Algorithms for Geometric Covering and Piercing Problems

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Vignettes

- **Discrete Unit Disk Cover Problem**
  - Various settings
  - Strip-based decomposition of the plane
  - Hardness of problem even in a strip

- **Hausdorff Core Problem**
  - General simple polygons
  - Polygons with a single reflex vertex

- **MST with Neighborhoods Problem**
  - Parameterized algorithms
  - Hardness of problem
The Discrete Unit Disk Cover (DUDC) Problem

- \( m \) unit disks \( D \) with centrepoints \( Q \), \( n \) points \( P \).
- Select a minimum subset of \( D \) which covers \( P \).
Applications
Line-separated DUDC

- $Q$ and $P$ are separable by $\ell$.
- Exact $O(mn + n \log n)$ time algorithm.
Strip-Separated DUDC

- All points $P$ contained in strip between lines $\ell_1, \ell_2$, and disks $D$ centred outside strip.
- Exact $O(m^2n + n \log n)$ time algorithm.

[Ambühl et al., 2006]
Assisted LSDUDC

- \( P \) on one side of \( \ell \), \( D = U \cup L \) above and below \( \ell \).
- All points in \( P \) covered by \( U \).
- Want \( D' \subseteq D \) of minimum cardinality.
- Best known is 2-approximation.
- Hardness open.
Strip-Based Decomposition

- DUDC algorithm: work from top to bottom, then bottom to top, then cover what remains.
- Uses assisted LSDU DC and WSDU DC algorithms.
Within-Strip DUDC

- $m$ unit disks $D$ with centre points $Q$, $n$ points $P$.
- Strip $s$, defined by $\ell_1$ and $\ell_2$, of height $h$ which contains $Q$ and $P$.
- Select a minimum subset of $D$ which covers $P$. 

\[ \ell_1 \quad \ell_2 \quad s \quad h \]
Within-Strip DUDC

- If easy, implies a simple PTAS for DUDC based on shifting grids…
- MAX independent set of unit disk graph is easy in strips of fixed height!
- Within-Strip Discrete Unit Square Cover is easy in strips of fixed height!
Hardness of WSDUDC (1/2)

- Reduce from Vertex Cover on planar graphs of max degree 3.
Hardness of WSDUDC (2/2)

- Each edge has even number of added points:

- Convert to instance of WSDUDP:
3-Approximate WSDUDC Algorithm

- Find the rectangle of height $2h$ circumscribed by each disk, then take the intersection with the strip:

- Rectangles define *intervals*, the remainder are *gaps*. 
Covering Gaps

- All disk centres are outside of the gap.
- Instance of SSDUDC, so use $O(m^2n + n \log n)$ time algorithm to cover gap optimally.
- Optimality is lost when combining solutions from multiple gaps.
3-Approximate Algorithm

- Consider all single disks, as well as pairs and triples.
- Run a dynamic program from left to right on the strip.
- Runs in $O(m^6n)$ time.
- Each gap & interval is covered optimally!
Summary of DUDC Results

- LSDUDC algorithm, $O(mn + n \log n)$ time.
- 2-approximate assisted LSDUDC algorithm.
- Strip-based decomposition providing DUDC algorithms as follows:
  - 18-approximate, $O(mn + n \log n)$ time
  - 16-approximate, $O(m^2n + n \log n)$ time
  - 15-approximate, $O(m^6n + n \log n)$ time
- WSDUDC is NP-hard.
Trade-off in DUDC Algorithms

$c$-Approximation Algorithms, Running in $O(m^cn)$ Time

PTAS of Mustafa & Ray, 2010

Valid range for PTAS

Our results
Given a simple polygon $P$, a *Hausdorff Core* of $P$ is a convex polygon $Q$ contained in $P$ that minimizes the Hausdorff distance between $P$ and $Q$.

We denote the Hausdorff core as $H(P, Q)$.

Does there exist a convex polygon $Q$ contained in $P$ such that $H(P, Q) \leq k$?

Note: the $1$-centre of a polygon $P$ is the point $c$ which minimizes $H(P, c)$. 

The Hausdorff Core Problem
Shrinking Disks
Why is this challenging?
Hausdorff Core Approximation Algorithm for Simple Polygons

- Approximation scheme based upon a discretization of the search space.

Intuition:
1. Divide the disks into discrete segments.
2. Grow the disks slightly such that at least one segment of each expanded disk will be contained in the optimal solution.
3. Check for a solution among all intervals on all disks.
A Simpler Problem

- What if the polygon that we want to approximate has only one reflex vertex?
One Vertex Solution

- Find angle where max distance is minimized.

\[ \angle \ell_1 \ell_2 = 33^\circ \]
\[ \angle \ell_1 \ell_3 = 70^\circ \]
\[ \angle \ell_1 \ell_4 = 107^\circ \]
\[ \angle \ell_1 \ell_5 = 131^\circ \]
\[ \angle \ell_1 \ell_6 = 145^\circ \]
\[ \angle \ell_1 \ell_7 = 162^\circ \]
Summary of Hausdorff Core Results

- FPTAS for Hausdorff core on simple polygons:
  - $O((n^3 + n^2 \varepsilon^{-6}) \cdot \log(\varepsilon^{-1}))$ time.
- $O(n^3)$ time exact algorithm for polygons containing a single reflex vertex.
The (Minimum Weight) MST with Neighborhoods Problem
The Maximum Weight MST with Neighborhoods Problem

max-MSTN
Parameterized Algorithms

- $k$ = separability of the instance
  - min distance between any two disks $\geq kr_m$

$k = 0.25$

$r_m$
Parameterized max-MSTN Algorithm

- \( \left( 1 - \frac{2}{k+4} \right) \) – factor approximation by choosing disk centres
Parameterized max-MSTN Algorithm

- \( \left(1 - \frac{2}{k+4}\right) \) – factor approximation by choosing disk centres

Consider this edge \( r_i - r_j \)

\[
\frac{d + r_i + r_j}{d + 2r_i + 2r_j} \geq \frac{kr_m + r_i + r_j}{kr_m + 2r_i + 2r_j} \geq \frac{kr_m + r_m + r_m}{kr_m + 2r_m + 2r_m} = \frac{k + 2}{k + 4} = 1 - \frac{2}{k + 4}
\]
Hardness of MSTN

Reduce from planar 3-SAT (with spinal path)

Create instance of MSTN (resp. max-MSTN) so that:
- Weight of the optimal solution may be precomputed for any instance;
- Weight of solution corresponding to a non-satisfiable instance is greater than (resp. less than) the optimal solution by a significant amount.
Summary of MST Results

- **MSTN**
  - NP-hard
  - \( \left( 1 + \frac{2}{k} \right) \) – factor approximation

- **max-MSTN**
  - NP-hard
  - \( \left( 1 - \frac{2}{k+4} \right) \) – factor approximation

- **2-GMST**
  - NP-hard in 2 dimensions
  - Exact solution if topology is known
Nice Results and Open Problems

- Best known fast DUDC approximation algorithm.
- Hardness of WSDUDEC.
- **OPEN** Is assisted LSDUDEC NP-hard?
- **OPEN** Is there a nice PTAS for WSDUDEC?

- First known solutions for the Hausdorff Core problem.
- **OPEN** Are there classes of polygons for which finding the Hausdorff Core is easy?

- Best known results for MSTN.
- First results for max-MSTN.
- Hardness of 2-GMST.
- **OPEN** Is there a PTAS for max-MSTN (even on unit disks)?
- **OPEN** Is there a combinatorial approximation algorithm for 2-GMST (particularly with an approximation factor of <4)?
• Thanks!