“Rivers know this: there is no hurry. We shall get there some day…” from Winnie the Pooh

Write your name here (no need for student id, etc.) Elmer Hartman

- Do not open this booklet until instructed. You can use the last (blank) page if you needed more space.

- It is an open-book exam. Your can use any printed/written material from the course. You are not allowed to use laptops/cell-phones. Please turn off your cell phones and put them in your bags.

- Manage your time. We start the exam at 10:00 and end the exam at 11:20.

- If you are not sure about the answer to a true/false or short-answer question, work on other question and get back to it at the end. Do not leave true/false questions blank as there is no penalty for wrong answers. The questions are NOT ordered by their level of difficulty.

- Do not waste your time writing lengthy answers. You can be succinct and yet precise. Note that your time is limited.

- There are more important things in life than this exam. Also, there are more important components to this course than this exam. So, relax and don’t be stressed.
Problem 1  True/False Questions [25 marks]

Indicate whether any of the following statements is true or false. There is no need to justify your answers.

a) Greedy algorithm is the optimal deterministic algorithm for online bipartite matching.
   True [ ] False [ ]  Answer: True, we saw in the class that the Greedy algorithm has a competitive ratio of 2 while no online algorithm can have a better competitive ratio.

b) When all items are in the range \((1/4, 1/3]\), Next Fit has competitive ratio of 1.
   True [ ] False [ ]  Answer: True. In this setting, any bin of Next Fit and Opt has exactly three items in it, i.e., cost of Next Fit and Opt are both \(\lceil n/3 \rceil\) for \(n\) items.

c) Advice of size \(O(n \log \Delta)\) is sufficient for optimal edge-coloring of a graph with \(n\) edges and max-degree \(\Delta\).
   True [ ] False [ ]  Answer: True; for each edge, the advice indicates exactly what color it gets in the optimal solution (we will need \(\log(\Delta + 1)\) bits).

d) An analysis measure that compares an online algorithm with an optimal offline algorithm is a worst-case measure.
   True [ ] False [ ]  Answer: False; we saw average-case performance ratio for bin packing gives different results than competitive ratio.

e) Consider two bin packing algorithms A and B. If the competitive ratio of A is better than B then the expected waste of B is better than A.
   True [ ] False [ ]  Answer: It is false. It was mentioned in the class that there is not necessarily a trade-off between competitive ratio and average-case performance (e.g., slide 12 of lecture 18). In particular we saw algorithms like Harmonic-Match (algorithm A) that have a better competitive ratio than Best Fit (algorithm B) while their average-case performance is no worse than Best Fit.

f) For the edge-coloring problem, \(\Theta(\log n)\) bit of advice is sufficient to achieve an online algorithm with a competitive ratio better than all purely-online algorithms.
   True [ ] False [ ]  Answer: This is false (and a bit tricky). The idea is to send more stars compared to what we did in the lower bound argument. With \(O(\log n)\) bits of advice, we have \(O(n)\) algorithms; and by sufficient increase in the number of stars, two of these algorithms color \(\Delta\) stars similarly. The rest of the argument is the same as before. See https://arxiv.org/pdf/1410.7923.pdf for details.

g) The competitive ratio of a randomized algorithm is the expected ratio between the cost of the algorithm and that of an optimal offline algorithm, where the expectation is taken over all sequences.
   True [ ] False [ ]  Answer: This is false. The expectation is taken over all random bits for a single worst-case sequence.

h) A packing in which there is an empty space of at least 0.5 in each bin is always a valid solution for the fault-tolerant bin packing.
   True [ ] False [ ]  Answer: It is true; clearly, there is at most a space of 0.5 filled in each bin and hence any two bin have a shared replicas of total size at most 0.5, which is less than the empty space of each of them.

i) With \(\Theta(n)\) bits of advice, one can achieve optimal packing for any instance of online bin packing with \(n\) items.
   True [ ] False [ ]  Answer: This is false. As we saw in the class, advice of size \(\Omega(n \log n)\) is required to achieve an optimal packing assuming Opt opens \(\Theta(n)\) bins. Note the difference between optimal packing and competitive ratio of 1 (refer to Piazza for details).
j) Except for possibly one critical bin, the level of all critical bins in the final packing of ReserveCritical is at least 2/3, assuming a tiny bin is opened by the algorithm.

True [ ] False [ ]  **Answer:** True. Consider one (the first) bin which has an empty space of 1/6 or more in the non-reserved space. Every bin after that receives tiny items of size larger than 1/6 and its level will be at least 2/3 (counting the item of size larger than 1/2 placed in the reserved space).

k) Randomization helps improving the competitive ratio of online bin packing algorithms

True [ ] False [ ]  **Answer:** False; we saw the lower bound nature of bin packing algorithms is not about the online constraint but the sequential constraint.

l) In order to prove an algorithm A is the optimal online algorithm (among all online algorithms, and with respect to competitive ratio) for a given problem, it suffices to prove the following two statements: I) for any input sequence, the ratio between the cost of A and OPT is at most c. II) there are sequences for which the cost of A is c times more than the cost of OPT.

True [ ] False [ ]  **Answer:** This is False. The above statements show that the competitive ratio of A is indeed c. But, to prove that A is optimal algorithm for the problem, in part II, “A” should be replaced by “any online algorithm”.

**Notes:** All parts have two marks except the last question which has three marks. The last question concerns a big picture about analysis of online algorithms, and it was important that you know it.
Problem 2  Short Answer Questions [28 marks]

Provide short answers to the following questions.
You need to show your work but not formal justification is required.
Don’t waste time by writing long answers.

a) Show the final packing of Harmonic Match for the following instance of the bin packing problem. Assume the number of classes in $k = 100$. You do not need to show the intermediate packings.

$$\sigma = (0.4, 0.7, 0.55, 0.35, 0.45, 0.18)$$

**Answer:** First bin contains 0.4 and 0.45. Second bin contains 0.7. Third bin contains 0.55 and 0.35. Fourth bin contains 0.18.

b) Indicate the cost of the Best-Fit algorithm for the following instance of server-renting problem:

$$\sigma = ((0.8, 1, 9), (0.7, 2, 8), (0.25, 3, 5), (0.15, 4, 6), (0.05, 5, 7), (0.95, 6, 7), (0.1, 7, 9))$$

**Answer:** After placing 4 items (at the end of timestep 4), two bins are opened: bin $B_1$ includes items with sizes 0.8 and 0.15 while $B_2$ includes 0.7 and 0.25. At time $t = 5$, the next item of size 0.05 could be placed in either $B_1$ or $B_2$ (since both have the same level). In either case, the next item 0.95 is placed a new bin $B_3$. By the end of time $t = 6$, items 0.15 and 0.25 are released and the level of $B_2$ (which is 0.7 or 0.75) will be less than that of $B_1$ (which is 0.8 or 0.85). Hence the last item 0.1 is placed in $B_1$.
The cost of $B_1$ will be $9 - 1 = 8$ while that of $B_2$ will be $8 - 2 = 6$ and that of $B_3$ is $7 - 6 = 1$. The total cost would be 15.
c) Consider the following variant of the first fit algorithm for bin packing. For placing an item $x$, the algorithm considers the set of bins which have enough space for $x$. If more than one bin has enough space, it places $x$ in the second bin (in the order they are opened). When only one item has enough space, the algorithm places $x$ in that bin. If no item has enough space, a new bin is opened. Verify whether the following statement is correct or incorrect. Briefly justify your answer:

Statement: Sequence $\sigma = \{0.5, \epsilon, 0.5, \epsilon, \ldots, 0.5, \epsilon\}$ shows that the competitive ratio of the above algorithm is at least 2.

Answer: The statement is wrong. The first bin will include the first two items. The second bin includes the second 0.5 and all $\epsilon$’s. The remaining bins will have two items of size 0.5. The c.r. is indeed 1.

Consider the following packing for an instance of fault-tolerant bin packing. Indicate whether this packing is valid or invalid. Briefly justify your answer.

Answer: It is invalid. If $S_3$ fails, there will be an overflow at $S_4$. 

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e) Consider a bin packing algorithm named FF-Har, which packs items of size larger than 1/3 using First Fit and separately from other items. For items of size smaller than or equal to 1/3, the algorithm uses Harmonic algorithm with parameter $K = 20$. Show the final packing of the algorithm for sequence $(0.45, 0.21, 0.3, 0.7, 0.22, 0.25, 0.55)$ (the intermediate packings are not required).

Answer: Here it is:

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f) An intelligent student named Eva is doing research on a maximization online problem $P$. She has been able to reduce the binary guessing problem into $P$ by showing that any online algorithm for $P$ has to make ‘guesses’ such that each correct guess has a benefit of 8 and each wrong guess has a benefit of 2 for the algorithm. What lower bound for competitive ratio of online algorithms with sublinear advice can be deduced from Eva’s reduction? A short justification is sufficient.

Answer: No online algorithm can guess more than half of bits correctly. In that case, the benefit of the algorithm will be at most $(n/2) \times 2 + (n/2) \times 8 = 5n$ for $n$ guesses. The benefit of $\text{Opt}$ will be a $8n$ (it guesses all bits correctly). The lower bound for competitive ratio will be $\frac{8n}{5n} = \frac{8}{5}$. That is, with sublinear advice, one cannot achieve a competitive ratio better than $\frac{8}{5}$ for $P$.

g) Indicate the output of the Rank algorithm for the following instance of the bipartite matching problem. The number on the left show the indices in the random permutation. Assume vertices on the right appear from top to bottom. Write your answers by indicating pairs of matched vertices. No justification is required.

Answer: $(A, a), (E, b), (C, c), (F, d), (D, c)$
Problem 3  Bin Packing [10 marks]

Consider instances of the bin packing problem in which all items are in the range \((1/7, 1/4)\). Use a weighting-function argument to provide an upper bound for the competitive ratio of the Harmonic Algorithm for these instances of the problem.

Answer:

In this range, there are only three classes of the Harmonic algorithm, that is class 4 (items in the range \((1/5, 1/4]\)), class 5 (items in the range \((1/6, 1/5]\)), and class 6 (items in the range \((1/7, 1/6]\)). As before, the weight of items in classes 4, 5, and 6 are respectively \(1/4\), \(1/5\), and \(1/6\). Since each bin of class 4 (resp. 5 and 6) includes 4 (resp. 5 and 6) items (except possibly for the last bin of each class), the weight of all bins, except possibly three, of Harmonic is 1. As before, the density of items in class 4 is more than those in class 5 which indeed have more density that items in class 6. So, the weight of a bin of Opti s maximized if there are 4 items of class 4 (of size \(1/5 + \epsilon\)) with total weight of \(4 \times (1/4) = 1\), and one item of class 5 of size \(1/6 + \epsilon\) and weight \(1/5\). Hence, the total weight of all items in an optimal packing is no more than \(1+1/5 = 1.2\). This shows an upper bound of 1.2 for the competitive ratio.
Problem 4 Bipartite Matching with Advice [10 marks]

Use a reduction from Binary Guessing problem to make a statement like the following for the online bipartite matching problem:

In order to achieve a competitive ratio better than 4/3 for an \( n \times n \) bipartite graph, advice of size \( \Omega(n) \) is required.

As a guideline, your answer should indicate I) How the bipartite graph is formed (what the input graph and its phases are) II) what the binary guesses are translated to in the bipartite matching instance III) How many binary guesses the bipartite guessing algorithm makes (for an \( n \times n \) graph). IV) How you deduce the lower bound of 4/3 for competitive ratio.

Answer:

I) The graph is similar to the lower bound graph we saw in the slides. Each pair of consecutive vertices on the right form either a cross or a parallel gadget with vertices of the same rank on the left. The first edge that appears from each gadget is shared between the two gadgets. An optimal algorithm matches two edges in each gadget. II) Upon arrival of the first (common) edge of each gadget, an online algorithm has to make a guess for the type of the gadget; if it matches the endpoints of the common edge, the guess is ‘parallel’ and if it does not match, the guess is ‘cross’. III) There are \( n/2 \) gadgets, hence, there will be \( n/2 \) guesses. IV) From binary guessing lemma, we know that, with sublinear advice, an online algorithm cannot guess more than half of bits correctly. That is, for at least half of gadgets, the algorithm matches only one edge while \( \text{OPT} \) matches two edges for each gadget. So, in the best case for the algorithm, for half of gadgets it matches 2 edges \((n/4 \times 2)\) and for the other half, it matches 1 edge \((n/4 \times 1)\). This sums to \( 3n/4 \). As mentioned earlier, \( \text{OPT} \) matches \( n \) edges, and the competitive ratio will be at least 4/3 with sublinear advice.
Use this blank page if you needed more space for your answers.