COMP 3170 - Analysis of Algorithms & Data Structures

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CLRS 1.1, 1.2, 2.2, 3.1, 4.3, 4.5
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Picture is from the cover of the textbook CLRS.
Asymptotic Analysis
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E.g., sorting is a problem; a set of numbers form an instance of that and ‘solving’ involves creating a sorted output.
Algorithms (review)

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- A **program** is an implementation of an algorithm using a specific programming language.
Algorithms & models of computation

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- For a given problem (e.g., sorting) there can be several algorithms (e.g., Quicksort, Mergesort), and for a given algorithm (e.g., Quicksort) there can be several programs.
  - Our focus in this course is on algorithms (not programs).
  - How to implement a given algorithm relates to the art of **performance engineering** (writing a fast code)
Given a problem $P$, we need to

- Design an algorithm $A$ that solves $P$ (*Algorithm Design*)

Verify correctness and efficiency of the algorithm (*Algorithm Analysis*)

If the algorithm is correct and efficient, implement it.

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Algorithm Evaluation

- How should we evaluate different algorithms for solving a problem?
  - In this course we are mainly concerned with amount of time it takes to solve a problem (this is called running time).
  - We can think of other measures such as the amount of memory that is required by the algorithm.
  - Other measures include amount of data movement, network traffic generated, etc.
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The amount of time/memory/traffic required by an algorithm depend on the size of the problem.

- Sorting a larger set of numbers takes more time!
Running Time of Algorithms

- How to assess the running time of an algorithm?
- **Experimental analysis:**
  - Implement the algorithm in a program
  - Run the program with inputs of different sizes
  - Experimentally measure the actual running time (e.g., using `clock()` from `time.h`)
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**Shortcomings of experimental studies:**
- We need to implement the program (what if we are lazy and those engineering are hard to employ?)
- We cannot test all input instances for the problem. What are the good samples? (remember the Morphy law)
- Many factors have impact on experimental timing, e.g., hardware (processor, memory), software environment (operating system, compiler, programming language), and human factors (how good was the programmer?)
Computational Models

- We need to assess time/memory requirement of algorithms using models that
  - Take into account all input instances
  - Do not require implementation of the algorithms
  - Are independent of hardware/software/programmer

In order to achieve this, we:
- Express algorithms using pseudo-codes (don't worry implementation)
- Instead of measuring time in seconds, count the number of primitive operations
  This requires an abstract model of computation
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The random access machine (RAM):
- Has a set of memory cells, each storing one ‘word’ of data.
- Any access to a memory location takes constant time.
- Any primitive operation takes constant time.
- The running time of a program can be computed to be the number of memory accesses plus the number of primitive operations.

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Observation

RAM is a simplified model which only provides an approximation of a ‘real’ computer.
First, calculate the ‘cost’ (sum of memory accesses and primitive operations) for each line

- E.g., in line 5, there are 3 memory accesses and 3 primitive operations
Next, find the number of times each line is executed

- This depends on the input, we may consider best or worst case input
- Let $t_j$ be number of times the while loop is executed for inserting the $j$’th item.
  - In the best case, $t_j = 1$ and in the worst case $t_j = j$.
- Summing up all costs, in the best case we have
  \[ T(n) = (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8) = an - b \]
  for constant $a$ and $b$
- In the worst case, we have
  \[ T_n = \alpha n^2 + \beta n + \gamma \]
  for constant $\alpha, \beta, \gamma$
Primitive Operations

- RAM model implicitly assumes primitive operations have fairly similar running time.

- Primitive operations:
  - basic integer arithmetic (addition, subtraction, multiplication, division, and modulo)
  - bitwise logic and bit shifts (logical AND, OR, exclusive-OR, negation, left shift, and right shift)

- Non-primitive operations:
  - exponentiation, radicals (square roots), logarithms, trigonometric functions (sine, cosine, tangent), etc.
Asymptotic Notations

**Statement**

*So, we can express the cost (running time) of an algorithm A for a problem of size n as a function $T_A(n)$.*

- How do we compare two different algorithms? say $T_A(n) = \frac{1}{1000}n^3$ and $T_B(n) = 1000n^2 + 500n + 200$.
- Summarize the time complexity using asymptotic notations!
- Idea: assume the size of input grows to infinity; identify which component of $T_A(n)$ contributes most to the growth of $T_A(n)$.
- As $n$ grows:
  - constants don’t matter (e.g., $T_A(n)$)
  - low-order terms don’t matter (e.g., $T_B(n)$)
Informally $T_B(n) = O(T_A(n))$ means $T_B$ is asymptotically smaller than or equal to $T_A$.

Is it sufficient to define $O$ so that we have $T_B(n) < T_A(n)$?

- No because the inequality might not hold for small values of $n$ which we don’t care about
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\[
f(n) \in O(g(n)) \iff \exists M > 0, \exists n_0 > 0 \text{ s.t. } \forall n > n_0, f(n) \leq M \cdot g(n).
\]

ignore low-order terms

ignore constants
Let \( f(n) = 1000n^2 + 1000n \) and \( g(n) = n^3 \). Prove \( f(n) \in O(g(n)) \)