Interference Minimization in $k$-Connected Wireless Networks

Stephane Durocher†  Sahar Mehrpour‡

Abstract

Given a set of positions for wireless nodes, the $k$-connected interference minimization problem seeks to assign a transmission radius to each node such that the resulting network is $k$-connected and the maximum interference is minimized. We show there exist sets of $n$ points on the line for which any $k$-connected network has maximum interference $\Omega(\sqrt{kn})$. We present polynomial-time algorithms that assign transmission radii to any given set of $n$ nodes to produce a $k$-connected network with maximum interference $O(\sqrt{kn})$ in one dimension and $O(\min\{k\sqrt{n}, k\log \lambda\})$ in two dimensions, where $\lambda$ denotes the ratio of the longest to shortest distances between any pair of nodes.

1 Introduction

1.1 Interference Minimization and $k$-Connectivity

A network must be connected if a multi-hop communication channel is required between every pair of nodes. Various secondary objectives can be considered in addition to the connectivity requirement, often resulting in an optimization problem to construct a network that meets both criteria. Common additional objectives include minimizing the maximum or average power consumption, sender-receiver route length, node degree, ratio of route length to Euclidean distance, and, of particular relevance to wireless networks, interference [11]. By increasing or decreasing its transmission power, a wireless node increases or decreases its transmission range. If wireless signal strength is assumed to fade uniformly in all directions, then the range within which transmission exceeds a given minimum threshold corresponds to a disk centred at the point of transmission; we refer to the disk’s radius as the transmitting node’s transmission radius. Under the receiver-based interference model [16], two nodes $p_1$ and $p_2$ can communicate if they lie mutually in each other’s transmission ranges, and any node $q_1$ that lies in the transmission range of a node $q_2$ receives interference from $q_2$, regardless of whether $q_1$ can communicate with $q_2$. Given a set of node positions as input, the objective of the interference minimization problem is to assign a transmission radius to each node to produce a connected network that minimizes the maximum interference among all nodes. The interference minimization problem has been examined extensively under the receiver-based interference model over the past decade (e.g., [2, 3, 5, 7, 11, 13, 16–18]).

Maintaining network connectivity is critical to preserving multi-hop communication channels between all pairs of nodes. Connectivity alone is insufficient to preserve communication in case of node failure: a connected network can become disconnected when even a single node fails. Guaranteeing network connectivity in the presence of node failure requires multiple disjoint routes joining every pair of nodes, i.e., redundancy in the network’s connectivity. A network is $k$-connected if it remains connected whenever fewer than $k$ nodes are removed. The factor $k$ parameterizes the network’s degree of connectivity. In this work, we examine interference minimization on $k$-connected networks. Given a set of node positions, the $k$-connected interference minimization problem is to assign a transmission radius to each node to produce a $k$-connected network while minimizing the maximum interference at any node. To the authors’ knowledge, this is the first work to examine interference minimization in $k$-connected networks.

1.2 Definitions

We represent the position of a wireless node by a point $p_i \in \mathbb{R}^d$. The set $P = \{p_1, \ldots, p_n\} \subseteq \mathbb{R}^d$ represents positions for a set of $n$ nodes, along with a corresponding function, $r : P \to \mathbb{R}^+$, that associates a positive real transmission radius with each node. Communication in a wireless network is often modelled by a symmetric disk graph (SDG); the symmetric disk graph of $P$ with respect to $r$ is an undirected graph with vertex set $P$ and edge set $\{(p, q) \mid \{p, q\} \subseteq P \land r(p) \geq \text{dist}(p, q) \land r(q) \geq \text{dist}(p, q)\}$, where $\text{dist}(u, v)$ denotes the Euclidean distance between the points $u$ and $v$ in $\mathbb{R}^d$ [1]. In this paper we focus on point sets in one or two dimensions ($d \in \{1, 2\}$).

von Rickenbach et al. [16] introduced the receiver-centric interference model. In this model, the interference at the node $p \in P$, denoted $I(p)$, is the number of nodes in $P$ whose transmission range covers $p$. That is, $I(p) = \{|q \mid q \in P \land \text{dist}(p, q) \leq r(q)\}$. The maximum interference for the set of points $P$ with transmission radii given by $r$ is the maximum $I(p)$ over all
For a given graph $G$ on the point set $P$, let $I(G) = \max_{p \in P} I(p)$. The interference minimization problem is to assign transmission radii (i.e., to define the function $r$) for a given set of points $P \subseteq \mathbb{R}^d$ such that the corresponding symmetric disk graph $G$ is connected and $I(G)$ is minimized.

A graph $G$ is connected if there is a path (a sequence of adjacent vertices) joining every pair of vertices in $G$. A graph $G$ is $k$-connected if there are $k$ disjoint paths between every pair of vertices in $G$ or, equivalently, if the removal of any $j$ vertices does not disconnect $G$, for all $j < k$. The $k$-connected interference minimization problem is to assign transmission radii (i.e., define the function $r$) for a given set of points $P \subseteq \mathbb{R}^d$ such that the corresponding symmetric disk graph $G$ is $k$-connected and $I(G)$ is minimized. Let $\text{OPT}_k(P)$ denote the minimum maximum interference among all $k$-connected networks on $P$.

Given a set $P \subseteq \mathbb{R}^d$, let $\text{MST}(P)$ denote its Euclidean minimum spanning tree, $\text{DT}(P)$ its generalized Delaunay triangulation, and $\lambda = \frac{d_{\text{max}}}{d_{\text{min}}}$ the ratio of the maximum and minimum distances between any two points in $P$, i.e., $d_{\text{max}} = \max_{(p,q) \in P} \text{dist}(p,q)$ and $d_{\text{min}} = \min_{(p,q) \in P} \text{dist}(p,q)$. A set $P = \{p_1, \ldots, p_n\}$ of $n$ points in $\mathbb{R}$ ordered such that $p_i < p_j$ for all $i < j$ contains an exponential chain of size $m$ if there exist $m$ integers $1 \leq a_1 < a_2 < \cdots < a_m \leq n$ (or $n \geq a_1 > a_2 > \cdots > a_m \geq 1$) such that $\text{dist}(p_{a_i}, p_{a_{i+1}}) \geq \text{dist}(p_{a_i}, p_{a_i})$ for all $i \in \{1, \ldots, m\}$. That is, the transmission range of $p_{a_i}$ in $\text{MST}(P)$ covers $\{p_{a_{i+1}}, \ldots, p_{a_m}\}$. For example, the set $\{2^i \mid i \in \{0, \ldots, m\}\}$ forms an exponential chain of size $m$. See Figure 1. Given a set $P \subseteq \mathbb{R}$ of $n$ node positions and an assignment of transmission radii corresponding to the symmetric disk graph $G$ on $P$, von Rickenbach et al. [16] define a hub node as any vertex of $G$ that has at least one neighbour to its right; a non-hub node in $P$ has all of its neighbours to its left. For networks in $\mathbb{R}^2$, a subset $H \subseteq P$ may be identified as a set of hubs, where these hub nodes provide a connected or $k$-connected backbone to which non-hub nodes connect.

Recall the definition of an $\epsilon$-net [8]. Given a set $P$ of points in $\mathbb{R}^2$ and a family $\mathcal{R}$ of regions (ranges) in $\mathbb{R}^2$, the pair $(P, \mathcal{R})$ is a range space. For any given $\epsilon \in (0, 1)$, an $\epsilon$-net of the range space $(P, \mathcal{R})$ is a subset $S \subseteq P$ such that for any region $R \in \mathcal{R}$, if $|R \cap P| \geq \epsilon n$, then $R \cap S \neq \emptyset$. As do Halldörsson and Tokuyama [7], our algorithm uses the set $\mathcal{R}$ of ranges consisting of all equilateral triangles with one edge parallel to the $x$-axis.

### 1.3 Overview of Results

We begin with a discussion of related work in Section 2. In Section 3 we establish a lower bound of $\Omega(\sqrt{n} \log n)$ on the worst-case maximum interference among all $k$-connected networks on a given set of $n$ points in $\mathbb{R}$. This bound applies to point sets in $\mathbb{R}^d$ for any $d \geq 1$ and any $1 \leq k < n$, and improves on the lower bounds of $\Omega(k)$ due to $k$-connectivity and $\Omega(\sqrt{n})$ for maximum interference in a connected network [16]. In Section 4 we generalize a technique introduced by von Rickenbach et al. [16] and apply it to give an $O(n \log(n/k))$-time algorithm that assigns transmission radii to any set of $n$ nodes in $\mathbb{R}$ to give a $k$-connected network with maximum interference $O(\sqrt{kn})$ for any $1 \leq k < n$, asymptotically matching our lower bound; interestingly, the dependence on $k$ is $O(\sqrt{k})$, as opposed to being linear in $k$. In Section 5 we generalize techniques introduced by Halldórrsson and Tokuyama [7] and apply them to develop two algorithms that assign transmission radii to any set $P$ of $n$ nodes in $\mathbb{R}^2$ to give $k$-connected networks with maximum interference $O(k \log \lambda)$ and $O(k(\sqrt{n}))$, respectively, in $O(n \log \lambda)$ and $O(nk + n log n + k^2 \sqrt{n} \log n)$ time, respectively. We conclude with a discussion and directions for future research in Section 6.

### 2 Related Work

Buchin [3] showed that finding an optimal solution to the interference minimization problem is NP-complete in two dimensions. At present, the problem’s complexity remains open in one dimension.

Several studies examine the interference minimization problem in one dimension, also known as the highway model. von Rickenbach et al. [16] gave an $O(n \sqrt{n})$-time approximation algorithm and showed a tight asymptotic bound of $\Theta(\sqrt{n})$ on the worst-case minimum maximum interference of any set $P$ of $n$ points in $\mathbb{R}$. Their approximation algorithm applies one of two strategies, $\text{MST}(P)$ or a hub backbone, whichever has lower interference. $\text{MST}(P)$ provides low interference when $P$ is near to being uniformly distributed. If $P$ contains an exponential chain of size $m$, then $I(\text{MST}(P)) = \Omega(m)$ [16]. The hub strategy of von Rickenbach et al. [16] selects every $\sqrt{n}$th node as a hub according to their ordering on the line, forms a connected backbone network on the hubs (e.g., their MST), and connects each non-hub node to its nearest hub, giving a network with maximum interference $O(\sqrt{n})$ for any set of $n$ points in $\mathbb{R}$. Tan et al. [18] gave an algorithm that finds an optimal solution for any set $P$ of $n$ points in $\mathbb{R}$ in $O(n^{3.5+\text{OPT}_1(P)})$ time.

The interference problem has also been examined extensively in two dimensions. Halldórrsson and Tokuyama [7] used $\epsilon$-nets to define a backbone of $O(\sqrt{n})$ hub nodes, resulting in a network with maximum interference $O(\sqrt{n})$ for any set of $n$ points in $\mathbb{R}^2$. See Section 2.1 for a detailed description. Halldórrsson and Tokuyama [7] present a second algorithm using a quadtree decomposition that guarantees maximum interference $O(\log \lambda)$ for any set of points $P$ in $\mathbb{R}^2$. As
the quadtree is constructed, each non-empty square $B_i$ of width $w_i$ contains some set $P_i \subseteq P$. A representative point $p \in P_i$ is selected arbitrarily and its transmission radius is set to $\max(\sqrt{2}w_i, \text{dist}(p,q))$, where $q$ is the representative of the parent square to $B_i$. The square $B_i$ is divided into four squares of width $w_i/2$ and $P_i \setminus \{p\}$ is partitioned accordingly. The recursion terminates when $P_i = \emptyset$. Still in $\mathbb{R}^2$, Holec [9] used linear programming to give an algorithm with maximum interference $O(OPT_1(P)^2 \log n)$. Aslanayan and Rolim [2] also proposed an algorithm that finds a connected network by applying an approximation algorithm for a variant of the minimum membership set cover problem.

In addition to the worst-case results described above, the interference minimization problem has been examined in the randomized setting. Kranakis et al. [13] proved that $\text{MST}(P)$ has maximum interference $\Theta((\log n)^{1/2})$ with high probability for any set $P$ of $n$ points selected uniformly at random in $[0,1]$. Khabbazian et al. [11] showed that $\text{MST}(P)$ has maximum interference $O(n)$ with high probability for any set $P$ of $n$ points selected uniformly at random in $[0,1]^2$; Devroye and Morin [5] improved these results to show that $\text{MST}(P)$ has maximum interference $\Theta((\log n)^{1/2})$ with high probability and, furthermore, that $\text{OPT}_1(P) \in O((\log n)^{1/3})$ and $\text{OPT}_1(P) \in \Omega((\log n)^{1/4})$ with high probability, showing that for nearly all point sets $P$, MST($P$) does not minimize interference.

### 2.1 $O(\sqrt{n})$ Interference in $\mathbb{R}^2$

We include a detailed overview of the algorithm of Halldórsson and Tokuyama [7] using $\epsilon$-nets, which will be important to describe our algorithm presented in Section 5.2. Given a set $P$ of $n$ points in $\mathbb{R}^2$, the algorithm selects an $\epsilon$-net $H \subseteq P$ of size $O(\epsilon^{-1})$ to serve as a set of hubs. Hubs are connected by MST($H$), and each non-hub node (the set $P \setminus H$) connects to its nearest hub in $H$. Each node receives interference from at most $|H| \in O(\epsilon^{-1})$ hubs and $O(n\epsilon)$ non-hub nodes. Consequently, the resulting network has maximum interference $O(n + \epsilon^{-1})$, which corresponds to maximum interference $O(\sqrt{n})$ when $\epsilon = n^{-1/2}$.

Halldórsson and Tokuyama [7] describe the following algorithm to find an $\epsilon$-net $H \subseteq P$ of size $O(\epsilon^{-1})$. The algorithm begins by greedily constructing a maximal family of disjoint subsets $\{P_1, \ldots, P_l\}$ such that for each $i$, $P_i \subseteq P$, $|P_i| = cn/5$, and there exists a range $R \in \mathcal{R}$ such that $R \cap P_i = P_i$. Select any range $R_0 \in \mathcal{R}$ such that $P \subseteq R_0$, and let $V(R_0)$ denote the set of three vertices on its boundary. Let $\tilde{P} = V(R_0) \cup \bigcup_{i=1}^l P_i$. Two nodes $\{p, q\} \subseteq \tilde{P}$ form a generalized Delaunay pair with respect to $\mathcal{R}$ if there exists a range $R \in \mathcal{R}$ such that $p$ and $q$ are on the boundary of $R$ and $R \cap \tilde{P} = \{p, q\}$. Construct $\text{DT}(\tilde{P})$ by adding an edge between all generalized Delaunay pairs in $\tilde{P}$. Consider a set of colours $\{c_1, \ldots, c_{l+3}\}$. For each $i$, assign each $p \in P_i$ the colour $c_i$, and colour the points in $V(R_0)$ distinctly using the three remaining colours. A corridor refers to a maximal chain of 2-coloured triangles in $\text{DT}(\tilde{P})$. Each corridor is greedily partitioned into subcorridors such that the union of the Delaunay triangles in each subcorridor contains $cn/5$ nodes of $P$. The set of endpoints of subcorridors corresponds to the set $H$ of hubs. Since each corridor contains $O(cn)$ points of $P$, the number of subcorridors and, therefore, $|H|$ are $O(\epsilon^{-1})$.

### 3 Lower Bounds

We show the following lower bound:

**Theorem 1** For every $n$ and every $k$, $1 \leq k \leq n$, there exists a set of $n$ points $P \subseteq \mathbb{R}$ such that every $k$-connected network on $P$ has maximum interference $\Omega(\sqrt{n})$.

**Proof.** Consider the set $P = \{p \mid p = 2^i, i \in \{0, \ldots, n-1\}\}$ that forms an exponential chain of size $n$ on the line. Consider any $k$-connected network on $P$. Let $H$ denote the set of hub vertices and let $S$ denote the set of non-hub vertices, where $|H| + |S| = n$. Since the network is $k$-connected, all vertices have between $k$ and $\Delta$ neighbours, where $\Delta$ denotes the maximum vertex degree. Consequently, the first $k$ vertices on the left of the chain are hubs and, furthermore, these $k$ vertices form a clique. Every hub interferes with the leftmost node in the exponential chain. Therefore, the interference at the first node (and, therefore, the maximum interference) is at least $|H| - 1$. Similarly, the maximum interference is at least $\Delta$. That is,

$$I(G) \geq \max\{|H| - 1, \Delta\}. \quad (1)$$

Let $E_{S \rightarrow H}$ denote the set of edges that join a non-hub vertex to a hub vertex. Similarly, let $E_{H \rightarrow H}$ denote the set of edges joining pairs of hubs. This gives,

$$k|S| \leq |E_{S \rightarrow H}|. \quad (2)$$

Since the first $k$ hubs form a clique, there are $k\choose2$ edges among these. So we have,

$$k\choose2 \leq |E_{H \rightarrow H}|. \quad (3)$$

The number of edge endpoints at a hub is bounded by

$$|E_{S \rightarrow H}| + 2|E_{H \rightarrow H}| \leq |H|\Delta \Rightarrow k|S| + 2k\choose2 \leq |H| \cdot I(G) \text{ (by (1), (2) and (3))} \Rightarrow k(n - |H|) + k(k - 1) \leq |H| \cdot I(G) \Rightarrow k(n + k - 1) \leq |H|(I(G) + k)$$
\[
\leq (I(G) + 1)(I(G) + k) \quad \text{(by (1)}) \\
\Rightarrow \quad I(G) \geq \frac{\sqrt{(4n - 6)k + 5k^2 + 1} - (k + 1)}{2}. \quad (4)
\]

Next we show that \( I(G) \in \Omega(\sqrt{n}k) \) for all \( n \geq 5 \). The result holds trivially for \( n \in O(1) \) and, specifically, for \( n < 5 \). Assume

\[
n \geq 5 \\
\Rightarrow \quad 3n + k \geq 14 \quad \text{(since \( k \geq 1 \)} \\
\Rightarrow \quad 3nk + k^2 \geq 14k \\
\Rightarrow \quad 4nk + k^2 \geq 14k + 3 \quad \text{(by (5) since \( k \geq 1 \)} \\
\Rightarrow \quad 4nk - 6k + 5k^2 + 1 \geq 4k^2 + 8k + 4 \\
\Rightarrow \quad \sqrt{(4n - 6)k + 5k^2 + 1} \geq \frac{2(k + 1)}{2} \\
\Rightarrow \quad -(k + 1) \geq -\frac{\sqrt{(4n - 6)k + 5k^2 + 1}}{2} \\
\Rightarrow \quad I(G) \geq -\frac{\sqrt{(4n - 6)k + 5k^2 + 1} - 2}{4} \quad \text{(by (4))} \\
\quad \geq \frac{\sqrt{2nk + 5k^2}}{4} \quad \text{(by (5))} \\
\in \Omega(\sqrt{n}k). \quad \Box
\]

As we show in Theorem 2, the lower bound of Theorem 1 is asymptotically tight.

### 4 k-Connected Networks in One Dimension

In this section, we present an algorithm that constructs a \( k \)-connected network on any set \( P \) of \( n \) points in \( \mathbb{R} \). Our algorithm generalizes the hub technique applied in the algorithm of von Rickenbach et al. [16] to construct a connected network with maximum interference \( O(\sqrt{n}) \), as discussed in Section 2.

Instead of every \( \sqrt{n} \)th node as in [16], we select every \( \sqrt{n/(2k + 1)} \)th node as a hub, resulting in \( \lceil\sqrt{n/(2k + 1)}\rceil \) hubs. Specifically, select the \( i \)th node as a hub if \( i = \lfloor j \sqrt{n/(2k + 1)} \rfloor \) for some \( j \in \mathbb{Z} \) (where nodes are numbered \( i = 0, \ldots, n - 1 \)). Set each hub node’s transmission radius to its furthest point in \( P \) (forming a clique on the hubs). Finally, set each non-hub node’s transmission radius to the further of the \( k \)th hub to its left and the \( k \)th hub to its right.

**Theorem 2** Given any set \( P \) of \( n \) points in \( \mathbb{R} \) and any \( k < n \), transmission radii corresponding to a \( k \)-connected network on \( P \) with maximum interference \( O(\sqrt{kn}) \) can be found in \( O(n \log(n/k)) \) time.

**Proof.** First we show that the network produced is \( k \)-

connected.

\[
n > k \\
\Rightarrow \quad n > \frac{k}{2 + 1/k} \\
\Rightarrow \quad \sqrt{n(2k + 1)} > k, \\
\Rightarrow \quad \lceil\sqrt{n(2k + 1)}\rceil > k.
\]

Therefore, there are at least \( k \) hubs. Since the hubs form a clique and each non-hub node is connected to \( k \) hubs, the network is \( k \)-connected.

Next we bound the maximum interference. Choose any point \( p \in P \). The interference at \( p \), denoted \( I(p) \), is the sum of the interference it receives from hub and non-hub nodes. Hub nodes define a partition of non-hub nodes into \( \lceil\sqrt{n/(2k + 1)}\rceil \) intervals. Suppose the hub at the left end of each interval belongs to that interval. Let \( I_i \) denote the interval that contains \( p \) where intervals are numbered in order from the left. Let \( h_l \) and \( h_r \) denote the respective hubs at the left and right extremities of \( I_i \). Three types of non-hub nodes interfere with \( p \): nodes in \( I_i \), nodes in \( I_j \) for \( j < i \) that are connected to \( h_r \), and nodes in \( I_j \) for \( j > i \) that are connected to \( h_l \). Since each non-hub node connects to its \( k \) nearest hubs, \( p \) may receive interference from non-hub nodes in \( k \) intervals on each side, or \( 2k \) total intervals, corresponding to at most \( 2k\sqrt{n/(2k + 1)} \) non-hub nodes in other intervals. In addition, \( p \) may receive interference from non-hub nodes within its own interval. Finally, \( p \) receives interference from at most \( \lceil\sqrt{n(2k + 1)}\rceil \) hubs. Summing these gives

\[
I(p) \leq \left\lceil\sqrt{n(2k + 1)}\right\rceil + 2k\sqrt{n/(2k + 1)} + \left\lfloor\sqrt{n/(2k + 1)}\right\rfloor \\
< \sqrt{n(2k + 1)} + (2k + 1)\sqrt{n/(2k + 1)} + 3 \\
= 2\sqrt{n(2k + 1)} + 3 \\
in O(\sqrt{kn}).
\]

The hubs can be identified in \( O(n \log(n/k)) \) time by near-sorting \( P \), e.g., by a partial execution of deterministic quicksort to partition \( P \) into blocks of size \( \sqrt{n/(2k + 1)} \) that returns the partition pivots in sorted order. The list of hubs is traversed in \( O(\sqrt{n/k}) \) time to assign a transmission radius to each hub, corresponding to the further of the leftmost or rightmost points in \( P \). Non-hub nodes are examined in block sequence, in arbitrary order within a given block. Each non-hub’s transmission radius is set to the maximum distance of its \( k \)th hub to the left and its \( k \)th hub to the right in \( O(n) \) total time, achieved by simultaneously traversing the list of hubs and referring to the \( (i-k) \)th and \( (i+k) \)th hubs, where \( i \) denotes the block index. The total time is dominated by near-sorting, resulting in \( O(n \log(n/k)) \) time in the worst case. \( \Box \)
This guaranteed $O(\sqrt{k\lambda})$ maximum interference matches the lower bound of $O(\sqrt{k\lambda})$ established in Theorem 1, showing that our algorithm is asymptotically optimal in the worst case. Previously, we knew $I(G) \in \Omega(\sqrt{n})$ in the worst case, implied by $k = 1$ [16], and $I(G) \in \Omega(k)$, since every node in a $k$-connected graph has at least $k$ neighbours. Furthermore, $I(G) \to n - 1$ as $k \to n - 1$. The interesting implication of Theorem 2, however, is for values of $k$ between these two extrema: that the worst-case maximum interference’s dependence on $k$ is sublinear for all values of $k$.

5 k-Connected Networks in Two Dimensions

In this section we present two algorithms that generalize techniques applied in algorithms of Halldórsson and Tokuyama [7] described in Section 2. Given a set $P$ of $n$ points in $\mathbb{R}^2$, our algorithms construct respective $k$-connected networks on $P$ with maximum interference $O(k \log \lambda)$ and $O(k \sqrt{\lambda})$, for any $k$.

5.1 Quadtree Decomposition

**Theorem 3** Given any set $P$ of $n$ points in $\mathbb{R}^2$ and any $k < n$, transmission radii corresponding to a $k$-connected network on $P$ with maximum interference $O(k \log \lambda)$ can be found in $O(n \log \lambda)$ time, where $\lambda = d_{\max}/d_{\min}$ is the ratio of the maximum and minimum distances between any two points in $P$.

**Proof.** Let $B_0$ be an axis-parallel square of minimum width $w_0 \leq d_{\max}$ that contains $P$. Select any set of $k$ points $R_0 \subseteq P$ as representatives for $B_0$ and set their transmission radii to $\sqrt{2w_0}$. Divide $B_0$ into four sub-squares of width $w_0/2$ and partition $P \setminus R_0$ accordingly. This procedure is applied recursively as follows. Each non-empty square $B_i$ of width $w_i$ contains some set $P_i \subseteq P$. Select a representative set $R_i \subseteq P_i$ arbitrarily, where $|R_i| = \min\{k, |P_i|\}$. Set the transmission radius of each $p \in R_i$ to $\max_{q \in B_i} \text{dist}(p,q)$, where $B_i$ is the parent square to $B_i$ (i.e., $q$ is one of the corners of $B_j$). The square $B_i$ is divided into four squares of width $w_i/2$ and $P_i \setminus R_i$ is partitioned accordingly. The recursion terminates when $|P_i| \leq k$.

The first $k$ representatives form a $k$-clique. Each remaining node is connected to the $k$ representatives of its parent square. Consequently, any node forms a $k$-connected graph with its ancestors in the quadtree. Therefore, the entire network is $k$-connected.

The width of the root square is at most $d_{\max}$. The width of the lowest leaf square in the quadtree is at least $d_{\min}/(2\sqrt{2})$. Therefore, the height of the quadtree is at most $\lceil \log(2\sqrt{2}\lambda) \rceil = \lceil 3/2 + \log \lambda \rceil$. Each representative interferes with at most 32 cells at its level in the quadtree; see Figure 2. Therefore, each node $p \in P$ receives interference from at most 32k nodes at each level of the tree, for a total interference of at most $32k[3/2 + \log \lambda] \in O(k \log \lambda)$.

At each node of the quadtree, $k$ representatives are selected and have their transmission radii assigned, and the set $P_i$ is partitioned into four subsets in $O(|P_i|)$ time. Since the quadtree’s height is $O(\log \lambda)$, the total time is $O(n \log \lambda)$.

$\square$

5.2 $O(k \sqrt{n})$ Interference

In this section we describe an algorithm that constructs a $k$-connected network with maximum interference $O(k \sqrt{n})$ for any given set $P$ of $n$ points in $\mathbb{R}^2$. We assume a non-degeneracy condition on points, specifically, that no two points lie on the same line forming an angle of $0$, $\pi/3$, or $2\pi/3$ with the $x$-axis.

This algorithm first selects a set $H$ of $O(k \sqrt{n})$ hubs by finding an $((k \sqrt{n})^{-1})$-net of size $O(k \sqrt{n})$ on $P$ as in the algorithm of Halldórsson and Tokuyama [7] described in Section 2.1. Consequently, any range containing at least $O(\sqrt{n}/k)$ points of $P$ must contain a hub. Next, a $k$-connected backbone is built on the hubs. Finally, each non-hub node is connected to its $k$ nearest hubs.

It suffices to $k$-connect the hubs by forming a clique on the hubs. Although the hubs could be $k$-connected by applying the algorithm recursively, this does not lead to any asymptotic reduction in the maximum interference. Connecting hubs by a tree, such as the MST or the local neighbourhood graph, does not guarantee $k$-connectivity after non-hubs connect to their $k$ nearest hubs. For small $k$ (e.g., $k \leq 3$) the Delaunay triangulation provides a good strategy for $k$-connecting hubs, but a more general strategy is required for larger $k$.

We analyze the maximum interference of the resulting network. Consider an arbitrary point $p \in P$. Divide the plane around $p$ into six cones $R_1(p), \ldots, R_6(p)$ such that for each $i$, $R_i(p)$ is the cone consisting of all rays with apex $p$ and angle in $[(i-1)/3, i\pi/3)$. Without loss of generality, we consider the cone $R_1(p)$; analogous results apply to the remaining cones. Let $h_1, \ldots, h_k$ denote the $k$ hubs nearest to $p$ in $R_1(p)$ ordered by increasing distance to $p$. Let $l_\alpha(p)$ denote the line through $p$ with angle $\alpha$.

**Lemma 4** No point in $R_1(p) \cap (P \setminus H)$ lies on the right of $l_{2\pi/3}(h_k)$ and interferes with $p$.

**Proof.** For the sake of contradiction, assume such a point $q$ exists. Consequently, the transmission radius of $q$ is at least $\text{dist}(p,q)$, and so, $q$ is connected to some hub $h \in H$ where $\text{dist}(p,q) < \text{dist}(q,h)$. However, $\text{dist}(q,h_i) < \text{dist}(p,q) < \text{dist}(q,h)$ for all $i \in \{1, \ldots, k\}$, contradicting the fact that $q$ is connected to its $k$ nearest hubs.

$\square$

**Lemma 5** There are $O(k \sqrt{n})$ nodes in the area enclosed by $l_0(p)$, $l_{\pi/3}(p)$, and $l_{2\pi/3}(h_k)$.
Proof. We decompose the range enclosed by $l_0(p)$, $l_{\pi/3}(p)$, and $l_{2\pi/3}(h_k)$ into smaller regions and count the vertices in each region. The first region is the range enclosed by $l_0(p)$, $l_{\pi/3}(p)$, and $l_{2\pi/3}(h_1)$. As this range contains no hub, it contains at most $c\sqrt{n}/k$ nodes of $P$, for some fixed $c \in \mathbb{R}^+$. For each $i \in \{1, \ldots, k-1\}$, let $Q_i$ denote the isosceles trapezoidal region enclosed by $l_0(p)$, $l_{\pi/3}(p)$, $l_{2\pi/3}(h_i)$, and $l_{2\pi/3}(h_{i+1})$. We identify ranges in $R$ that contain no hub whose union covers $Q_i$. Let $H'_i$ be a list of the $i$ nearest hubs to $p$ in descending order according to their distance to $l_{\pi/3}(p)$. For each $j$, let $h'_j$ denote the first hub in the list $H'_i$. Let $A_1$ be the range enclosed by $l_0(p)$, $l_{\pi/3}(h'_1)$, and $l_{2\pi/3}(h_k)$. For $j \geq 2$, let $H'_j = H'_{j-1} \setminus \{h'_{j-1}\}$ and all hubs below $l_0(h'_{j-1})$. If $H'_j \neq \emptyset$, let $A_j$ be the range enclosed by $l_0(h'_{j-1})$, $l_{\pi/3}(h'_j)$, and $l_{2\pi/3}(h_i)$. Otherwise, $A_{j-1}$ is the final range necessary to cover $Q_i$, and we let $A_{j-1}$ be the range enclosed by $l_0(h'_{j-1})$, $l_{\pi/3}(p)$, and $l_{2\pi/3}(h_k)$. This procedure selects at most $i + 1$ ranges whose union covers $Q_i$, each of which contains no hub in its interior. See Figure 3.

Along with the first range, the region $\bigcup_{i=1}^{k-1} Q_i$ is exactly the entire region enclosed by $l_0(p)$, $l_{\pi/3}(p)$, and $l_{2\pi/3}(h_k)$. Since each $Q_i$ can be covered by $i + 1$ ranges, each of which contains no hub in its interior, the entire range can be covered by $3k/2 + k^2/2$ ranges. Since each empty range contains at most $c\sqrt{n}/k$ nodes of $P$, the region enclosed by $l_0(p)$, $l_{\pi/3}(p)$, and $l_{2\pi/3}(h_k)$ contains at most $ck\sqrt{n} \in O(k\sqrt{n})$ nodes of $P$. □

Theorem 6 Given any set $P$ of $n$ points in $\mathbb{R}^2$ and any $k < n$, transmission radii corresponding to a $k$-connected network on $P$ with maximum interference $O(k\sqrt{n})$ can be found in $O(nk + n\log n + k^3\sqrt{n}\log n)$ time.

Proof. We first argue that the resulting network is $k$-connected. The clique of hubs is $k$-connected. Each non-hub node is connected to $k$ hubs. Therefore, the entire network is $k$-connected.

Next we bound the maximum interference. By Lemmas 4 and 5, for any node $p \in P$, $O(k\sqrt{n})$ non-hub nodes interfere with $p$ in each of the six cones around $p$. There are $O(k\sqrt{n})$ hubs, each of which may interfere with $p$. Therefore, $I(p) \in O(k\sqrt{n})$.

Finally we analyze the algorithm’s running time. Since this algorithm require running part of the algorithm of Halldórsson and Tokuyama [7] described in Section 2.1, we begin by analyzing the time it takes to build the $\epsilon$-net.

Greedyly constructing the maximal family of disjoint subsets can be achieved in $O(n\log n)$ time. Similarly, the generalized Delaunay triangulation can be constructed in $O(n\log n)$ time [6] after constructing the $\Theta$-graph (e.g., see [4, 10, 15]). Finding corridors, sub-corridors, and their endpoints can done greedily in $O(n)$ time.

In our algorithm we form a clique on the set $H$ of hubs, which can be done in $O(|H|\log |H|)$ time by finding the convex hull of the hubs and setting the transmission radius of each hub to the distance to its furthest hub in $O(\log |H|)$ time per hub using binary search on the boundary of the convex hull, or $O(|H|\log |H|)$ total time. In the final step, we set the transmission radius of each non-hub node to the distance to its $k$th nearest hub. To do so we compute a $k$-nearest neighbour Voronoi diagram of the set $H$ of hubs in $O(k^2|H|\log |H|)$ time [14], upon which a point location data structure (e.g., [12]) is constructed in $O(|H|(|\log k + \log |H|)|)$ time and applied in $O(\log k + |H|)$ time per non-hub node, or $O(nk + n\log |H|)$ total time. Thus, the running time is dominated by the larger of $O(nk)$, $O(n\log n)$, and $O(k^2|H|\log |H|)$. Since $|H| \leq O(k\sqrt{n})$, this gives a total running time of $O(nk + n\log n + k^3\sqrt{n}\log n)$. □

6 Discussion and Directions for Future Research

We showed asymptotically tight upper and lower bounds of $\Theta(\sqrt{kn})$ on the worst-case maximum interference for $k$-connected networks in one dimension. The lower bound $\Omega(\sqrt{kn})$ applies in two dimensions, where we showed an upper bound of $O(k\sqrt{n})$, leaving open the question of whether a $k$-connected network with lower maximum interference can be found. In particular, is maximum interference $O(\sqrt{kn})$ always achievable in two dimensions?

von Rickenbach et al. [16] gave a polynomial-time algorithm that builds a connected network with interference at most $O(n^{3/4} \cdot \text{OPT}_1(P))$ for any set $P$ of $n$ points on the line. Their algorithm constructs a network either by applying the hub strategy or returning MST($P$), whichever has lower maximum interference. To bound the approximation factor they rely on a pair of lemmas showing that $\text{OPT}_1(P) \in O(\sqrt{n})$ and $\text{OPT}_1(P) \in \Omega(\sqrt{T(MST(P))})$. A natural direction for future research is to determine whether this approximation algorithm can be generalized to build a $k$-connected network in one dimension. Instead of connecting to the nearest neighbours to the left and right as in a one-dimensional MST, we can consider the graph MST$_k(P)$, in which each point connects to its $k$ nearest neighbours to the left and $k$ nearest neighbours to the right. In Theorem 2 we showed the generalization of the first lemma, i.e., that $\text{OPT}_k(P) \in O(\sqrt{n})$. It remains open whether the second lemma generalizes. I.e., is $\text{OPT}_k(P) \in \Omega(\sqrt{T(MST_k(P))})$ for any set $P \subseteq \mathbb{R}$?

Finally, Buchin [3] showed that the problem of finding a connected network that minimizes maximum interference for a given set of $n$ points in two dimensions is NP-complete. The complexity of the interference minimization problem in one dimension remains an important open question.
References


A Appendix: Figures

Figure 1: The first five points of $P_1$ form an exponential chain of size 5, where $a_1 = 5, a_2 = 4, \ldots, a_5 = 1$. An exponential chain need not be a perfect geometric sequence, nor need its points be consecutive. For example, $a_1 = 3, a_2 = 6, a_3 = 8, a_4 = 9, a_5 = 10$ (red points) is an exponential chain of size 5 in $P_2$. The exponential chain property holds for $i = 1$ in $P_2$ since $\text{dist}(p_{a_1}, p_{10}) = \text{dist}(p_{1}, p_{2}) = 15 \geq \max_{2 \leq j \leq 9} \text{dist}(p_{a_1}, p_{a_j}) = \text{dist}(p_1, p_{10}) = 14$; it also holds for all $i \in \{2, 3, 4\}$.

Figure 2: A point $p$ is selected as a representative for a quadtree cell, denoted by the smaller bold green square. Conse- quently, $p$’s transmission range interferes with at most 32 cells of the quadtree at its level.

Figure 3: (a) The shaded region is the trapezoid $Q_6$. (b) Four hubs that determine the ranges used to cover $Q_6$. (c) The five empty ranges whose union covers $Q_6$. 