Explanation Supporting Kurowski's Proof

In my thesis I had a discussion that M. Kurowski, in his paper "1.235n lower bound on the number of points needed to draw all n-vertex planar graphs", Information Processing Letters, Volume 92 Issue 2, 31 October 2004 Pages 95-98, did not take the isomorphism into account to prove the lower bound. I compiled my confusions in a draft on the webpage (which is also attached at the end of this draft). Recently, Cardinal et al. have proved that no point set of 15 points are universal for all plane 3-trees with 15 vertices (http://arxiv.org/abs/1209.3594). In CCCG 2013, I had a chance to discuss about M. Kurowski's proof with J. Cardinal and V. Kusters, and depending on that discussion, I convinced myself with the correctness of the proof, as explained below.

Let S be the set of labeled plane 3-trees with $n > 3$ vertices that are constructed by starting with a triangle and then by repeatedly inserting a vertex one some innerface or outerface (making the new vertex adjacent to the vertices incident to that face). For some $G \in S$, G is a graph with fixed embedding, and the labels of the vertices of G correspond to the insertion order. Note that some graphs of S may be isomorphic.

Let P be a set of n points that is universal for all the planar 3-trees of n vertices. For each graph G in S, we can assign a unique labeling of the points in P, as follows.

- If G admits an embedding on S, then the labeling of the points of P is obtained from the labeling of G and the embedding of G on S.
- If G does not admit any point-set embedding on S, then since P is universal, there must be a graph $G' \in S$ which admits a point-set embedding Γ on S. We now for each vertex u in Γ, label the corresponding point with the label of u in G. Figure 1 shows an example where $n = 6$. The graph G of Figure 1(a) does not admit a point set embedding on the point set P of Figure 1(c). The graph G' of Figure 1(b), which is isomorphic to G, admits a point-set embedding on P. Figure 1(c) illustrates a point-set embedding Γ of G'. Figure 1(d) shows the unique labeling of P.

Therefore, if P, where $|P| \geq n$, is a universal point set of all the plane 3-trees of S (i.e., for all planar 3-trees with n vertices), then each of the graphs in S must correspond to an unique labeling of the points of P. Consequently, the number of possible labelings of the points in P must be larger than the cardinality of the set S.

M. Kurowski showed that for sufficiently large n, $|P|$ can be as large as 1.235n, whereas the possible labelings for P is still smaller than the cardinality of S. Therefore, 1.235n is a lower bound on the number of points needed to draw all n-vertex planar graphs. I hope this note might be helpful to better understand M. Kurowski's proof on the lower bound on universal point set.

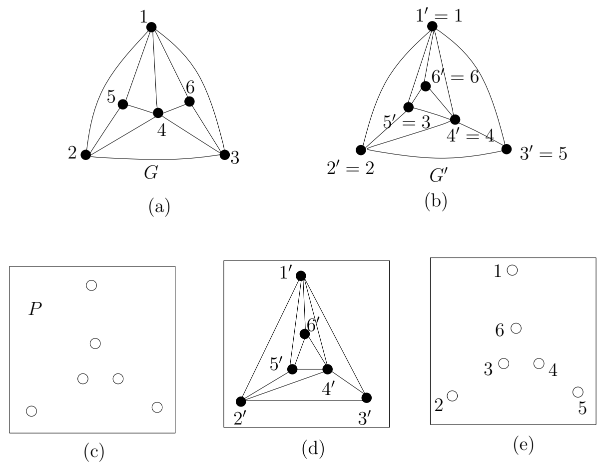

Figure 1: (a) G. (b) G'. (c) P. (d) Point-set embedding of G' on P. (e) A labeling that is assigned to P for G.

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These are some discussion on Kurowski's paper "A $1.235n$ lower bound on the number of points needed to draw all n-vertex planar graphs", Information Processing Letters, Volume 92 Issue 2, 31 October 2004 Pages 95-98.

According to Chrobak et al.'s proof, no set of 38 points is universal for all graphs with 38 vertices. If Kurowski's argument is correct, then by $19! > 3 \times 5 \times ... \times 31$ we obtain that no set of 19 points is universal for all graphs with 19 vertices. This has recently been improved by Dr. Cardinal et al. in this article (arxiv.org/abs/1209.3594).

In my thesis I analyzed that Kurowski's proof does not consider isomorphism. But one may claim that the idea of Kurowski's proof is to precisely count labeled planar 3-trees, and not only the nonisomorphic ones. This allows the use of the simpler formulas on both sides, for the number of graphs and the number of embeddings.

Let the set of "well-defined set of labeled planar 3-trees with n vertices" of Kurowski's proof be S. Then Kurowski's proof implies that no fixed point set of $n = 19$ points can support all graphs of S. Suppose someone CLAIMs that all plane 3-trees with n vertices can be embedded on a fixed point set P of 19 points. Let that set of all non-isomorphic plane 3-trees be S'.

To contradict the CLAIM lets try to prove that if all the graphs of S' admits embedding on P, then all the graphs of S also admits embedding on P. Here the graphs of S' are unlabeled, and we can choose their embeddings to embed them on P. Let E' be the set of all embeddings we used. Let E be the set of those embeddings for the graphs in S. Then we can label the embeddings of E' in all possible ways to get all the embeddings of E. But there may exist an embedding Γ in E that has not been generated (because, we could choose the embeddings for the graphs of S' freely, and it may be the case that we did not use all the structures of S). Now, existence of such Γ helps Kurowski's proof to have a bound on S, but I am afraid that this does not imply any bound on S'.

At this point one can say that I am using variable embedding for the graphs of S' and fixed embeddings for the graphs of S. But Kurowski's proof considers variable embeddings for the graphs of S.

If Kurowski's proof considers variable embeddings of S, then include all such embeddings in S such that S now contains only fixed embeddings. Observe that we are now allowing more embeddings in S, but the embedding set E' for the graphs of S' is small. Therefore, it is unlikely that the E' can be labeled to generate all possible variable embeddings E of S.

It seems to me that to prove the correctness of Kurowski's proof, one need to show that S is a subset of the embeddings that are generated by all possible labellings of the embeddings of E', which may not be trivial to prove.

[I would be very happy to know if anybody have any feedback regarding that proof (jyoti@cs.umanitoba.ca). -Debajyoti]

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