness of Step 1. For any maximal outerplanar graph G with $n \geq 6$ vertices we construct another maximal outerplanar graph G' with n vertices such that $\mathcal{N}(G') \geq \mathcal{N}(G)$ and the number of vertices of degree three in the weak dual of G' is less than the number of vertices of degree three in the weak dual of G.

We first examine the properties of Hamiltonian paths in G. Let the number of vertices of degree three in T be x, where $x \geq 1$. Let abc be a face of G such that no edge of abc is an outer edge. Then the dual node v of abc must be a vertex of degree three in T. See Figure 2. Let G_1 and G_2 be the two split graphs of G with respect to the separating edge $\{a, b\}$, where G_1 contains the vertex c. Let p be any vertex in G_2 other than a, b. Since $\{a, b\}$ is a separating edge in G, any Hamiltonian path starting at p must contain a subpath, which is a Hamiltonian path of $G_1, G_1 - \{a\}$ or $G_1 - \{b\}$.

For convenience, we redefine T as an ordered rooted tree, where the root corresponds to some face in G_2. We now compute $h(G_1)_a, h(G_1)_b$, and then $h(G_1 - \{b\})_a$, $h(G_1 - \{a\})_b$. We need the following lemma, whose proof is omitted due to space constraints.

Lemma 1. *Let G be a maximal outerplanar graph and let (a, b) be an outer edge of G. Then there exists a Hamiltonian path in $G - \{a\}$ starting from b that ends at a vertex of degree two in G.*

Let G_L (resp., G_R) be the subgraph of G that contains the left (resp., right) subtree of v as its weak dual. We now compute $h(G_1)_a$ and $h(G_1)_b$ considering the following cases.

(a) The Hamiltonian paths that start at a, visit the vertices in G_L along the outer face ending at c, and then visit G_R starting at c.
(b) The Hamiltonian paths that start at a, visit the vertices in G_R along the outer face starting at b and ending at c, and then visit the vertices in $G_L - \{a, c\}$.
(c) The Hamiltonian paths that start at b, visit the vertices in G_R along the outer face ending at c, and then visit the vertices in G_L starting at c.
(d) The Hamiltonian paths that start at b, visit the vertices in G_L along the outer face starting at a and ending at c, and then visit the vertices in $G_R - \{b, c\}$.

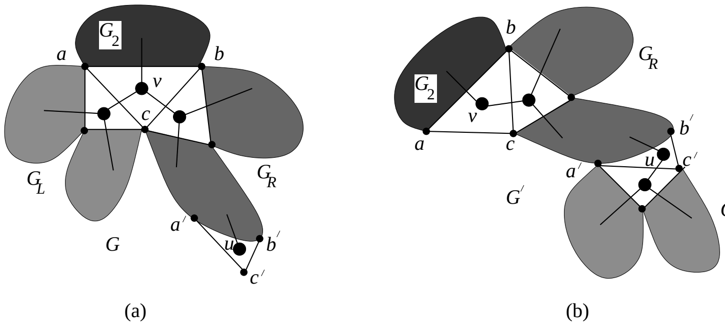

Fig. 2. Illustration for (a) G, and (b) G'.

Therefore, $h(G_1)_a = h(G_R)_c + h(G_L - \{a\})_c$ and $h(G_1)_b = h(G_L)_c + h(G_R - \{b\})_c$.

In the following we construct the graph G'. By Lemma 1, at least one Hamiltonian path in $G_R - \{b\}$ starts from c and ends at a vertex of degree two in $G_R - \{b\}$. Let that vertex be c' and let the vertex just before c' on that path be a'. Let u be the dual node of the face $a'b'c'$ of G. See Figure 2(a). Take a copy of G, remove all the vertices of G_L other than a and c from that copy. Let the resulting graph be X. Now take a copy of G_L and merge the vertices a and c of G_L with the vertices a' and c' of X, respectively, and then remove any resulting multi-edges. We denote the resulting graph by G'. See Figure 2(b). Let the weak dual of G' be T'. Observe that the construction of G' can be described by an operation on T as follows: remove the left subtree of v and add that subtree as a subtree of u. We call this operation *child swap*. Since u and v are vertices of degree two in T', the number of vertices of degree three in T' is $x - 1$.

Let G'_1 be the subgraph of G', where the weak dual of G'_1 is the subtree of T' rooted at v. Observe that the two split graphs of $G'_1 - \{a\}$ with respect to $\{a', c'\}$ consist of a copy of G_L and a copy of G_R. For simplicity, we use the same notation, i.e., G_L and G_R, to denote those split graphs. We now compute $h(G'_1)_a$ and $h(G'_1)_b$ considering the following cases.

(a) The paths that start at a and then visit the vertices in G_R starting at c. For every such path that does not end at c', we replace the subpath $\langle a', c' \rangle$ (resp., $\langle c', a' \rangle$) with the outer face of G_L starting at a' and ending at c' (resp., starting at c' and ending at a'). For every path that ends at c', we extend that path along the outer face of $G_L - \{a'\}$. Therefore, the number of such Hamiltonian paths is at least $h(G_R)_c$.

(b) The paths that start at a and then visit the vertices in G_R starting at b. For every such path we replace the subpath $\langle a', c' \rangle$ (resp., $\langle c', a' \rangle$) with the outer face of G_L starting at a' and ending at c' (resp., starting at c' and ending at a'). If we visit c after b, then by construction of G', at least one Hamiltonian path in $G_R - \{b\}$ that starts from c must end at c'. Recall that the path visits a' just before c'. Therefore, we can take the sequence a to a' of that path and extend it in $h(G_L)_{a'}$ ways. Otherwise, we start from b and visit the outer face of $G'_1 - \{a\}$ ending at c. Therefore, the number of Hamiltonian paths is at least $h(G_R - \{b\})_c - 1 + h(G_L)_{a'} + 1 = h(G_R - \{b\})_c + h(G_L)_{a'}$.

(c) The paths that start at b, visit a, then visit the vertices in $G_R - \{b\}$ starting at c. As in Case 2, i.e., (b), these paths can be extended to at least $h(G_R - \{b\})_c - 1 + h(G_L)_{a'}$ Hamiltonian paths in G'_1.

(d) The Hamiltonian path that starts at b and then visits the outer face of G'_1 ending at a. This Hamiltonian path is unique by Fact 1.

Before the child swap operation we relabel the vertices a, c of G in the following way so that after child swap $h(G_L)_{a'} \geq h(G_L)_{c'}$ holds. If $h(G_L)_a < h(G_L)_c$, we swap the labels of the vertices a and c. In the case when we do not change labels, $h(G_L)_{a'} = h(G_L)_a > h(G_L - \{a\})_c$. Otherwise, $h(G_L)_{a'} = h(G_L)_c > h(G_L)_a > h(G_L - \{a\})_c$. Therefore, $h(G'_1)_a \geq h(G_R)_c + h(G_R - \{b\})_c + h(G_L)_{a'} \geq h(G_1)_a$, and $h(G'_1)_b \geq h(G_R - \{b\})_c + h(G_L)_{a'} \geq h(G_1)_b$.

Similarly, we can compute that $h(G_1 - \{a\})_b = h(G_L - \{a\})_c$, $h(G_1 - \{b\})_a = h(G_R - \{b\})_c$, $h(G'_1 - \{a\})_b \geq h(G_R - \{b\})_c + h(G_L)_{a'}$, and $h(G'_1 - \{b\})_a \geq h(G_R - \{b\})_c$. Therefore, $h(G'_1 - \{b\})_a \geq h(G_1 - \{b\})_a$ and $h(G'_1 - \{a\})_b \geq h(G_1 - \{a\})_b$.

Recall that for any vertex $p \in V(G_2 - \{a, b\})$, any Hamiltonian path starting at p must contain a subpath, which is a Hamiltonian path of $G_1, G_1 - \{a\}$ or $G_1 - \{b\}$. We have proved that in each of these cases, the number of such subpaths in G' is greater than or equal to the number of such subpaths in G. Therefore, for any vertex $p \in V(G_2 - \{a, b\})$, $h(G')_p \geq h(G)_p$. We use the above technique to prove the following theorem.

Theorem 1. *For any positive integer $n \geq 6$, there exists an outerplanar graph G such that the weak dual of G is a path and $\mathcal{N}(G)$ is the maximum over all possible outerplanar graphs of n vertices.*

Proof (Outline). Let Y be a graph whose weak dual is not a path and $\mathcal{N}(Y)$ is the maximum over all possible outerplanar graphs of n vertices. Suppose for a contradiction that there is no graph G whose weak dual is a path and $h(G)_x \geq \mathcal{N}(Y)$, for some vertex x in G.

Let T be the weak dual of Y. Since T is not a path, there is at least one node v of degree three in T. Let abc be the face of Y that corresponds to v, where the vertices a, b, c appear on the face abc in clockwise order. Let S_{ab} be the set of outer vertices between a and b in clockwise order on the outer face of Y. Define sets S_{bc} and S_{ca} in a similar way. Let w be an ace vertex of Y.

If w belongs to S_{ab}, S_{bc} or S_{ca}, then by the child swap operation we can construct a graph Y' from Y such that $h(Y')_w \geq \mathcal{N}(Y)$ and the number of vertices of degree three in the weak dual of Y' is one less than that of T. Otherwise, $w \in \{a, b, c\}$. Also in this case, we can prove that $h(Y')_w \geq \mathcal{N}(Y)$; a detailed proof is omitted due to space constraints. We apply the process repeatedly on the resulting graph to construct a graph G whose weak dual is a path and $h(G)_x \geq \mathcal{N}(Y)$, for some vertex x in G, a contradiction. □

We now prove the correctness of Step 2. Let G be a maximal outerplanar graph with $n \geq 3$ vertices and let T be its weak dual. Let T be a path $\langle r, u_1, u_2, \ldots, u_{n-3} \rangle$ rooted at r. By definition, u_1 is the left child of r. For each i, $1 \leq i \leq n-4$, assume that u_{i+1} is the left child of u_i if i is even, and right child of u_i otherwise. We then call G a *ZigZag outerplanar graph*. See Figure 3(d). Let G' be another outerplanar graph of n vertices such that the weak dual T' of G' is a path $v_1, v_2, \ldots, v_{n-2}$ rooted at v_1. If G' is not a ZigZag outerplanar graph, then there is a subpath $\langle v_{i-1}, v_i, v_{i+1} \rangle$, $1 < i < n-2$, such that either each of v_i and v_{i+1} is the left child of their parents, or both of them are the respective right children of their parents. We call such a subpath a *repeated ancestry*. Without loss of generality, suppose that both of v_i and v_{i+1} are the respective left children of their parents and the child of v_{i+1} is a right child, if any. We construct a graph G'' from G', by applying one of the following *flip* operations such that the number of repeated ancestries in the weak dual T'' of G'' is at least one less than the number of repeated ancestries in T'. See Figures 3(a)–(c).

Child flip: Let Y be the subgraph of G' that contains the subtree rooted at v_i as its weak dual. This operation takes a copy of G' and replaces the subgraph Y with a mirror copy of Y. This construction can be described by T' as follows: for each node y in the subtree rooted at v_i, this operation flips the left-right order of the child of y. See Figure 3(b), where $v = v_i$ and $w = v_{i+1}$.

Parent flip: Let (a, e) be the common edge of the faces that correspond to the vertices v_{i-1} and v_i of T'. Let the two split graphs with respect to $\{a, e\}$ be Y and Z, where the weak dual T_1 of Y is rooted at v_i. Let z be the leaf of T_1 and let the vertices of Y on the face corresponding to z be a', b', c' in clockwise order such that c' is a vertex of degree two. If z is the left child of its parent then we connect the weak dual of Z rooted at v_{i-1} as a right subtree of z, by merging vertices e, a to the vertices b', c', respectively. Otherwise, we merge the vertices e, a to the vertices a', c', respectively. See Figure 3(c), where $u = v_{i-1}$, $v = v_i$ and z is the right child of its parent.

Lemma 2. *Let G be an outerplanar graph, where the weak dual of G is a path with at least one repeated ancestry. Then there exists an outerplanar graph G' that can be obtained from G by a single flip operation such that $\mathcal{N}(G') \geq \mathcal{N}(G)$.*

We apply flip operations on G repeatedly using Lemma 2. Since at each step the number of repeated ancestries decreases, we finally obtain a ZigZag outerplanar graph. Consequently, we have the following theorem.

Theorem 2. *Let G be a ZigZag outerplanar graph with $n \geq 6$ vertices. Then $\mathcal{N}(G)$ is the maximum among all possible outerplanar graphs with $n \geq 6$ vertices.*

We now prove the correctness of Step 3. Let G be a ZigZag graph with n vertices. See Figure 3(d). We now give a bound on $\mathcal{N}(G)$. For any $n \geq 6$, a ZigZag graph of n vertices has exactly two vertices of degree two and exactly two vertices of degree three; all the other vertices are of degree four. Let a, y and b, x be the vertices of degree two and degree three in G, respectively. Using the zigzag structure of G, it is straightforward to observe that $h(G)_a = h(G)_y$ and $h(G)_b = h(G)_x$, independent of the parity of n. Let $\{h_n\}_i, i \in \{2, 3, 4\}$, be the maximum number of Hamiltonian paths in a ZigZag graph of n vertices starting from any vertex of degree i.

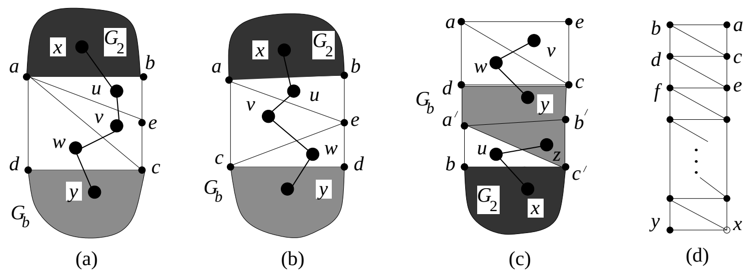

Fig. 3. (a) G'. (b) G'', which is obtained by a child flip on G'. (c) G'', which is obtained by a parent flip on G'. (d) A ZigZag graph.

We first compute $\{h_n\}_2$. Without loss of generality we compute the number of Hamiltonian paths starting at a in G. Any Hamiltonian path that starts from a, chooses either b or c as the next vertex to visit. If the next vertex is b, then the number of such Hamiltonian paths will be equal to the number of Hamiltonian paths in the ZigZag graph $G - \{a\}$ starting at a vertex of degree two, which is $\{h_{n-1}\}_2$. If the next vertex is c, then there are two ways to choose the next vertex. If we visit vertex b and then vertex d, then the number of

Hamiltonian paths will be equal to the number of Hamiltonian paths in the ZigZag graph $G - \{a, b, c\}$ starting at a vertex of degree two, which is $\{h_{n-3}\}_2$. Otherwise, we have to visit e after c and then we can complete a Hamiltonian path in only one way, i.e., by visiting the vertices along the outer face. Therefore, $\{h_n\}_2 = \{h_{n-1}\}_2 + \{h_{n-3}\}_2 + 1$. The solution to this recurrence is bounded by $O(\alpha^n)$, where $\alpha \approx 1.46557$ is the unique real root of $\alpha^3 = \alpha^2 + 1$. This recurrence establishes a lower bound of $\Omega(\alpha^n)$ on the maximum number of Hamiltonian paths in an outerplanar graph of n vertices, as follows.

Observe that $\{h_n\}_2 > \{h_{n-1}\}_2 + \{h_{n-3}\}_2$. We claim that $\{h_n\}_2 > \alpha^{n-1}$. The case when $n \in \{6, 7, 8\}$ is straightforward since $\{h_6\}_2 = 9 > \alpha^5, \{h_7\}_2 = 14 > \alpha^6$ and $\{h_8\}_2 = 21 > \alpha^7$. Assume that for all k, where $8 < k < n$, $\{h_k\}_2 > \alpha^{k-1}$. Now $\{h_n\}_2 > \{h_{n-1}\}_2 + \{h_{n-3}\}_2 > \alpha^{n-2} + \alpha^{n-4} = \alpha^{n-4}(\alpha^2 + 1) = \alpha^{n-1}$. Consequently, $\{h_n\}_2 \in \Omega(\alpha^n)$.

We compute $\{h_n\}_3$ and $\{h_n\}_4$, and prove that $\{h_n\}_2 > \{h_n\}_3$ and $\{h_n\}_2 > \{h_n\}_4$. We thus have the following theorem.

Theorem 3. *The number of Hamiltonian paths in any outerplanar graph with n vertices is $O(n\alpha^n)$. Furthermore, there exists a maximal outerplanar graph with n vertices that contains $\Omega(\alpha^n)$ Hamiltonian paths.*

4 Hamiltonian Cycles in Planar Graphs

In this section we modify the idea of the proof of Buchin et al. [2] to obtain an improved upper bound on the number of Hamiltonian cycles in planar graphs.

Since the number of simple cycles in a planar graph G is an upper bound on the number of Hamiltonian cycles in G, we first find a recurrence relation for the number of simple cycles in G using a similar argument as in [2, Lemma 1]. We then simplify that recurrence relation to obtain an upper bound on the number of Hamiltonian cycles. Unlike Buchin et al., we impose some restrictions on the cycles that we count, as shown in the following lemma. We will need the concept of *cycle-path*, which is a simple path in G that can be completed to a simple cycle in G.

Lemma 3. *Let $G = (V, E)$ be a maximal plane graph with $n \geq 3$ vertices. For each vertex $v \in V$, partition the edges incident to v into two non-empty sets s_v and s'_v, which are local to v, such that the edges in each set appear consecutively around v in clockwise order. Let $H(G)$ be the number of restricted Hamiltonian cycles in G, where a Hamiltonian cycle h is called restricted if for every vertex v, the two edges that are incident to v in h do not belong to the same set s_v or s'_v. Then $H(G) = O(n2^n)$.*

Proof. A cycle-path P of G is called restricted if for every internal vertex v of P, the two edges that are incident to v in P do not belong to the same set s_v or s'_v. Figure 4(a) illustrates a restricted cycle-path.

Every edge $e \in E$ can have two orientations, which we denote by e' and e''. We first count the number of cycle-paths starting at a fixed edge $e \in E$.

Let P' and P'' be the upper bounds on the total number of restricted cycle-paths starting from e with orientation e' and e'', respectively. Then P' and P'' must be the same by the symmetry of the edge orientations. To simplify the explanation, assume that the total number P of restricted Hamiltonian cycles starting from e is $\min\{P', P''\}$. Without loss of generality we give the starting edge the orientation e'.

We associate restricted cycle-paths with the nodes of a tree. The root of the tree contains the path of length one corresponding to the starting edge. The children of a tree node contain paths starting with the path stored in the predecessor plus an additional edge. Every restricted cycle-path is stored in only one tree node. The children of a tree node ϕ are defined as follows: If the oriented path arrives to a vertex v, and the last edge of the oriented path belongs to s_v (respectively, s'_v), then the children of that tree node consist of the edges in s'_v (respectively, s_v).

No matter which child we choose to continue the path, we will mark all the faces incident to v so that we can avoid reconsidering these faces while continuing the path. Let k_v be the number of unmarked faces that lie among the edges corresponding to the children of ϕ. Then the number of children of ϕ is $k_v + 1$. Figure 4(b) gives an example of ϕ and its children. It is now straightforward to verify that P' can be expressed by the following recurrence:

$$P'(n, f) \leq (k_v + 1) \cdot P'(n - 1, f - k_v) + 1, \tag{1}$$

where f is the number of faces in G and $P'(i, j)$ is the number of cycle-paths in G with i unvisited nodes and j unmarked faces.

Since we want to maximize the number of nodes in the recursion tree, we can assume that the k_vs for all v within a level l of the tree are equal [2, Lemma 1]. Let k_l be the number k_v for the vertices v on level l.

$P' = P'(n - 2, 2n - 6)$ will give us the number of nodes in the tree, where $P'(0, \cdot) = P'(\cdot, 0) = 1$. Observe that for each oriented cycle, we can define another cycle with the opposite orientation. Define a term k'_l analogous to the term k_l for the cycle with opposite direction. Then all k_ls and k'_ls have to be non-negative numbers. Now the k_ls and k'_ls along a cycle have to fulfill the condition $\sum_{l \leq L}(k_l + k'_l) \leq 2n-6$, where $L \leq n-1$. We now bound the number of restricted Hamiltonian cycles P as follows:

$$P \leq \min\{P', P''\} \leq \min\left\{ \sum_{L=1}^{n-1} \prod_{l \leq L} (k_l + 1), \sum_{L=1}^{n-1} \prod_{l \leq L} (k'_l + 1) \right\}. \tag{2}$$

We are interested in a set k_l which maximizes (1). Due to the convexity of $\prod_{1 \leq l \leq n-1}(k_l+1)$ (respectively, $\prod_{1 \leq l \leq n-1}(k'_l+1)$), the maximum will be attained when all k_l (respectively, all k'_l) are equal. To maximize Equation (2), we now need to maximize k_l or k'_l. Since $\sum_{1 \leq l \leq n-1}(k_l + k'_l) \leq 2n - 6$ holds and we are taking $\min\{P', P''\}$, Equation (2) is maximized when k_l and k'_l are equal[1].

[1] Since we are bounding only the number of restricted Hamiltonian cycles, we can safely ignore the effect of restricted cycle-paths that are not Hamiltonian.

Consequently, $k_l = k'_l = \frac{2n-6}{2(n-1)}$ and

$$P \leq \prod_{l=1}^{n-1}(k_l + 1) = \prod_{l=1}^{n-1}(\frac{2n-6}{2(n-1)} + 1). \quad (3)$$

We ignore the summation since P is an upper bound only on the number of restricted Hamiltonian cycles. Therefore, the exponential growth of the maximum number of restricted Hamiltonian cycles that contains the edge e is 2^n. ☐

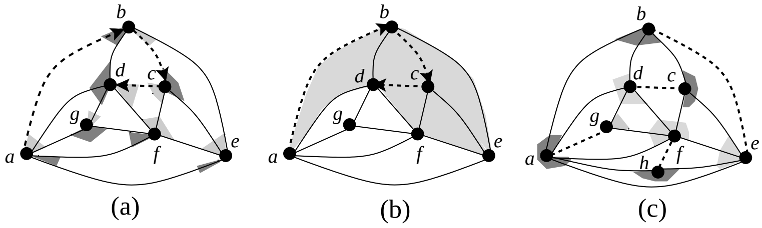

Fig. 4. (a) A restricted cycle-path in G, which is shown in dashed line. For each vertex v, the sets s_v and s'_v are shown in dark-gray and light-gray, respectively. Assume that $\phi = (a, b, c, d)$. Then (d, b), (d, a) and (d, g) are the candidates for the children of ϕ. (b) The faces that are marked are shown in light-gray. Since the faces incident to (d, b) are already marked, we only consider (a, b, c, d, a) and (a, b, c, d, g) as the children of ϕ. Therefore, $k_v + 1 = 2$. (c) Illustration for the proof of Theorem 4, where the edges in M are shown with dashed lines.

Theorem 4. *The exponential growth of the maximum number of Hamiltonian cycles in a planar graph with n vertices is 2.2134^n.*

Proof. First consider the case when n is even. Any Hamiltonian cycle in a maximal planar graph G with an even number of vertices splits into two non-intersecting perfect matchings. We now count the number of ways that a perfect matching M in G can be extended to a Hamiltonian cycle. Let e be an edge of M, where x and y are the end vertices of e. We define two sets s_e and s'_e, which are local to e, such that s_e and s'_e consists of the edges incident to x and y, respectively. We define such pair of sets for every edge in M, as shown in Figure 4(c). Observe that each edge in M plays the role of a single vertex and hence the number of ways that M can be extended to a Hamiltonian cycle is equal to the number of restricted Hamiltonian cycles in G. We count these restricted Hamiltonian cycles in a way similar to Lemma 3 by modifying the parameters n, f, k_l, k'_l as follows.

For each edge e in M we mark the faces adjacent to e. Since for any two edges $\{e_1, e_2\} \subseteq M$, the pair of faces incident to e_1 and the pair of faces incident to e_2 are different, the number of unmarked faces in G is $(2n - 4) - n = n - 4$. Since each edge in M play the role of a single vertex, $k_l = k'_l = \frac{n-4}{2.(n/2)}$. Therefore, the the number of ways that M can be extended to a Hamiltonian cycle is

$$O(n) \cdot \prod_{l=1}^{n/2-1} (k_l + 1) = O(n) \cdot \prod_{l=1}^{n/2-1} (\frac{n-4}{n} + 1) = O(n2^{n/2}). \quad (4)$$

Observe that the upper bound on the number of Hamiltonian cycles in G is the number of perfect matchings $\mathcal{M}$ in G times $O(n2^{n/2})$. Since $\mathcal{M} \leq 6^{n/4}$ [2], the exponential growth of the maximum number of Hamiltonian cycles in a planar graph with n vertices is $6^{n/4} \cdot 2^{n/2} < 2.2134^n$. The case when n is odd can be dealt in a similar way as in [2, Theorem 4]. □

5 Conclusion

In this paper we have given an 2.2134^n upper bound on the number of Hamiltonian cycles in planar graphs. We have also proved an $O(n\alpha^n)$ upper bound and an $\Omega(\alpha^n)$ lower bound on the number of Hamiltonian paths in outerplanar graphs, where $\alpha \approx 1.46557$. It would be interesting to examine whether the techniques of this paper can be extended to establish similar results for planar graphs with bounded treewidth.

We have proved the 2.2134^n upper bound on the number of Hamiltonian cycles in a planar graph under certain assumptions on the recursion tree determined by Equation (1). It would be nice to have an alternative proof that achieves the same upper bound, but does not use any such assumptions.

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