Practical Discrete Unit Disk Cover Using an Exact Line-Separable Algorithm

Francisco Claude, Reza Dorrigiv, Stephane Durocher, Robert Fraser, Alejandro López-Ortiz, Alejandro Salinger
Outline

- Past work
- Line-separable problem
- Combining our algorithm with the previous work
- Conclusions & Future work
Discrete Unit Disk Cover (DUDC)

- Given $m$ unit disks $D$ and $n$ points $Q$ in the plane, the discrete unit disk cover problem is to select a minimum subset of the disks to cover the points.
Applications

- Wireless networks
  - Select minimum number of servers from a set of candidate sites.

- Natural Resources
  - Position water bombers at airports so that active fires are within a maximum distance.

- Climatology
  - Position weather stations at candidate sites so that all cities are covered by radar.
About DUDC

- NP-Hard (Johnson, 1982)
- Approximation algorithms:
  - 108-approximate (Călinescu et al., 2004)
  - 72-approximate (Narayanappa & Voytechovsky, 2006)
  - 38-approximate (Carmi et al., 2007)
  - $(1+\varepsilon)$-approximate (Mustafa & Ray, 2009)
    - Uses $\varepsilon$-net based local improvement approach.
    - $O(m^{257}n)$ time in the worst case (2-approximation).
- This paper:
  - 22-approximate, $O(m^2n^4)$ algorithm.
Line-separated DUDC

- Disk centres and points in $Q$ are now separable by a line, which we will call horizontal.
- Formally, given sets of points $P=\{p_1,\ldots,p_m\}$ and $Q=\{q_1,\ldots,q_n\}$, where $D=\{d_1,\ldots,d_m\}$ is the set of unit disks centred at the points in $P$, find $D' \subseteq D$ of minimum cardinality such that all points in $Q$ are covered by unit disks.
Simplification Rules

Observations:

1. If a disk $d_1$ covers no points from $Q$, we can remove it.
2. If a disk $d_1$ is dominated by a disk $d_2$, then we can remove $d_1$ from the problem instance.
3. If a point $q_1$ is only covered by a disk $d_1$, then $d_1$ must be part of the solution.
Greedy Step

- Simplification rules are not always sufficient.

In this example, \( q_i \in d_j, \forall \ i \neq j \).

If no more simplification rules can be applied, then the leftmost disk is added to the solution set (disks are ordered by leftmost intersection with \( l \)).
The Greedy Algorithm

Algorithm 1 GREEDY (D, Q)

\[
\begin{align*}
D & \leftarrow \text{sortLeftToRight}(D) \quad // \text{sort in increasing order of left intersection with } l \\
D' & \leftarrow \emptyset \\
\text{while } D \neq \emptyset \text{ do do} \\
\quad & \text{SIMPLIFICATION } (D, Q, D') \quad // \text{SIMPLIFICATION possibly modifies } D, Q \text{ and } D' \\
\quad d_\ell & \leftarrow \text{leftmost disk in } D \\
\quad D' & \leftarrow D' \cup \{d_\ell\} \\
\quad D & \leftarrow D \setminus d_\ell \\
\quad Q' & \leftarrow \{q \in Q \mid q \text{ is contained in } d_\ell\} \\
\quad Q & \leftarrow Q \setminus Q' \\
\text{end while} \\
\text{return } D'
\end{align*}
\]
Correctness of GREEDY (1 of 4)

- It needs to be proven that $D'$, the Greedy algorithm solution, is the minimum sized set of disks needed to cover the set of points in $Q$.
- Assume the opposite is true, and that some algorithm OPT returns a solution $D_{OPT}$, such that $|D_{OPT}| < |D'|$.
- Call $d_1$ the first disk selected by GREEDY but not by OPT.
- Define $C \subseteq Q$ as the set of points covered by $d_1$. 
Correctness of GREEDY (2 of 4)

- Suppose disk $d_0$ in $D_{OPT}$ covers exactly $C$ as well.
  - $d_1$ is not dominated by any other disk.
  - Replace $d_0$ with $d_1$, and the difference between GREEDY and OPT moves to the right.
- Otherwise, $C$ is covered by multiple disks in the OPT solution.
  - Choose two disks $d_2$ & $d_3$ from $D_{OPT}$ such that each covers some strict subset of $C$ and $d_1 \cap d_2 \neq d_1 \cap d_3$.
  - We wish to prove that $d_1 \cup d_3$ covers all points covered by $d_2 \cup d_3$ (assume w.l.o.g. that $d_2$ precedes $d_3$ on $l$, $d_2 < d_3$).
Correctness of GREEDY (3 of 4)

- Let $l_i$ ($r_i$) be the left (right) intersection of $d_i$ with $l$.
- If $d_2 < d_1$, then $d_1$ dominates $d_2$ (otherwise $d_2$ would be selected by GREEDY).
  - Again, replace $d_2$ with $d_1$ in OPT, pushing the difference to the right.
- Otherwise, $d_1 < d_2 < d_3$.
- $(d_1 \cap d_3) \setminus d_2 \neq \emptyset$.
- Let $R = d_2 \setminus d_1$, show $R \subseteq d_3$.
- Define $x$ as shown.
Correctness of GREEDY (4 of 4)

- $r_1$ and $r_2$ both lie between $l_3$ and $r_3$ on line $l$, thus both $r_1$ and $r_2$ are contained in $d_3$.
- $x$ lies on the boundary of region $(d_1 \cap d_3) \setminus d_2$, so $x \in d_3$.
- The boundary of $R$ consists of arcs of unit disks joining $x$, $r_1$, $r_2$, thus $R$ is contained in the 1-hull of $\{x, r_1, r_2\}$.
- Since $\{x, r_1, r_2\} \subseteq d_3$, it follows that $R \subseteq 1$-hull$\{x, r_1, r_2\} \subseteq d_3$. 
Complexity of GREEDY

- A naïve implementation of the algorithm requires $O(m^3n)$ time:
  - $O(m^2n)$ time to find all dominance relations for the simplification steps;
  - $O(m)$ iterations of the simplification-greedy loop.
- This is improved to $O(m^2n)$ overall worst case time by encoding all the dominance relations into a graph (our work), or with dynamic programming (Ambühl et al., 2006).
- This has been further refined to $O(n(\log n + m))$ time in the journal version of this work, and is the product of joint work with Das and Nickerson.
Minimum Assisted Cover

- This variant of the line-separated DUDC problem has all points $Q$ on one side of the line $l$, and disks $D$ centred both above and below $l$ ($D = U \cup L$).
- Want $G \subseteq D$ of minimum cardinality.
- Perform GREEDY on $U$ to get $D'$. Use Carmi et al.'s minimum assisted cover algorithm on $D'$ & $L$ to get $E$. 
How good is $E$?

- $|E| = |E_U| + |E_L| \leq |\text{ac}(D', G_L)| + |G_L|$
  - $\text{ac}(D', G_L)$ is the smallest subset of $D'$ such that $\text{ac}(D', G_L)$ is a cover when assisted by $G_L$.
  - $E$ is the minimum size assisted cover based on $D'$.
- To show: $2|G_U| \geq |\text{ac}(D', G_L)|$
  - Given a disk $d \in G_U$, there are 2 cases:
    1. $d$ is above the lower boundary of $\text{ac}(D', G_L)$.
    2. Otherwise, $d$ contains one or more arc segments of the lower boundary of $\text{ac}(D', G_L)$.
Approximation Ratio

1. d is above the lower boundary of ac(D’,G_L).
   - In this case, two disks from ac(D’,G_L) can be used to cover d.

2. Otherwise, d contains one or more arc segments of the lower boundary of ac(D’,G_L).
   - Denote V the arc segments covered by d.
   - Let W={v_l,d,v_r}.
   - Since W dominates V, there at most one fully covered arc segment in V.
   - v_l and v_r must contain points not covered by d, so assign each to the left-most covering disk in G.
   - At most two disks from V are assigned to d.

Therefore, each disk in G_U has at most two associated disks in ac(D’,G_L):
\[ 2|G_U| \geq |ac(D’,G_L)|. \]
This allows us to prove an approximation ratio of 2:
\[ 2|G| = 2(|G_U| + |G_L|) \geq 2|G_U| + |G_L| \geq |ac(D’,G_L)| + |G_L| \geq |E_U| + |E_L| = |E|. \]
Carmi et al. use the assisted line separated DUDC algorithm as a subroutine in their work.

Our algorithm provides a 2-approximation (the original paper had a 4-approximation).

There are eight applications of this algorithm, and one other technique used has a 6-approximation, so the total approximation factor is \(8 \cdot 2 + 1 \cdot 6 = 22\).

The running time of our technique is \(O(m^2n)\) (computing the assisting cover adds nothing), and it is used a constant number of times.

Carmi et al. search for the DUDC of all \(3/2 \times 3/2\) squares, which in the worst case can take \(O(m^2n^4)\) time.
Conclusions

- An exact line-separable discrete unit disk cover algorithm is used to provide a $22$-approximate solution to the discrete unit disk cover problem in $O(m^2n^4)$ worst case time.

Future work:
- Can the approximation factor be improved further along these lines?
- Can the running time be improved?
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