COMP 7720 - Online Algorithms

Online Bin Packing

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Review & Plan
Today’s objectives

- Competitive ratio of Next Fit and Worst Fit
- Lower bound for competitive ratio of any algorithm
- Lower bound for competitive ratio of Best Fit and First Fit
(our beautiful) Bin Packing problem
Bin Packing Problem

- The input is a multi-set of items of various sizes in range (0,1].
- The goal is to pack these items into a minimum number of bins of uniform capacity.

  - E.g., $S = \{0.1, 0.2, 0.2, 0.3, 0.3, 0.4, 0.4, 0.4, 0.5, 0.5, 0.5, 0.5, 0.6, 0.8, 0.8, 0.9\}$
Offline Bin Packing

- The problem is NP-Hard
  - reduction from the partition problem
  - Partition: decide whether a multiset $S$ of positive integers can be partitioned into two subsets $S_1$ and $S_2$ s.t.
    - sum of the numbers in $S_1 = \text{sum of the numbers in } S_2 = X$
    - $S = \{3, 1, 3, 2, 3, 2, 3, 3, 4, 1\} \rightarrow S_1 = \{3, 2, 3, 3\} \quad S_2 = \{1, 3, 2, 4, 1\}$
    - The answer to partition is yes, if the items/integers can be placed in 2 bins of size $X$

- It is NP-hard to see whether a set can be packed in two or three bins!
Online Bin Packing

- The input is a **sequence** of items of various sizes which are revealed in a sequential, online manner.

- Item sizes are in range \((0, 1]\), and the goal is to pack these items into a minimum number of bins of uniform capacity.

- An online algorithm places items into bins, one by one, with no knowledge of future items.

- Decisions of an online algorithm are irrevocable.
Next Fit Algorithm

- Next Fit: Maintain one open bin at any given time.
- Place an incoming item in the open bin if there is space there; otherwise close the bin and open a new bin.

< 0.9 0.3 0.8 0.5 0.1 0.1 0.3 0.2 0.4 0.2 0.4 0.5 0.5 0.8 0.6 0.4 0.5 >
First Fit Algorithm

- First Fit: place an incoming item in the first bin which has enough space for the item.
- Open a new bin if such bin does not exist.

< 0.9  0.3  0.8  0.5  0.1  0.1  0.3  0.2  0.4  0.2  0.4  0.5  0.5  0.8  0.6  0.4  0.5 ... >
Best Fit Algorithm

Place an incoming item in the bin with the highest level (used space) which has enough space for the item.
Harmonic Algorithm

- Harmonic Algorithm classes: \( (\frac{1}{2}, 1], (\frac{1}{3}, \frac{1}{2}], \ldots, (\frac{1}{K}, \frac{1}{K-1}], (0, \frac{1}{K}] \).
- Place members of each class separately from others.

Harmonic \( K = 4 \)

\[
< 0.9 \ 0.3 \ 0.8 \ 0.5 \ 0.1 \ 0.1 \ 0.3 \ 0.2 \ 0.4 \ 0.2 \ 0.4 \ 0.5 \ 0.5 \ 0.8 \ 0.6 \ 0.4 \ 0.5 \ldots >
\]
Analysis Measures

- Competitive ratio of $A$ is the maximum value of $A(\sigma)/\text{Opt}(\sigma)$ among all sequences $\sigma$.
  - We are interested in the asymptomatic competitive ratio where $\text{Opt}(\sigma)$ is arbitrary large.
Competitive Analysis of Next Fit

- Competitive ratio of NextFit is at most 2.
  - In the final packing, total size of items in each two consecutive bins is larger than 1.
  - The cost of NextFit for serving $\sigma$ is smaller than $2S(\sigma)$ where $S(\sigma)$ is the total size of items in $\sigma$.
    - Assume $\text{cost}(\text{NextFit}) = k$
    - Each two consecutive bins have total size $> 1 \rightarrow$ total-size $S(\sigma)$ of items in $\sigma$ is more than $k/2$
  - $\text{OPT}(\sigma) \geq S(\sigma)$: Even when $\text{OPT}$ packs items tightly (with no wasted space), $S(\sigma)$ bins are required.
Competitive Analysis of Next Fit

- Competitive ratio of NextFit is at least 2.
  - Consider sequence $\sigma = \langle 0.5, \epsilon, 0.5, \epsilon, \ldots \rangle$.
  - The cost of NextFit for serving $\sigma$ is roughly $n/2$ ($n$ is the length of $\sigma$).
  - The cost of $OPT$ is roughly $n/4$.

\[ \begin{array}{l}
\text{NextFit} \\
5 \quad 5 \quad 5 \quad 5 \quad 5 \quad 5
\end{array} \quad \begin{array}{l}
\text{OPT} \\
5 \quad 5 \quad 5
\end{array} \]

**Theorem**

*Competitive ratio of NextFit is exactly 2.*
Competitive Analysis Of Other Algorithms

- Competitive ratio of First Fit and Best Fit are both 1.7
  - We see the proof later

- Any **Almost Any Fit** strategy has a competitive ratio of 1.7
  - Any Fit algorithm: algorithm which avoid opening new bin when one of the currently open bins have enough space
  - Almost Any Fit algorithm: an AnyFit algorithm which avoid WorstFit strategy (i.e., avoid placing item in the least full bin)

- Competitive ratio of Harmonic is $1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{42} + \ldots \approx 1.691$
An Easy Lower Bound ...

- Consider the following input
  \[ \sigma = \left( \frac{1}{2} - \epsilon, \frac{1}{2} - \epsilon, \ldots, \frac{1}{2} - \epsilon \right) \cup \left( \frac{1}{2} + \epsilon, \frac{1}{2} + \epsilon, \ldots, \frac{1}{2} + \epsilon \right) \]
  \[ m \text{ items} \]

- Consider the sub-sequence formed by the first \( m \) items
  - Cost of \( \text{OPT} \) for the sub-sequence is \( m/2 \)
  - Cost of Alg is \( \alpha m \) for some \( \alpha \) so that \( 1/2 \leq \alpha \leq 1. \)
  - Competitive ratio will be at least \( \frac{\alpha m}{m/2} = 2\alpha. \)

- Consider the whole sequence \( \sigma \)
  - Cost of \( \text{OPT} \) is \( m \)
  - Alg has opened \( \alpha m \) bins for the first \( m \) items, out of which \( m - \alpha m \) bins have two item
  - So, \( \alpha m - (m - \alpha m) = 2\alpha m - m \) bins have one item
  - Alg has to open \( m - (2\alpha m - m) = 2m - 2\alpha m \) new bins for second half.
  - The total cost will be \( \alpha m + 2m - 2\alpha m = 2m - \alpha m \)
  - The competitive ratio will be at least \( \frac{2m-\alpha m}{m} = 2 - \alpha. \)
An Easy Lower Bound

Consider the following input
\[ \sigma = \langle 1/2 - \epsilon, 1/2 - \epsilon, \ldots, 1/2 - \epsilon, 1/2 + \epsilon, 1/2 + \epsilon, \ldots, 1/2 + \epsilon \rangle \]

To summarize, any sequence that opens \( \alpha m \) bins for the first half has competitive ratio at least \( \max\{2\alpha, 2 - \alpha\} \)

- The value of \( \max\{2\alpha, 2 - \alpha\} \) is minimized for \( \alpha = 2/3 \), and the competitive ratio will be at least 4/3.

**Theorem**

*No online bin packing algorithm (deterministic or randomized) can have a competitive ratio better than 4/3.*
An Easy Lower Bound

- Input:
  \[ \sigma = \langle \frac{1}{2} - \epsilon, \frac{1}{2} - \epsilon, \ldots, \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon, \frac{1}{2} + \epsilon, \ldots, \frac{1}{2} + \epsilon \rangle \]
  \( m \) items
  \( m \) items

- No algorithm can be within a ratio less than 4/3 of \( \text{OPT} \).

- The worst-case sequence that we formed is **fixed** and not **adversarial**

- What if the online algorithm knows the whole input?
  - The lower bound still holds even if the algorithms knows the whole input
Shortcomings of Online Algorithms

- Online algorithms have two shortcomings against $\text{OPT}$
  - I) **Online constraint**: online algorithms do not know the future requests/items
    - We do not know the future items in bin packing, future points in clustering, etc.
    - Often randomization helps to cope with this!
  - II) **sequential constraint** online algorithms have to build their solution sequentially
    - They cannot change their previous decisions, e.g., an item placed in a bin, two points placed in the same cluster, etc.
    - Even if the algorithms knows the whole input, adversary just needs to decide where to end the input
    - Randomization does not help!
    - This is the main problem of online bin packing algorithms!
A better lower bound

- \( \sigma_1 = \langle \frac{1}{2} - \epsilon, \ldots \frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon, \ldots, \frac{1}{2} + \epsilon \rangle \rightarrow \)
  \( m \) items
  competitive ratio > 4/3

- \( \sigma_2 = \langle \frac{1}{6} - \epsilon, \ldots \frac{1}{6} - \epsilon, \frac{1}{3} - \epsilon, \ldots \frac{1}{3} - \epsilon, \frac{1}{2} + 2\epsilon, \ldots, \frac{1}{2} + 2\epsilon \rangle \rightarrow \)
  \( m \) items
  competitive ratio > 3/2

- More complicated sequences results better lower bounds

Theorem

No online bin packing algorithm can have a competitive ratio better than 1.54037.

In the next class, we study the weighting technique to prove that Best Fit and First Fit have competitive ratio 1.7.