COMP 2140 - Data Structures

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Topic 4 - Recursion

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Based on notes by S. Durocher.
Overview

- what is recursion?
- recursion vs. iteration
- analyzing the running time of recursive algorithms
Recursion

- The term **recursion** refers to a method that calls itself.
- Recursion is a powerful programming technique that results in efficient algorithms with concise descriptions.
Recursion example

Suppose we replace every line segment by eight shorter line segments according to the following geometric rule:
Recursion example

- Suppose we replace every line segment by eight shorter line segments according to the following geometric rule:

- By applying this rule recursively we obtain the following “fractal”:
Fractals are beautiful “creatures” often built on the recursion principle:
Iteration versus Recursion

- An iterative algorithm for solving a problem $P$ makes use of a loop to compute a sequence of analogous steps that solve $P$. 

\[ n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1. \]
Iteration versus Recursion

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- A recursive algorithm for solving a problem $P$ computes one step and calls itself to solve the remaining subproblem.

$$n! = \begin{cases} 
1 & n \leq 1 \\
(n(n - 1))! & n > 1 
\end{cases}$$
**Iteration versus Recursion**

- an iterative algorithm for computing n!:
  ```java
  int factorialIterative(int n) {
      int result = 1;
      for (int i = 1 ; i <= n ; i++)
          result *= i;
      return result;
  }
  ```
Iteration versus Recursion

- An iterative algorithm for computing \( n! \):

```c
int factorialIterative(int n) {
    int result = 1;
    for (int i = 1; i <= n; i++)
        result *= i;
    return result;
}
```

- A recursive algorithm for computing \( n! \):

```c
int factorialRecursive(int n) {
    int result = 1;
    if (n > 1) result = n * factorialRecursive(n - 1);
    return result;
}
```
Tracing a Recursive Function Call

```java
factorialRecursive(5) = 5 * factorialRecursive(4);
factorialRecursive(4) = 4 * factorialRecursive(3);
factorialRecursive(3) = 3 * factorialRecursive(2);
factorialRecursive(2) = 2 * factorialRecursive(1);
factorialRecursive(1) = 1;
```

For the factorial problem, both iterative and recursive functions run in time linear to $n$. 
The Fibonacci numbers are the sequence

0, 1, 1, 2, 3, 8, 13, 21, 34, 55, ...
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0, 1, 1, 2, 3, 8, 13, 21, 34, 55, …

The first two numbers in the sequence are 0 and 1.

Each succeeding number (third, fourth, …) is defined as the sum of the two numbers that precede it in the sequence.
Fibonacci numbers

- The Fibonacci numbers are the sequence
  
  \[0, 1, 1, 2, 3, 8, 13, 21, 34, 55, \ldots\]

- The first two numbers in the sequence are 0 and 1.
- Each succeeding number (third, fourth, ...) is defined as the sum of the two numbers that precede it in the sequence.
- That is, Fibonacci numbers are defined recursively:

\[
\text{fib}(n) = \begin{cases} 
0 & n = 0 \\
1 & n = 1 \\
\text{fib}(n) + \text{fib}(n - 1) & n > 1 
\end{cases}
\]
A Recursive Solution

```c
int fib(int n) {
    int result = 0;

    if (n == 1)
        result = 1;
    else if (n > 1)
        result = fib(n-1) + fib(n-2);

    return result;
}
```

Unlike our solution for computing factorial numbers, every call to fib generates *two recursive calls*. 
Recursion Tree

\[
\begin{align*}
\text{fib}(5) & : 5 \\
\text{fib}(4) & : 3 \\
\text{fib}(3) & : 2 \\
\text{fib}(2) & : 1 \\
\text{fib}(1) & : 1 \\
\text{fib}(0) & : 0
\end{align*}
\]
This implementation of Fibonacci numbers is inefficient because it recomputes values that have already been found.
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A more efficient solution can be obtained by storing Fibonacci numbers that have already been computed in an array.
int fib2(int n) {
    int [] F = new int [n+1];
    for (int i = 0 ; i < n + 1 ; i++) F[i] = -1;
    return fibMemo(n, F);
}

int fibMemo(int n, int [] F) {
    if (n <= 1) F[n] = n;
    else if (n > 1 && F[n] == -1)
        F[n] = fibMemo(n-1, F) + fibMemo(n-2, F);
    return F[n];
}
Efficient Fibonacci Computation

```java
int fib2(int n) {
    int [] F = new int [n+1];
    for (int i = 0; i < n + 1; i++) F[i] = -1;
    return fibMemo(n, F);
}

int fibMemo(int n, int [] F) {
    if (n <= 1) F[n] = n;
    else if (n > 1 && F[n] == -1)
        F[n] = fibMemo(n-1, F) + fibMemo(n-2, F);
    return F[n];
}
```

- fib2 is a public function:
  - It calls its private fibMemo function.
  - fib2 just initializes all values to -1
  - A value of -1 for entry $F[i]$ means the Fibonacci number of $i$ is not computed yet.
Recursion Tree

- Blue numbers indicate the order at which the functions are called.
- When the value of $F[i]$ is not -1 (is already computed), the recursion functions are not called (the calculated number is returned).
Recursion Tree for Efficient Recursion Trees

For large values of $n$, the tree looks like this:

- Nodes with the same color are calling the same function (with the same parameter)
A tail call is a method $B$ called inside a method $A$, whose return value is immediately returned by the calling method $A$.

```java
int methodA(int x) {
    int y = x + 2;
    return methodB(y);
}
```

No code is executed in the calling method $A$ after returning from the method call to $B$. 
A **tail call** is a method $B$ called inside a method $A$, whose return value is immediately returned by the calling method $A$.

```java
int methodA(int x) {
    int y = x + 2;
    return methodB(y);
}
```

- No code is executed in the calling method $A$ after returning from the method call to $B$.
- Advantage: no need to store local variables on the call stack. A tail recursive algorithm uses tail calls.
Tail Recursion

- A tail call to the same function!
  - Tail recursion often requires an additional parameter that accumulates a partial solution.
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- Tail recursion often requires an additional parameter that accumulates a partial solution.
- E.g.: summing integer values in a linked-list using tail recursion:
  - Here partialSum accumulates partial solution!

```java
int listSumTail(Node node) {
    return listSumTailHelper(0, node);
}

int listSumTailHelper(int partialSum, Node node) {
    int result = partialSum;

    if (node != null)
        result = listSumTailHelper(partialSum + node.getValue(),
                                   node.getNext());

    return result;
}
```
Tail Recursion

- A tail call to the same function!
  - Tail recursion often requires an additional parameter that accumulates a partial solution.
- E.g.: summing integer values in a linked-list using tail recursion:
  - Here `partialSum` accumulates partial solution!

```java
int listSumTail(Node node) {
    return listSumTailHelper(0, node);
}

int listSumTailHelper(int partialSum, Node node) {
    int result = partialSum;
    if (node == null) return partialSum;
    else if (node != null) {
        result = listSumTailHelper(partialSum + node.getValue(), node.getNext());
    }
    return result;
}
```
Tail Recursion

summing integer values in a linked-list without using tail recursion:

```java
int listSum(Node node) {
    int result = 0;

    if (node != null)
        result = node.getValue() + listSum(node.getNext());

    return result;
}
```
Tail Recursion

- summing integer values in a linked-list using tail recursion:

- summing integer values in a linked-list without using tail recursion:
Tail Recursion Summary

- When writing a recursive code, consider applying the tail recursion.
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- Tail recursion can significantly improve the memory usage of your algorithm (in the stack memory).
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- When writing a recursive code, consider applying the tail recursion.
- Tail recursion can significantly improve the memory usage of your algorithm (in the stack memory).
- The time complexity does not change (asymptotically).
- You often need to pass partial solutions in the recursive calls.
Recursion vs. Iteration

- Every recursive algorithm can be written iteratively.
Recursion vs. Iteration

- Every recursive algorithm can be written iteratively.
- A recursive algorithm (what does it do?)

```java
void recursiveFn(int n) {
    System.out.println("n = " + n);
    if (n > 1)
        recursiveFn(n/2);
}
```
Recursion vs. Iteration

- **Every recursive algorithm can be written iteratively.**

- A recursive algorithm (what does it do?)

```java
void recursiveFn(int n) {
    System.out.println("n = " + n);
    if (n > 1)
        recursiveFn(n/2);
}
```

- Iterative version of the same algorithm

```java
void iterativeFn(int n) {
    for (int i = n ; i >= 1 ; i /= 2)
        System.out.println("n = " + i);
}
```
Recursion vs. Iteration

- Every recursive algorithm can be written iteratively.
- A recursive algorithm (what does it do?)

```java
void recursiveFn(int n) {
    System.out.println("n = " + n);
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- Iterative version of the same algorithm

```java
void iterativeFn(int n) {
    for (int i = n ; i >= 1 ; i /= 2)
        System.out.println("n = " + i);
}
```

- These two algorithms run in the same time!
Recursion vs. Iteration

- What is the running time of iterativeFn?

On the first iteration:
\[ i = n \]

On the second iteration:
\[ i = n / 2 \]

On the third iteration:
\[ i = n / 4 \]

On the fourth iteration:
\[ i = n / 8 \]

... 

On the \( k \)th iteration:
\[ i = n^{2k-1} \]

... 

On the last iteration:
\[ i = 1 \]

The loop stops when:
\[ i < 1 \]

\[ n^{2k-1} < 1 \]

\[ \log n < k - 1 \]

\[ k > \log n + 1 \]

Therefore, the loop iterates \( \log n + 1 \) times!

Remember! \( \log n \) always means \( \log_2 n \)
Recursion vs. Iteration

- What is the running time of iterativeFn?
- How many times does the for loop iterate?
  - On the first iteration $i = n$
  - On the second iteration $i = n/2$
  - On the third iteration $i = n/4$
  - On the fourth iteration $i = n/8$
  - ... 
  - On the $k$th iteration $i = \frac{n}{2^{k-1}}$
  - ... 
  - On the last iteration $i = 1$

Therefore, the loop iterates $\log n + 1$ times!

Remember! $\log n$ always mean $\log_2(n)$
Recursion vs. Iteration

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  - ...  
  - On the $k$th iteration $i = \frac{n}{2^{k-1}}$
  - ...  
  - On the last iteration $i = 1$
- The loop stops when $i < 1$
  \[
  \frac{n}{2^{k-1}} < 1 \rightarrow n < 2^{k-1} \rightarrow \log n < k - 1 \rightarrow k > \log n + 1
  \]
Recursion vs. Iteration

- What is the running time of iterativeFn?
- How many times does the for loop iterate?
  - On the first iteration \( i = n \)
  - On the second iteration \( i = n/2 \)
  - On the third iteration \( i = n/4 \)
  - On the fourth iteration \( i = n/8 \)
  - \( \ldots \)
  - On the \( k \)th iteration \( i = \frac{n}{2^{k-1}} \)
  - \( \ldots \)
  - On the last iteration \( i = 1 \)
- The loop stops when \( i < 1 \)
  - \( \frac{n}{2^{k-1}} < 1 \rightarrow n < 2^{k-1} \rightarrow \log n < k - 1 \rightarrow k > \log n + 1 \)
- Therefore, the loop iterates \( \log n + 1 \) times!
  - Remember! \( \log n \) always mean \( \log_2(n) \)
Determining the Running Time of Recursive Algorithms

- We don't need to rewrite a recursive algorithm as an iterative algorithm to find its running time.
- A recurrence relation can be used to express the running time of a recursive algorithm.
- Solving recurrence relations requires using a proof by induction.
- Mathematical induction and recurrence relations are covered in COMP 2080 and COMP 2130.
Next topic is sorting.

What is a **Comparison Based Sorting**?