Binomial Heaps

CLRS 6.1, 6.2, 6.3

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Priority queues

- A **priority queue** is an abstract data type formed by a set $S$ of key-value pairs.

- **Basic operations** include:
  - **insert** ($k$) inserts a new element with key $k$ into $S$.
  - **get-Max** which returns the element of $S$ with the largest key.
  - **extract-Max** which returns the element of $S$ with the largest key and delete it from $S$.

- We are often given the whole data and need to **build** the data structure based on it.
  - Any data structure for a priority queue should be **constructed** efficiently.
Priority queue implementation

- What is a good implementation (data structure) for priority queues?
- You have seen **binary heaps** before: get-Max runs in $O(1)$ and extract-Max and insert both take $\Theta(\log n)$ for $n$ keys.
- Is a balanced binary search tree a good implementation of a priority queue?
  - with a little augmentation, get-Max runs in $O(1)$ and extract-Max and insert both can run in $\Theta(\log n)$.
- The problem with BSTs: it is costly to build them
  - How long does it take to form a BST from a given set of items?
  - It takes $\Omega(n \log n)$; otherwise you can sort them in $o(n \log n)$ by building the BST and doing an inorder traverse in $O(n)$.
  - We know we cannot comparison-sort in $o(n \log n)$ and hence cannot build the tree in such time.
Binary heaps

- A **heap** is a **tree** data structure
- For every node $i$ other than the root, we have $\text{key}[\text{parent}[i]] \geq \text{key}[i]$.
- A **binary** heap is a complete binary tree which can be stored using an array.
  - build-heap takes $\Theta(n)$ time
  - insert, extract-Max take $\Theta(\log n)$
  - get-Max takes $O(1)$
Binary heaps

- Suppose multiple priority queues on different servers.
- Occasionally a server must be rebooted, requiring two priority queues to be **merged**.
- With a typical binary heap, merging requires concatenating arrays and **re-running** build-heap; this takes $\Theta(n)$ :-(
- When implementing an abstract data type always consider if you need it to be **mergable** or not.
Rethinking about Data Structure

- We would like to build a data structure for priority queues that:
  - supports insert, extract-Max, get-Max, and build efficiently (as in binary heaps)
  - merging two priority queues takes $o(n)$
- Solution: **binomial heaps** which are mergable heaps that efficiently support

  - $\text{insert}(H, x)$
  - $\text{extract-Max}(H)$
  - $\text{get-Max}(H)$
  - $\text{build}(A)$
  - union($H_1, H_2$) (merge)
  - $\text{increase-key}(H, x, k)$
  - $\text{delete}(H, x)$
A **binomial tree** is an ordered tree defined recursively

- children of each node have a specific ordering (similar to ‘left’ and ‘right’ child in binary trees).

The base case for a binomial tree $B_0$ is a single node.

To build $B_k$, we take two copies of $B_{k-1}$ and let the first child of the root of the second copy be the root of the first copy.
Fun with Binomial Trees

- Fun 1: The children of the root of the binomial tree $B_k$ are the binomial trees $B_{k-1}, \ldots, B_0$.
  - Induction: assume it is true for all binomial trees $B_i$ with $i \leq k - 1$ (base easily holds).
  - The tree $B_k$ has its first child as $B_{k-1}$ (recursive construction).
  - With respect to other children, it is a binomial tree $B_{k-1}$ and hence has children $B_{k-2}, \ldots, B_0$ by induction hypothesis.

![Diagram of binomial tree]

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Fun with Binomial Trees

- Fun 2: $B_k$ has $2^k$ nodes:
  - The recursion is $N(B_k) = 2N(B_{k-1}), N(B_0) = 1$

- $B_k$ has height $k$:
  - The recursion is $h(B_k) = h(B_{k-1}) + 1$

- Within $B_k$ there are $\binom{k}{i}$ nodes at depth $i$.
  - The recursion is $ch(k, i) = ch(k - 1, i - 1) + ch(k - 1, i)$
  - Solving this recursion gives $ch(k, i) = \binom{k}{i}$. To get an idea of the proof, note that $\binom{k}{i} = \binom{k-1}{i-1} + \binom{k-1}{i}$
Binomial Heaps

Definition

A **binomial heap** is a set of binomial trees such that:

- each binomial tree is heap-ordered \( key[parent[i]] \geq key[i] \)
- for each \( k \) there is at most one binomial tree of order \( k \)
Number of Trees in Binomial Heaps

- How many trees are in a binomial heap of \( n \) nodes?
  - Let \( x \) be the number of trees
  - We express the number of nodes as a function of \( x \)
  - The number of nodes is minimized when there is one tree of order \( i \) for any \( i \in [0, x - 1] \) (note that no two trees of same order can exist).
    - Recall that a binomial tree of order \( i \) has \( 2^i \) nodes.
    - We have \( n \geq 1 + 2 + \ldots + 2^{x-1} = 2^x - 1 \), i.e., \( x \leq \log(n + 1) \)

- A binomial heap storing \( n \) keys has at most \( \log(n + 1) \) binomial trees.
Finding Max in Binomial Heaps

- For `get-Max()` operation, just follow the links connecting roots of binomial trees
  - The maximum element in all the heap is the max node, hence root, in one of the trees
  - E.g., max in the below heap is \( \max\{11, 99, 40\} = 90 \)
- There are \( \log(n + 1) \) trees and hence the time complexity is \( \Theta(\log n) \).
  - It is a bit worse than \( O(1) \) of `get-Max()` in binary heaps
Merging of Two Binomial Heaps

- Union operation: we want to merge two heaps of sizes $n_1$ and $n_2$.
  - Similar to merge operation in merge sort, follow the links connecting roots of the heaps, and ‘merge’ them into one list (i.e., one heap).
  - If two trees of same order $i$ are visited, merge them into a binomial tree of order $i + 1$
    - It is possible by the definition of binomial tree.
    - The tree with the smaller key in its root becomes a child of the other tree.
    - Two trees can be merged in $O(1)$.
  - When 3 trees of order $i$, merge the 2 older trees (keep the new one).
Merging of Two Binomial Heaps

- There is an analogy with **binary** addition: add bits and carry
  - Read from the least significant to the most significant bit (right to left)
  - $111 + 011 = 1010$; “1010” means 1 tree of order 3, 0 tree of order 2, 1 tree of order 1, and 0 tree of order 0.
What is time complexity of merge?

- Each merge operation takes $O(1)$.
- For each tree rank, there will be at most one merge
- The total time complexity is $O(\log(n_1) + \log(n_2)) = O(2 \log(\max\{n_1, n_2\})) = O(\log n)$ where $n$ is the size after the merge.

It is possible to merge two binomial heaps in $O(\log n)$ where $n$ is the number of keys after the merge.
Insert Operation

To insert a new key $x$ to the priority queue:
- Create a new binomial heap of size 1 (order 0) with the new key
- Return the union of the old heap with the new one (e.g., Insert(40))
- The time complexity is similar to merge.

It is possible to insert a new item to a binomial heap in $O(\log n)$, which is as good as binary heaps.
Extract-Max Operation

- To extract max, first search and find the maximum.
  - Assuming max is in a binomial tree of order $k$, its children are $k$ binomial trees of order $1, 2, \ldots, k - 1$
  - Delete max and create a new binomial heap formed by these trees.
  - Merge the old heap and the new one.
  - The time complexity is $O(\log n)$ for finding the max and $O(\log n)$ for merging the two heaps, i.e., $O(\log n)$ in total

- It is possible to extract maximum element in a binomial heap in $O(\log n)$, which is as good as binary heaps
Bionmial Heaps Review

- Get-Max can be done in $\Theta(\log n)$ (a bit slower than $\Theta(1)$ of binary heaps).
- Merge can be done in $\Theta(\log n)$ (much better than $\Theta(n)$ of binary heaps).
- Insert and Extract-Max can be done in $\Theta(\log n)$ (similar to binary heaps)
Increase Key

- Increase\((a,x)\): assume you are given a pointer to a key \(a\) and want to increase it by \(x\).
  - Note that if the pointer is not given, you need to search for the key, which takes \(\Theta(n)\) in any heap (heaps are NOT good for searching).

- Increase the key and ‘float’ it upward until \(key[parent[i]] \geq key[i]\) (e.g., increase ’8’ to ’68’).

- Time is proportional to the height of a binomial tree, i.e., the order of the tree
  - Recall that a binomial tree of order \(k\) has \(2^k\) nodes, so, the order and hence the height of any tree in the heap is \(O(\log n)\).

Increase the key of a given node can be done in time \(\Theta(\log n)\).
Delete

Delete(a): assume you are given a pointer to a key a and want to delete it

- Call Increase-key to set the key to \( \infty \).
- Call Extract-Max to remove the largest item; this would remove our node from the heap

Time is \( O(\log n) \) for Increase-key and \( O(\log n) \) for Extract-Max.

Deleting a given node can be done in time \( O(\log n) \).
Binomial Heaps Summary

- Given a key (a pointer to its node), we can increase or delete that node in $O(\log n)$.

**Theorem**

*Priority queries can be implemented with binomial tree so that Get-Max, Merge, Extract-Max, Increase (with given pointer) and delete (with given pointer) can all be performed in $O(\log n)$.*