Problem 1  [4+4+4+4+4=20 marks]

Provide a complete proof of the following statements from first principles (i.e., using the original definitions of order notation).

Ex.)  $15n^3 + 10n^2 + 20 \in O(n^3)$

Consider $M := 15 + 10 + 20 = 45$ and $n_0 := 1$. Then $0 \leq 12n^3 + 11n^2 + 10 \leq Mn^3$ for all $n \geq n_0$.

a) $n^2 + \frac{3n^2}{2+\cos(n)} \in O(n^2)$

b) $2n^2(\log n) \in \Omega(n(\log n)^2)$

c) $10n^2/(n - 10) \in \Theta(n)$

d) $1396n \in o(n \log n)$

e) $n^n \in \omega(n^{20})$

Problem 2  [4+4+4=12 marks]

For each pair of the following functions, fill in the correct asymptotic notation among $\Theta$, $o$, and $\omega$ in the statement $f(n) \in \cup(g(n))$. Provide a brief justification of your answers. In your justification you may use any relationship or technique that is described in class.

Ex.) $f(n) = n^{3.5}$ and $g(n) = n^3 \log(n)$.

We have $\lim_{n \to \infty} \frac{n^{3.5}}{n^3 \log n} = \lim_{n \to \infty} \frac{\sqrt[n]{n}}{\log n} = \lim_{n \to \infty} (n^{-1/2})^{\frac{1}{2}} = \lim_{n \to \infty} \frac{n^{1/2} \ln n}{2} = \infty$. Hence we have $f(n) = \omega(g(n))$.

a) $f(n) = n(\log n)^3$ versus $g(n) = n^2$
b) \( f(n) = \sqrt{n} \) versus \( g(n) = (\log n)^4 \)

c) \( f(n) = n^3(8 + 2\cos 2n) \) versus \( g(n) = n^2 + 2n^3 + 3n \)

**Problem 3  [6+6=12 marks]**

Prove or disprove each of the following statements. To prove a statement, you should provide a formal proof that is based on the definitions of the order notations. To disprove a statement, you can either provide a counter example and explain it or provide a formal proof. All functions are positive functions.

Ex.) \( f(n) \notin o(g(n)) \) and \( f(n) \notin \omega(g(n)) \) \( \Rightarrow \) \( f(n) \in \Theta(g(n)) \)

False. Counter example: Consider \( f(n) := n \) and \( g(n) := \begin{cases} 1 & n \text{ odd} \\ n^2 & n \text{ even} \end{cases} \). To prove the claim false it will be sufficient to show that \( f(n) \notin O(g(n)) \) and \( f(n) \notin \Omega(g(n)) \), since then the antecedent of the implication is satisfied while the consequent is not.

If \( f(n) \in O(g(n)) \), then there exist constants \( n_0 > 0 \) and \( c > 0 \) such that \( f(n) \leq cg(n) \) for all \( n \geq n_0 \). But for any odd number \( n_1 > c \) we have \( f(n_1) = n_1 > c = cg(n_1) \), showing that \( f(n) \notin O(g(n)) \).

Similarly, if \( f(n) \in \Omega(g(n)) \), then there exists constants \( n_0 > 0 \) and \( c > 0 \) such that \( cg(n) \leq f(n) \) for all \( n \geq n_0 \). But for any even number \( n_1 > 1/c \) we have \( cg(n_1) = cn_1^2 \geq n_1 = f(n_1) \), showing that \( f(n) \notin \Omega(g(n)) \).

a) \( f(n) \in \Theta(g(n)) \) and \( h(n) \in \Theta(g(n)) \) \( \Rightarrow \) \( \frac{f(n)}{h(n)} \in \Theta(1) \)

b) \( f(n) \in \Theta(g(n)) \) \( \Rightarrow \) \( 2^f(n) \in \Theta(2^g(n)) \)

**Problem 4  [7 marks]**

Analyze the following piece of pseudocode and give a tight (\( \Theta \)) bound on the running time as a function of \( n \). Show your work. A formal proof is not required, but you should justify your answer.

```plaintext
1. bee ← 0
2. for i ← 1 to 2n^2 do
3.     bee ← bee × 4
4. for j ← 1396 to 2018 do
5.     for k ← 4i to 6i do
6.         bee ← bee × k
```

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Problem 5  [4+4+4=12 marks]

For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply. For all cases, we have $T(x) = 1$ when $x \leq 100$ (base of recursion).

Ex.) $T(n) = 2T(n/2) + \frac{n}{\log n}$.

We have $n^{\log_b a} = n$ and hence $f(n) = O(n^{\log_b a})$. But we cannot state that $f(n) = O(n^{\log_b a - \epsilon})$, i.e., the difference between $f(n)$ and $n^{\log_b a}$ is non-polynomial. Note that for any value of $\epsilon > 0$ we have $n/\log n \in \omega(n^{1-\epsilon})$. So, none of the cases of the Master theorem (specifically case one) can be applied. We cannot use Master theorem here.

a) $T(n) = 3T(n/2) + 20n$

b) $T(n) = 3T(n/3) + \sqrt{n}$

c) $T(n) = 4T(n/2) + 1984n^2$