COMP 3170 - Analysis of Algorithms & Data Structures

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Lecture 2 - Jan. 9, 2019
CLRS 1.1, 1.2, 2.2, 3.1, 4.3, 4.5
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Picture is from the cover of the textbook CLRS.
Asymptotic Analysis
Algorithms (review)

- An **algorithm** is a step-by-step procedure carrying out a computation to solve an arbitrary instance of a problem.
  - E.g., sorting is a problem; a set of numbers form an instance of that and ‘solving’ involves creating a sorted output.
- A **program** is an implementation of an algorithm using a specific programming language
- For a given problem (e.g., sorting) there can be several algorithms (e.g., Quicksort, Mergesort), and for a given algorithm (e.g., Quicksort) there can be several programs.
  - Our focus in this course is on algorithms (not programs).
  - How to implement a given algorithm relates to the art of **performance engineering** (writing a fast code)
Given a problem $P$, we need to

- Design an algorithm $A$ that solves $P$ (**Algorithm Design**)
- Verify **correctness** and **efficiency** of the algorithm (**Algorithm Analysis**)
- If the algorithm is correct and efficient, **implement** it
  - If you implement something that is not necessarily correct or efficient in all cases, that would be a **heuristic**.
How should we evaluate different algorithms for solving a problem?

- In this course we are mainly concerned with amount of time it takes to solve a problem (this is called running time)
- We can think of other measures such as the amount of memory that is required by the algorithm
- Other measures include amount of data movement, network traffic generated, etc.

The amount of time/memory/traffic required by an algorithm depend on the size of the problem

- Sorting a larger set of numbers takes more time!
Running Time of Algorithms

How to assess the running time of an algorithm?

**Experimental analysis:**

- Implement the algorithm in a program
- Run the program with inputs of different sizes
- Experimentally measure the actual running time (e.g., using `clock()` from time.h)

Shortcomings of experimental studies:

- We need to implement the program (what if we are lazy and those engineers are hard to employ?)
- We cannot test all input instances for the problem. What are the good samples? (remember the Morphy’s law)
- Many factors have impact on experimental timing, e.g., hardware (processor, memory), software environment (operating system, compiler, programming language), and human factors (how good was the programmer?)
Computational Models

- We need to assess time/memory requirement of algorithms using models that
  - Take into account all input instances
  - Do not require implementation of the algorithms
  - Are independent of hardware/software/programmer

- In order to achieve this, we:
  - Express algorithms using **pseudo-codes** (don’t worry about implementation)
  - Instead of measuring time in seconds, count the number of **primitive operations**
    - This requires an abstract **model of computation**
Random Access Machine (RAM) Model

- The random access machine (RAM):
  - Has a set of memory cells, each storing one ‘word’ of data.
  - Any access to a memory location takes constant time.
  - Any primitive operation takes constant time.
  - The running time of a program can be computed to be the number of memory accesses plus the number of primitive operations.

- Word-RAM is a RAM machine with the extra assumption that all values in our problem can ‘fit’ in a constant number of words (values are not too big).

- We often use Word-RAM model for analysis of algorithms

Observation

RAM is a simplified model which only provides an approximation of a ‘real’ computer
First, calculate the ‘cost’ (sum of memory accesses and primitive operations) for each line

- E.g., in line 5, there are 3 memory accesses and 3 primitive operations
Analysis of Insertion Sort under RAM

Next, find the number of times each line is executed

- This depends on the input, we may consider best or worst case input
- Let $t_j$ be number of times the `while` loop is executed for inserting the $j$'th item.
  - In the best case, $t_j = 1$ and in the worst case $t_j = j$.
- Summing up all costs, in the best case we have
  \[ T(n) = (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8) = an - b \]
  for constant $a$ and $b$
- In the worst case, we have $T_n = \alpha n^2 + \beta n + \gamma$ for constant $\alpha, \beta, \gamma$

\begin{verbatim}
INSERTION-SORT(A)
1 for j = 2 to A.length
2 key = A[j]
3 // Insert A[j] into the sorted sequence A[1..j-1].
4 i = j - 1
5 while i > 0 and A[i] > key
6 A[i + 1] = A[i]
7 i = i - 1
8 A[i + 1] = key
\end{verbatim}

<table>
<thead>
<tr>
<th>cost</th>
<th>times</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$ = 2</td>
<td>$n$</td>
</tr>
<tr>
<td>$c_2$ = 3</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$c_4$ = 2</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$c_5$ = 6</td>
<td>$\sum_{j=2}^n t_j$</td>
</tr>
<tr>
<td>$c_6$ = 4</td>
<td>$\sum_{j=2}^n (t_j - 1)$</td>
</tr>
<tr>
<td>$c_7$ = 2</td>
<td>$\sum_{j=2}^n (t_j - 1)$</td>
</tr>
<tr>
<td>$c_8$ = 3</td>
<td>$n - 1$</td>
</tr>
</tbody>
</table>
Primitive Operations

- RAM model implicitly assumes primitive operations have fairly similar running time

- Primitive operations:
  - basic integer arithmetic (addition, subtraction, multiplication, division, and modulo)
  - bitwise logic and bit shifts (logical AND, OR, exclusive-OR, negation, left shift, and right shift)

- Non-primitive operations:
  - exponentiation, radicals (square roots), logarithms, trigonometric functions (sine, cosine, tangent), etc.
Asymptotic Notations

Statement

So, we can express the cost (running time) of an algorithm A for a problem of size \( n \) as a function \( T_A(n) \).

- How do we compare two different algorithms? say \( T_A(n) = \frac{1}{1000} n^3 \) and \( T_B(n) = 1000n^2 + 500n + 200 \).

- Summarize the time complexity using asymptotic notations!

- Idea: assume the size of input grows to infinity; identify which component of \( T_A(n) \) contributes most to the grow of \( T_A(n) \).

- As \( n \) grows:
  - constants don’t matter (e.g., \( T_A(n) \))
  - low-order terms don’t matter (e.g., \( T_B(n) \))
Asymptotic Notations

- Informally $T_B(n) = O(T_A(n))$ means $T_B$ is asymptotically smaller than or equal to $T_A$.

- Is it sufficient to define $O$ so that we have $T_B(n) < T_A(n)$?
  - No because the inequality might not hold for small values of $n$ which we don’t care about
  - The two function might have constants we would prefer to ignore.

\[
  f(n) \in O(g(n)) \iff \exists M > 0, \exists n_0 > 0 \text{ s.t. } \forall n > n_0, \ f(n) \leq M \cdot g(n).
\]

ignore low-order terms \hspace{2cm} ignore constants
Let \( f(n) = 1000n^2 + 1000n \) and \( g(n) = n^3 \). Prove \( f(n) \in O(g(n)) \)