Review & Plan
Today’s plan

- Online bipartite matching (marriage problem)
- Review of the exam (how it will look)
- Concluding remarks
Online Bipartite Matching (Marriage problem)
Online Bipartite Matching

- Given a bipartite graph, create a matching of maximum size
- In the online setting, vertices in one side of the graph are given, and vertices in the other side arrive one by one
  - Upon arrival of vertex $x$, all edges connecting $x$ to its neighbors on the left are revealed
  - An online algorithm should match $x$ with another vertex, if possible, without any information about future vertices
- **Greedy algorithm**: match with any vertex on the right if possible!
Competitive ratio for max. problems

- In this example, \( \text{OPT} \) has a benefit of 4 (a perfect matching) while greedy has a benefit of 3!

- Matching is a maximization problem: we would like to maximize the benefit instead of minimizing the cost

- Competitive ratio is often defined as the maximum value of

\[
\frac{\text{Benefit(OPT)}}{\text{benefit(Alg)}}
\]

**Theorem**

*Greedy has a competitive ratio of at most 2.*
Greedy Bipartite Matching

- Greedy algorithm always creates a maximal matching
  - All neighbors of an unmatched vertex $u$ are matched (otherwise greedy would have matched $u$ with one of its neighbors).

- Consider a bipartite graph with $n$ vertices on the left.
  - If $X$ denote the number of vertices on left which are matched by $\text{OPT}$ and unmatched by greedy, all neighbours of these vertices are matched by greedy
    - Greedy matches $n - X$ vertices on the left
    - Greedy matches at least $X$ vertex on the right

- Size of greedy matching is at least $\max\{n - X, X\}$ and size of $\text{OPT}$ matching is at most $n$
  - Competitive ratio will be $\frac{n}{\max\{n - X, X\}} \leq \frac{n}{n/2} = 2$
Greedy Bipartite Matching

**Theorem**

*Greedy has a competitive ratio of at most 2.*

- We proved an upper bound before (i.e., we showed $c.r. \leq 2$).
- For the lower bound, consider the following example:
  - Greedy matches $n/2$ vertices while $\text{OPT}$ matches $n$ vertices.
Deterministic Bipartite Matching

**Theorem**

No deterministic online algorithm has a competitive ratio better than 2.

- Proof similar to the case of Greedy.
- So, Greedy is the best deterministic algorithm.
- A randomized algorithm which chooses random match has also a competitive ratio of 2 (is it good?).
Deterministic Bipartite Matching

**Rank Algorithm:**
- Initially, choose a random permutation of all vertices on left
- Upon arrival of a vertex $u$ on right:
  - Let $N(u)$ be the set of unmatched neighbors of $u$
  - If $N(u) \neq 0$, match $u$ to the vertex $v \in N(u)$ with minimum index in the permutation

**Theorem**

*Rank has a competitive ratio of $\frac{e}{e-1} \approx 1.58$*
Online Bipartite Matching Summary

- Greedy algorithm has a competitive ratio of 2 and it is the best that a deterministic algorithm can achieve.

- Rank is a simple randomized algorithm with competitive ratio of $e/(e - 1)$.
  - It is known that no randomized, online algorithm can achieve a competitive ratio better than $e/(e - 1)$.