COMP 7720 - Online Algorithms

Online Bin Packing

Shahin Kamali

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University of Manitoba
Review & Plan
Today’s objectives

- A review of weighting argument and competitive ratio of Best Fit/First Fit
- Worst-case vs Average case: practical algorithms
- Average-case analysis of Best Fit and other algorithms
**Harmonic Algorithm**

- Harmonic Algorithm classes: $\left( \frac{1}{2}, 1 \right], \left( \frac{1}{3}, \frac{1}{2} \right], \ldots, \left( \frac{1}{K}, \frac{1}{K-1} \right], (0, \frac{1}{K}]$.
- Place members of each class separately from others.

**Harmonic Algorithm for $K = 4$**

$$< 0.9 \ 0.3 \ 0.8 \ 0.5 \ 0.1 \ 0.1 \ 0.3 \ 0.2 \ 0.4 \ 0.2 \ 0.4 \ 0.5 \ 0.5 \ 0.8 \ 0.6 \ 0.4 \ 0.5 \ ... >$$

- $x > \frac{1}{2}$
- $\frac{1}{3} < x \leq \frac{1}{2}$
- $\frac{1}{4} < x \leq \frac{1}{3}$
- $x \leq \frac{1}{4}$
Weighting Technique in a Nutshell

- Step I: Define a weight function $w(x) \geq x$ for an item of size $x$
- Step II: Prove that any bin of the online algorithm has weight 1.
- Step III: Prove that it is not possible to place a total weight more than $J$ in any empty bin
- The competitive ratio will be $J$
Weighting Technique

- Define a weight for each item based on its size
- The weight of an item in class $i$ is $1/i$ when $i < k$
- The weight of an item of size $x$ in class $k$ is $\frac{k}{k-1}x$

$Harmonic \ K = 4$

$< 0.9 \ 0.3 \ 0.8 \ 0.5 \ 0.1 \ 0.1 \ 0.3 \ 0.2 \ 0.4 \ 0.2 \ 0.4 \ 0.5 \ 0.8 \ 0.6 \ 0.4 \ 0.5 \ ... >$

<table>
<thead>
<tr>
<th>$x &gt; \frac{1}{2}$</th>
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<td>0.6</td>
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<tr>
<td>0.2</td>
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<td>0.1</td>
<td>0.1</td>
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</table>

weight = 1 \hspace{1cm} weight = \frac{1}{2} \hspace{1cm} weight = \frac{1}{3} \hspace{1cm} weight = \frac{4}{3}x
Summary of Weighting Technique for Harmonic

- We define a weight of an item of class $i < k$ to be $1/i$ and the weight of an item of class $k$ to be $\frac{k}{k-1} \cdot x$.

- We showed the weight of all bins (except at most $k$ of them) is at least 1 in Harmonic’s packing.

- We showed the maximum weight of any bin is at most $J = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{42} + \ldots \approx 1.691$ when $k$ is large enough.
  - We often assume $k$ is a constant around 20.

- The competitive ratio of the algorithm will be at most $J$.
Competitive Analysis Of First Fit

- Competitive ratio of First Fit is 1.7
  - More precisely, for any sequence \( \sigma \), we have \( FF(\sigma) \leq \lceil 1.7 \cdot OPT(\sigma) \rceil \).
- Use a weighting method!

\[
W(\alpha) = \begin{cases} 
(6/5) \alpha & \text{for } 0 \leq \alpha \leq 1/6, \\
(9/5) \alpha - 1/10 & \text{for } 1/6 < \alpha \leq 1/3, \\
(6/5) \alpha + 1/10 & \text{for } 1/3 < \alpha \leq 1/2, \\
(6/5) \alpha + 4/10 & \text{for } 1/2 < \alpha \leq 1.
\end{cases}
\]

- Use case analysis to prove:
  - Total weight of all items in a bin of FF is at least 1
  - Total weight of items in any bin is at most 1.7
Any-Fit family of algorithms

- Any **Almost Any Fit** strategy has a competitive ratio of 1.7
  - Any Fit algorithm: algorithm which avoid opening new bin when one of the currently open bins have enough space
  - Almost Any Fit algorithm: an AnyFit algorithm which avoid Worst-Fit strategy (i.e., avoid placing item in the least full bin)

- Proof is similar to First Fit

- Best Fit has a competitive ratio of 1.7.
Analysis Measures

- Compare the performance of an online algorithm $A$ with an optimal offline algorithm $\text{OPT}$:
  - $\text{OPT}$ knows the whole sequence in the beginning.
  - $\text{OPT}$ can change its packing at any time.

- Competitive ratio of $A$ is the maximum value of $A(\sigma)/\text{OPT}(\sigma)$ among all sequences $\sigma$.
  - We are interested in the asymptomatic competitive ratio where $\text{OPT}(\sigma)$ is arbitrary large.

- Average case ratio of $A$ is the expected value of $A(\sigma)/\text{OPT}(\sigma)$.
  - Item sizes are generated randomly and independently, from an identical distribution (typically uniform distribution).

- Expected waste of $A$ is the expected value of $A(\sigma) - \text{OPT}(\sigma)$. 
Summary of Bin Packing Algorithms

- Average performance ratio, expected waste, and competitive ratios for different bin packing algorithms.

- Competitive ratio of any algorithm is at least 1.54037 (BalBek12)

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<td>1.3 CoHoSY80</td>
<td>Ω(n)</td>
</tr>
<tr>
<td>Best Fit (BF)</td>
<td>1.7 Johnso73</td>
<td>1 BeJLMM84</td>
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<td>→ T_∞ ≈ 1.691</td>
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<tr>
<td>Refined Harmonic</td>
<td>1.635 LeeLee85</td>
<td>1.2824 GuChXu02</td>
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<tr>
<td>Modified Harmonic</td>
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<td>1.189</td>
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Compromise between Competitive Ratio and Average-case Ratio

Is there an algorithm that performs as well as Best Fit while having better competitive ratio?

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Harmonic Match

Harmonic Match:

- An extension of the classes of Harmonic algorithm.
- Apply a relaxed variant of Best Fit on items of each class.

\[
\begin{align*}
\frac{1}{3} &< x \leq \frac{1}{2} \\
\frac{1}{4} &< x \leq \frac{1}{3} \\
\frac{1}{5} &< x \leq \frac{1}{4} \\
\frac{1}{k+1} &< x \leq \frac{1}{k}
\end{align*}
\]

\[
\begin{align*}
i = 1 &\quad \frac{1}{3} < x \leq \frac{1}{2} &\leftrightarrow& &\frac{1}{2} < x \leq \frac{2}{3} \\
i = 2 &\quad \frac{1}{4} < x \leq \frac{1}{3} &\leftrightarrow& &\frac{2}{3} < x \leq \frac{3}{4} \\
i = 3 &\quad \frac{1}{5} < x \leq \frac{1}{4} &\leftrightarrow& &\frac{3}{4} < x \leq \frac{4}{5} \\
\vdots \\
i = k-1 &\quad \frac{1}{k+1} < x \leq \frac{1}{k} &\leftrightarrow& &\frac{k-1}{k} < x \leq \frac{k}{k+1} \\
i = k &\quad x \leq \frac{1}{k+1} &\leftrightarrow& &x > \frac{k}{k+1}
\end{align*}
\]
Harmonic Match Algorithm

For placing an item of size $x$:

- If $x > 0.5$, open a new bin.
- If $x \leq 0.5$:
  - Use Best Fit strategy to place $x$ together with an item $y > 0.5$ of the same class.
  - If no such $y$ exists, place $x$ together with items of the same class using Next Fit strategy.
Packing of Harmonic Match is the same as Harmonic except that some items are ‘removed’ from Harmonic packing.
Competitive Analysis

- Harmonic is a \textit{monotone} algorithm.
  - Removing an item does not increase the number of bins opened by Harmonic.

\textbf{Theorem}

For any sequence, the number of bins opened by Harmonic Match is no more than that of Harmonic.

- Competitive ratio of Harmonic Match is the same as Harmonic, i.e., $T_{\infty} \approx 1.691$.
- Unlike Harmonic, First Fit and Best Fit are \textit{anomalous} in the sense that removing items might increase the cost of these algorithms.
Average-Case Analysis

- Consider **upright matching** problem.
  - We are given $n$ points in a $1 \times 1$ coordinate.
  - The goal is to match a maximum number of $\ominus$ with $\oplus$ points.
  - Each $\ominus$ point can be matched only to $\oplus$ points on its upright position.
  - Labels and positions of points are i.i.d. random variables.

- Greedy algorithm: process *ominus* points one by one from top to bottom
  - Match each $\ominus$ item with the left-most unmatched $\oplus$ item above it.

- It is known that Greedy matches all points except and expected number of $\Theta(\sqrt{n} \log^{3/4} n)$ points.
Consider a bin packing sequence of length $n$ with item sizes randomly distributed in $(0, 1]$.

Create an instance of upright matching:
- Items are mapped to points in the square.
- An item of size $\alpha > 0.5$ gets an $\oplus$ label and $x$-coordinate $2(1 - \alpha)$.
- An item of size $\alpha \leq 0.5$ gets an $\ominus$ label and $x$-coordinate $2\alpha$.
- $y$-coordinate of the item at index $i$ is set randomly in $\lfloor i/n \rfloor$, $\lceil i/n \rceil$

E.g., $\sigma = \langle 0.53, 0.69, 0.21, 0.78, 0.4 \rangle$
Reduction of bin packing to upright matching

- Points receive random labels (with a chance of 0.5 an item is larger than 0.5 (⊕) and with a chance of 0.5 it is ≤ 0.5 (⊖).

- Points x-coordinates are random
  - for an ⊕ point, item size x is random in $U(0.5, 1]$ and hence $2(1 - x)$ is random in $U[0, 1]$
  - for an ⊖ point, item size $x$ is random in $U(0, 0.5]$ and hence $2(x)$ is random in $U(0, 1]$

- Points y-coordinates are random
  - Exactly one point is distributed randomly in the interval $U[i/n, (i + 1)/n]$ on the y-axis.
Reduction of bin packing to upright matching

- So, an instance of bin packing can be reduced to upright matching.

- What is the equivalent of greedy algorithm?
  - An $\oplus$ point $y$ appears on the right of $x$ if sum of items $x$ and $y$ is less than 1.
    - $y$ is on right of $x$ $\rightarrow$ $2(1 - y) \geq 2x$ $\rightarrow$ $x + y \leq 1$

- Greedy matches each $\ominus$ point $p$ (item $x \leq 0.5$) with the leftmost $\oplus$ point (largest item $y$ so that $> 0.5$) that appears above (i.e., $y$ is before $x$ in the sequence) and on the right of $p$ (i.e., $x + y \leq 1$).
Reduction of bin packing to upright matching

- Greedy is equivalent to Almost Best Fit:
  - If $x > 1/2$, open a new bin for $x$.
  - If $x \leq 1/2$, place $x$ with an item $y \geq 0.5$ which best fits $x$ (i.e., largest such $y$ so that $x + y \leq 1$).
  - If no such $y$ exists, open a new bin for $x$.

- Almost Best Fit is similar to Best Fit except that:
  - It closes a bin right after it is opened if the bin is opened by an item of size $\leq 1/2$.
  - It closes a bin as soon as two items are placed in it.

- For any sequence, the cost of Best Fit is at most equal to Almost-Best-Fit.
Average-case analysis of Best Fit

- Number of unmatched point by greedy is expected to be $\Theta(\sqrt{n \log^{3/4} n})$.

- So, the number of bins with 1 item in Almost Best Fit (ABF) is at most $\Theta(\sqrt{n \log^{3/4} n})$ on expectation.

- The cost of ABF is at most $n/2 + \Theta(\sqrt{n \log^{3/4} n})$ for a sequence of length $n$ on expectation.

- The cost of $\text{OPT}$ is expected to be at least $n/2$ (since half items are expected to be larger than 0.5).

- Average case ratio of ABF (and hence BF) is at most
  \[
  \frac{n/2 + \Theta(\sqrt{n \log^{3/4} n})}{n/2} \approx 1 \text{ for large values of } n
  \]

- Expected waste of ABF (and hence BF) is at most
  \[
  E(ABF(\sigma) - \text{OPT}(\sigma)) = n/2 + \Theta(\sqrt{n \log^{3/4} n}) - n/2 = \Theta(\sqrt{n \log^{3/4} n})
  \]
The average-case analysis for Harmonic Match is similar to Best Fit; we repeat the same analysis for each class separately.

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Experimental Evaluation

- Experimental average-case performance of online algorithms for different distributions.

![Graph showing performance comparison]