COMP 7720 - Online Algorithms

Paging and $k$-Server Problem

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Review & Plan
Today’s objectives

- Caching problem & advice
- \( k \)-server problem
  - Introduction
  - Greedy algorithms
Caching Problem
Problem Definition

- There are two types of memory: a fast ‘cache’ of size $k$, and a slow memory of unbounded size.
  - The input is an online sequence of requests to pages of size 1.
- To serve a request to page $x$, it should be in the cache
  - In case $x$ is not in the cache, a fault of cost 1 happens
  - In case $x$ is in the cache, a hit of cost 0 happens
  - The goal is to minimize the total number of faults
- To bring $x$ to the cache, we might need to evict a page.
  - A caching algorithm is defined through its eviction policy.
Latest-In-Future (LIF) is the optimal offline algorithm.

No deterministic algorithm has a competitive ratio better than $k$.

An algorithm is marking if it maintains a ‘mark’ for each page.

- After a request to $x$ mark it.
- Always evict an unmarked page (if all marked, first unmark all pages and then evict one)

Any deterministic marking algorithm has a competitive ratio of $k$.

- Least-Recently-Used (LRU), and Flash-When-Full (FWF) both have competitive ratio $k$.

Fist-In-First-Out (FIFO) also has a competitive ratio of $k$. 
A randomized algorithm which randomly evict a page has a competitive ratio of $k$.

A marking algorithm that evicts an unmarked page uniformly at random has a competitive ratio of $H_k$

- $H_k = 1 + 1/2 + 1/3 + \ldots + 1/k$
- For large values of $k$, we have $H_k \approx \ln(k) \in \Theta(\log k)$.

In fact, no randomized algorithm can achieve a better competitive ratio (i.e., $o(\log k)$)
Caching & advice

- How many bits of advice are sufficient to achieve an optimal algorithm?
  - \( n \): the length of input sequence (number of requests).
  - \( k \): the size of the cache
  - Hint: an algorithm has to make at most \( n \) decisions about the page to be evicted.
    - one decision per fault.

- At most \( O(n \log k) \) bits are sufficient!
Caching & advice

- What does the advice encode?
- What is the size of advice? Assume Opt makes $m \leq n$ faults for the optimal algorithm. For each fault, the advice indicates which page should be evicted. There are $k$ pages in the cache, and the evicted page can be indicated in $\Theta(\log k)$ bits. The total number of bits will be $m \cdot O(\log k) \in O(n \log k)$.
- How the algorithm works, provided by these bits of advice? It just mimics Opt; whenever there is a fault, it reads the advice to see which page should be evicted.
- Why the algorithm has a competitive ratio of 1 (optimal here)? It works exactly like Opt.
Caching & advice (cntd.)

Theorem

There is an online algorithm that, provided with $O(n \log k)$ bits of advice, can achieve an optimal solution.

- It is a naive solution :'-)
- Can we achieve an optimal solution with a smaller number of bits of advice?
  - For many problems, the answer is no!
  - For caching problem, we can indeed do better.
Caching & advice (cntd.)

- Assume $\text{OPT}$ brings a page $x$ to the cache at time $t$.
  - Either $\text{OPT}$ evicts $x$ before the next access to $x \rightarrow x$ is mortal.
  - $\text{OPT}$ keeps $x$ in the cache until the next access to $x \rightarrow x$ is resident.
Caching & advice (cntd.)

- Assume $OPT$ brings a page $x$ to the cache at time $t$.
  - Either $OPT$ evict $x$ before the next access to it $\rightarrow x$ is mortal.
  - $OPT$ keeps $x$ in the cache until the next access to $x$ $\rightarrow x$ is resident

\[ a, c \text{ are residents} \quad b \text{ is mortal} \]

\[ \sigma = a \ b \ c \ b \ a \ d \ c \ e \ f \ a \ c \ d \ c \ f \ a \ b \ a \ e \]

\[
\begin{array}{ccc}
  a & d & c \\
\end{array}
\]
Caching & advice (cntd.)

- If $\text{OPT}$ has a hit for request $x \Rightarrow x$ has been resident in cache since its last access to $x$.

- If $\text{OPT}$ has a fault for request $x \Rightarrow$ either it is the first access to $x$ or $x$ has been mortal after its previous access (so that it is evicted at some point).

- Consider an algorithm $\text{ResMor}$ that evicts a mortal page if an eviction is required.
  
  - $\text{ResMor}$ always has the same resident pages as $\text{OPT}$ in its cache
  - The mortal pages might be different.

- $\text{OPT}$ and $\text{ResMor}$ have the same cost
  
  - Assume $\text{OPT}$ has smaller cost $\Rightarrow$ there is a request to $x$ that is a hit by $\text{OPT}$ and a miss for $\text{ResMor} \Rightarrow x$ is resident in $\text{OPT}$ and not in $\text{ResMore} \Rightarrow$ they maintain different resident pages ($\text{ResMor}$ has evicted a resident page at some point) $\Rightarrow$ contradiction
Caching & advice (cntd.)

- *ResMor* is an optimal algorithm that, instead of the whole sequence, only needs to know which pages are resident/mortal at each given time.

- Assume with each request, there is one bit of advice that indicates whether the requested page is resident or mortal after the request.

- We can think of *ResMor* as an online algorithm with $n$ bits of advice.
Caching & advice

- What does the advice encode?
- What is the size of advice?
- How the algorithm works, provided by these bits of advice?
- Why the algorithm has a competitive ratio of 1 (optimal here)? It maintains the same resident pages as Opt; so in case of a hit by Opt there will be a hit by the algorithm.
Advice complexity of paging

- With $n$ bits of advice, one can achieve an optimal algorithm.
- With roughly $\log(\frac{r+1}{r}) \cdot n$ bits, one can achieve a competitive ratio of $r$
  
  - With roughly $0.27n$ bits, one can achieve a competitive ratio of 2.
  - With roughly $0.24n$ bits, one can achieve a competitive ratio of 3.

- For a potential project, do a survey on advice complexity of paging, and try to deduce new results!
$k$-Server Problem
**$k$-sever problem**

- A **metric** is a set of points with a **distance** between each of pairs so that $d(x, y) \leq d(x, z) + d(z, y)$.
  - E.g., a connected, undirected graph or a set of points in plane
- We have a metric space of size $m$
  - $k < m$ servers in the graph
- A sequence of $n$ **requests** to the vertices of the graph
  - Each request should be served by a server
  - Requests appear in an online manner
- Minimize the total distance moved by servers

\[
\sigma = < S, M, K, A, D, B, D, B, D > \\
costs = 2, 0, 2, 1, 1, 1, 1, 1, 1
\]
The $k$-server problem

- What happens if we have a complete graph (a **uniform** metric)?
  - If there is a request to a vertex at which a sever is located → there is no cost; otherwise, there is a cost of 1 to move a server to requested vertex.
    - Think of vertices as pages; vertices with servers on them are pages in the cache → caching problem.

- Recall that for caching problem, we have:

  **Theorem**

  *No deterministic algorithm can achieve a competitive ratio better than $k$, and LRU and FIFO achieve this ratio.*

  *No randomized algorithm can achieve a competitive ratio that is asymptotically better than $\Theta(\log k)$ and Mark algorithm achieves this.*

- $k$-server problem has the **right level of difficulty** compared to paging (which is ‘too easy’) and Metrical Task Systems (another problem which is ‘too hard’).
Greedy Algorithm

- Move the closest server to serve each request.
- Is Greedy a good algorithm?
  - what about the input $\sigma = \langle B \ R \ B \ R \ldots \rangle$?
    - For $n$ requests, greedy incurs a cost of $n$
    - $\text{OPT}$ moves another server from $M$ to $T$ at a cost of 3 and incurs no cost.
    - Competitive ratio will be at least $\frac{n}{3}$ for this graph!
Greedy Algorithm

**Theorem**

For any graph of diameter $d$, the competitive ratio of greedy is at least $\frac{n}{2d}$.

- It holds for any graph, even a path!
- Consider two vertices $A$ and $B$ which are close to one server and further from other servers.
  - Greedy servers sequence $\langle A \ B \ A \ B \ \ldots \rangle$ by one server
Competitive analysis

- For general metrics
  - No deterministic online algorithm can be better than $k$-competitive. (we see the proof in the next class)
**Conjecture**

**Conjecture:** for any metric space, there is a deterministic algorithm with competitive ratio of $k$.

- $k$-server conjecture is one of the big open problems in the context of online algorithms.
  - Verified when $k = 2$, $m = k + 1$, $m = k + 2$, and trees.
- In the next class, we learn about **potential function** algorithm, which has a competitive ratio of $2k - 1$.