The whole purpose of education is to turn mirrors into windows.

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All problems are written problems; submit your solutions electronically only via Crowdmark. Most questions include an example accompanied with an answer which is aimed to provide some guideline on how the solutions should look like. Think of them as a tool for reviewing the material. Your solutions do not necessarily need to look like provided answers. There are 66 marks available. Please read [http://www.cs.umanitoba.ca/~kamalis/winter19/infoCOMP3170.pdf](http://www.cs.umanitoba.ca/~kamalis/winter19/infoCOMP3170.pdf) for guidelines on academic integrity.

Throughout the assignment, all logarithms are based 2 logarithms, i.e., \( \log x = \log_2(x) \).

Problem 1 \([4+4+4+4+4=20 \text{ marks}]\)

Provide a complete proof of the following statements from first principles (i.e., using the original definitions of order notation).

Ex.) \(2n^3 + 14n^2 + 28n \in O(2n^3)\)

Let \( M \) be any value larger than 22 and \( n_0 := 1 \). Then \( 2n^3 + 14n^2 + 28n \leq 44 \cdot n^3 < M(2n^3) \) for all \( n \geq n_0 \).

a) \( n + \frac{n^2}{4 + \sin(n)^3} \in O(n^2)\)

Answer: For any value of \( n \), we have \( \sin(n) \geq -1 \) and hence \( \frac{n^2}{4 + \sin(n)^3} \leq \frac{n^2}{3} \). So \( n + \frac{n^2}{4 + \sin(n)^3} \leq n + \frac{n^2}{3} \leq n^2 + \frac{n^2}{3} = \frac{4}{3}n^2 \). So, it suffices to have \( n_0 \geq 1 \) and \( M \geq \frac{4}{3} \).

b) \( n^6(\log n) \in \Omega(n^5(\log n)^2)\).

Answer: We need to provide \( n_0 \) and \( M \) s.t. for \( n > n_0 \) we have \( n^6 \log n \geq Mn^5(\log n)^2 \), i.e., \( n \geq M \log n \). Since \( n > \log n \) for all positive integers, it suffices to have \( n_0 \geq 1 \) and \( M \leq 1 \).

c) \( 10n^2/(n - 100) \in \Theta(n) \).

Answer: Assume \( n_0 \geq 200 \). For \( n > n_0 \), we have \( n/2 \leq n - 100 \leq n \), which implies \( 1/n \leq \frac{1}{n-100} \leq 2/n \); multiplying all by \( 10n^2 \), we get \( 10n \leq \frac{10n^2}{n-100} \leq 20n \). So is suffices to define \( M_1 \leq 10 \) and \( M_2 \geq 20 \).
d) $2019n \in o(n \log n)$

Answer: Given any value of $M$, we should provide $n_0$ so that $2019n < Mn \log n$, i.e., $2019/M < \log n$. For this to hold, it suffices to have $n > 2^{2019/M}$. So it suffices to define $n_0$ as $\max\{1, 2^{2019/M}\}$.

e) $n^{n/2} \in \omega(n^2)$

Answer: We should find an $n_0$ such that for $n > n_0$ we have $n^{n/2} > Mn^2$ which is equivalent to $\frac{2}{n} \log n > \log M + 2 \log n$, that is log $M < (n/2 - 2) \log n$. For $n \geq 6$ we have $n/2 - 2 \geq 1$. So, it suffices to sow $\log M < 6 \log n$. This holds as long as $n \geq 2^{\log M}$. So, we set $n_0 = \max\{6, 2^{\log M/6}\}$.

**Problem 2  [4+4+4=12 marks]**

For each pair of the following functions, fill in the correct asymptotic notation among $\Theta, o,$ and $\omega$ in the statement $f(n) \in \bigcap(g(n))$. Provide a brief justification of your answers. In your justification you may use any relationship or technique that is described in class.

Ex.) $f(n) = n(\log n)^3$ versus $g(n) = n^2$ We have $\lim_{n \rightarrow \infty} \frac{n(\log n)^{3}}{n^2} = \lim_{n \rightarrow \infty} \frac{(\log n)^{3}}{n} = \lim_{n \rightarrow \infty} \frac{3(\log n)^{2}}{n \ln^2} = \lim_{n \rightarrow \infty} \frac{6 \log n}{n(\ln 2)^2} = \lim_{n \rightarrow \infty} \frac{6}{n(\ln 2)^2} = 0$. Hence we have $f(n) = o(g(n))$. Note that we applied L’Hopital rule three times.

a) $f(n) = n^{3/2}$ and $g(n) = n \log(n)$.

Answer: We have $\lim_{n \rightarrow \infty} \frac{n^{3/2}}{n \log n} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{n} = \lim_{n \rightarrow \infty} \left(\frac{n}{\log n}\right)^{3/2} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{2} = \infty$. Hence we have $f(n) = \omega(g(n))$.

b) $2^n$ versus $3^{n/2}$.

Answer: We have $\lim_{n \rightarrow \infty} \frac{2^n}{3^{n/2}} = \lim_{n \rightarrow \infty} \frac{\ln 2}{\ln 3} \frac{2^n}{3^{n/2}} = \lim_{n \rightarrow \infty} \frac{2n \ln 2}{3^{n/2}} \approx 1.26 \lim_{n \rightarrow \infty} \frac{2^n}{3^{n/2}}$. So, if the value of the limit is $A$, it increases by a factor larger than 1 in each iteration, meaning that the limit goes to infinity. As such, we have $2^n \in \omega(3^{n/2})$.

c) $f(n) = n^3(5 + 4 \sin 2n)$ versus $g(n) = n^2 + 2n^3 + 3n$

Answer: For any value of $n$ we have $1 \leq 5 + 4 \sin 2n \leq 9$. Hence $n^3 \leq f(n) \leq 9n^3$. Similarly, for any value of $n$ we have $n^3 \leq n^2 + 2n^3 + 3n \leq 6n^3$, that is $n^3 \leq g(n) \leq 6n^3$. We can conclude $\frac{1}{6}g(n) \leq n^3 \leq f(n) \leq 9n^3 \leq 9g(n)$. So, it suffices to have $M_1 \leq 1/6$, $M_2 \geq 9$ and $n_0 = 1$.

**Problem 3  [5+5+5=15 marks]**

Prove or disprove each of the following statements. To prove a statement, you should provide a formal proof that is based on the definitions of the order notations. To disprove a statement, you can either provide a counter example and explain it or provide a formal proof. All functions are positive functions.
Ex.) \( f(n) \not\in o(g(n)) \) and \( f(n) \not\in \omega(g(n)) \Rightarrow f(n) \in \Theta(g(n)) \)

False. Counter example: Consider \( f(n) := n \) and \( g(n) := \begin{cases} 1 & n \text{ odd} \\ n^2 & n \text{ even} \end{cases} \). To prove the claim false it will be sufficient to show that \( f(n) \not\in O(g(n)) \) and \( f(n) \not\in \Omega(g(n)) \), since then the antecedent of the implication is satisfied while the consequent is not.

If \( f(n) \in O(g(n)) \), then there exist constants \( n_0 > 0 \) and \( c > 0 \) such that \( f(n) \leq cg(n) \) for all \( n \geq n_0 \). But for any odd number \( n_1 > c \) we have \( f(n_1) = n_1 > c = cg(n_1) \), showing that \( f(n) \not\in O(g(n)) \).

Similarly, if \( f(n) \in \Omega(g(n)) \), then there exists constants \( n_0 > 0 \) and \( c > 0 \) such that \( cg(n) \leq f(n) \) for all \( n \geq n_0 \). But for any even number \( n_1 > 1/c \) we have \( cg(n_1) = cn_1^2 > n_1 = f(n_1) \), showing that \( f(n) \not\in \Omega(g(n)) \).

a) \( f(n) \in \Theta(g(n)) \Rightarrow g(n) \in \Theta(f(n)) \)

Answer: \( f(n) \in \Theta(g(n)) \), for large values of \( n \) we have \( M_1g(n) \leq f(n) \leq M_2g(n) \) for some \( M_1 \) and \( M_2 \). This means we have \( \frac{1}{M_2}f(n) \leq g(n) \leq \frac{1}{M_2}f(n) \), which shows \( g(n) \Theta(f(n)) \).

b) \( f(n) \in \Theta(g(n)) \) and \( h(n) \in \Theta(g(n)) \Rightarrow \frac{f(n)}{h(n)} \in \Theta(1) \)

Answer: True. Proof: Assume that \( f(n) \in \Theta(g(n)) \) and \( h(n) \in \Theta(g(n)) \), and let \( n_1, n_2 > 0 \) and \( c_1, c_2, c_3, c_4 > 0 \) be such that \( c_1g(n) \leq f(n) \leq c_2g(n) \) for all \( n \geq n_1 \) and \( c_3g(n) \leq h(n) \leq c_4g(n) \) for all \( n \geq n_2 \). Since \( g \) and \( h \) are positive, for every \( n \geq n_2 \) we have

\[
\frac{1}{c_4g(n)} \leq \frac{1}{h(n)} \leq \frac{1}{c_3g(n)}.
\]

Let \( n_0 = \max\{n_1, n_2\} \). Then for every \( n \geq n_0 \) we have

\[
\frac{c_1g(n)}{c_4g(n)} \leq \frac{f(n)}{h(n)} \leq \frac{c_2g(n)}{c_3g(n)} \Rightarrow \frac{c_1}{c_4} \leq \frac{f(n)}{h(n)} \leq \frac{c_2}{c_3}.
\]

Selecting constants \( c'_1 = \frac{c_1}{c_4} \) and \( c'_2 = \frac{c_2}{c_3} \) we have \( c'_1 \leq \frac{f(n)}{h(n)} \leq c'_2 \) for every \( n \geq n_0 \). Thus, according to the definition of \( \Theta \) we have \( \frac{f(n)}{h(n)} \in \Theta(1) \).

c) \( f(n) \in o(g(n)) \Rightarrow \log(f(n)) \in o(\log(g(n))) \)

Answer: False. Counter example: Consider \( f(n) = n \) and \( g(n) = n^2 \). Then \( f(n) \in o(g(n)) \) but \( \log f(n) = \Theta(\log n) \) and \( \log g(n) = \Theta(\log n^2) = \Theta(2 \log n) = \Theta(\log n) \). So, \( \log(n) \in \Theta(\log g(n)) \).
Problem 4  [7 marks]

Analyze the following piece of pseudocode and give a tight (Θ) bound on the running time as a function of \( n \). Show your work. A formal proof is not required, but you should justify your answer.

```plaintext
1. dog ← 0
2. for i ← 1 to 2n do
3.    for j ← 1397 to 2019 do
4.       dog ← dog × 4
5.    for k ← i to i^2 do
6.       dog ← dog * k * k
```

Answer: The following calculation shows that the time complexity is \( \Theta(n^3) \).

\[
T(n) = \sum_{i=1}^{2n} \sum_{j=1397}^{2019} \sum_{k=i}^{i^2} c
\]

\[
= \sum_{i=1}^{2n} \sum_{j=1397}^{2019} c (i^2 - i) = \sum_{i>1} 622c (i^2 - i) \approx
\]

\[
\approx 622c \sum_{i>1} 2n - 622c \sum_{i>1} i = 622c \frac{2n(2n+1)(2n+2)}{6} - 622c \frac{2n^2}{2}
\]

\[
\approx 622c \frac{n^3 + o(n^3)}{6} - o(n^2)
\]

Problem 5  [4+4+4=12 marks]

For each of the following recurrences, give an expression for the runtime \( T(n) \) if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply. For all cases, we have \( T(x) = 1 \) when \( x \leq 100 \) (base of recursion).

Ex.) \( T(n) = 3T(n/3) + \sqrt{n} \) We have \( n^{\log_b a} = n \). Since \( f(n) = O(n^{1-\epsilon}) \) (for any \( \epsilon < 1/2 \)), we are again at case 1 and \( T(n) = \Theta(n) \).
a)  $T(n) = 4T(n/3) + 2019n$
Answer: We have $n^{\log_4 a} = n^{\log_3 4} = n^{1.26}$ and hence $f(n) = \Theta(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon$ (any $\epsilon < 0.26$ works). Hence we are at case I and we can state $T(n) = \Theta(n^{1.26})$.

b)  $T(n) = 9T(n/3) + 1984n^3$
Answer: We have $n^{\log_b a} = n^2$. Hence we have $f(n) \in \Omega(n^{2+\epsilon})$ for $\epsilon < 1$. We are in case 3 and need to check regularity condition. We have $af(n/b) = 1984 \cdot 9(n/3)^3 = (1/3)1984n^3$. So, for $c \in [1/3, 1)$, the regularity condition holds and we have $T(n) \in \Theta(n^3)$.

c)  $T(n) = 9T(n/2) + \frac{n^2}{\log n}$
Answer: We have $n^{\log_b a} > n^{2+\epsilon}$ for any $\epsilon \log 9 - 2 \approx 1.69$. Also $f(n) = O(n^2)$. So we are in case I and we have $T(n) \in \Theta(n^{\log 9})$. 