COMP 3170 - Analysis of Algorithms & Data Structures

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Not in CLRS; material from another textbook will be posted

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Skip Lists

- **Randomized** data structure for dictionary ADT
- A hierarchy of ordered linked lists
- A **skip list** for a set $S$ of items is a series of lists $S_0, S_1, \cdots, S_h$ such that:
  - Each list $S_i$ contains the special keys $-\infty$ and $+\infty$
  - List $S_0$ contains the keys of $S$ in nondecreasing order
  - Each list is a subsequence of the previous one, i.e., $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_h$
  - List $S_h$ contains only the two special keys

![Diagram of skip lists](image-url)
Skip Lists

- A skip list for a set $S$ of items is a series of lists $S_0, S_1, \cdots, S_h$
- A two-dimensional collection of positions: levels and towers
- Traversing the skip list: after($p$), below($p$)
Search in Skip Lists

\[\text{skip-search}(L, k)\]

\(L\): A skip list, \(k\): a key

1. \(p \leftarrow \text{topmost left position of } L\)
2. \(S \leftarrow \text{stack of positions, initially containing } p\)
3. \(\text{while } \text{below}(p) \neq \text{null} \text{ do}\)
4. \(p \leftarrow \text{below}(p)\)
5. \(\text{while } \text{key}(\text{after}(p)) < k \text{ do}\)
6. \(p \leftarrow \text{after}(p)\)
7. \(\text{push } p \text{ onto } S\)
8. \(\text{return } S\)

- \(S\) contains positions of the largest key \textbf{less than} \(k\) at each level.
- \(\text{after} (\text{top}(S))\) will have key \(k\), iff \(k\) is in \(L\).
- \textbf{drop down:} \(p \leftarrow \text{below}(p)\)
- \textbf{scan forward:} \(p \leftarrow \text{after}(p)\)
Search in Skip Lists

Example: Skip-Search(S, 87)
Insert in Skip Lists

- **Skip-Insert** \((S, k, v)\)
  - Randomly compute the height of new item: repeatedly toss a coin until you get tails, let \(i\) the number of times the coin came up heads
  - Search for \(k\) in the skip list and find the positions \(p_0, p_1, \cdots, p_i\) of the items with largest key less than \(k\) in each list \(S_0, S_1, \cdots, S_i\) (by performing \(\text{Skip-Search}(S, k)\))
  - Insert item \((k, v)\) into list \(S_j\) after position \(p_j\) for \(0 \leq j \leq i\) (a tower of height \(i\))
Insert in Skip Lists

Example: Skip-Insert(S, 52, v)
Coin tosses: H, T ⇒ i = 1
\textit{Skip-Search}(S, 52)
Insert in Skip Lists

Example: Skip-Insert($S, 100, v$)
Delete in Skip Lists

- **Skip-Delete** \((S, k)\)
  - Search for \(k\) in the skip list and find all the positions \(p_0, p_1, \ldots, p_i\) of the items with the largest key smaller than \(k\), where \(p_j\) is in list \(S_j\). (this is the same as Skip-Search)
  - For each \(i\), if \(\text{key}(\text{after}(p_i)) = k\), then remove \(\text{after}(p_i)\) from list \(S_i\)
  - Remove all but one of the lists \(S_i\) that contain only the two special keys
Delete in Skip Lists

Example: Skip-Delete($S, 65$)
Skip List Memory Complexity

- What is the expected height of a tower?
  - 1 if random flip sequence is $T$, 2 if it is $H, T$, 3 if it is $H, H, T$.
  - The chance of a tower having height $i$ is $\frac{1}{2^i}$.
    - For that the first $i - 1$ flips should be heads and the $i$'th one a tail.
  - The expected height of a tower will be $X = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + \ldots$
  - We have $X = 1/2 + 2/4 + 3/8 + 4/16 + 5/32 + 6/64 \ldots$, i.e., $X/2 = 1/4 + 2/8 + 3/16 + 4/32 + 5/64 + \ldots$;
    - So, $X - X/2 \leq 1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + \ldots = 1$,
    - i.e., $X = 2$.

- So, the expected height of a tower is 2, i.e., the expected size of the skip list is $2n \in \Theta(n)$.

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**Theorem**

*A skip list that includes $n$ keys is expected to have $\Theta(n)$ nodes.*
Skip List Height

How many levels are expected to be in a linked list of size $n$?

$$\text{Prob}(\text{max height} > c \log n) = \text{Prob}(\text{some element flipped} > c \log n \text{heads})$$

$$\leq n \cdot \text{Prob}(\text{element flipped} > c \log n \text{heads}) \text{ [Boole's ineq.]}$$

$$= n(1/2)^{c \log n} = n/n^c = \frac{1}{n^{c-1}}$$

With a chance of at least $1 - 1/n^{c-1}$, the height of the skip list is at most $c \log n$.

This can be used to show the number of levels in a skip list is expected to be $\Theta(\log n)$

**Theorem**

The height of a skip list on $n$ items is expected to be $\Theta(\log n)$. 
Search Time in Skip Lists

- How many nodes are visited for searching a key \( k \)?
- Think of backward moves from the lowest level that includes \( k \):
  - If it is possible to go up (the key appears in the next level), we go up (with a chance of 1/2).
  - If not, we stay in the same level and go left (again, with a chance of 1/2).
- Let \( C(j) \) be the maximum number of nodes to be visited when there are \( j \) levels above us.
- After a visiting a node at the current level (with cost 1) we have:
  \[
  C(j) \leq 1 + \frac{1}{2} \cdot C(j - 1) + \frac{1}{2} \cdot C(j)
  \]
  which gives \( C(j) \leq 2j \)
- From the previous slide, we know \( j \) is expected to be \( \Theta(\log n) \).
Search Time in Skip Lists

**Theorem**

The number of nodes visited when searching for an item in the skip list of \( n \) keys is expected to be \( \Theta(\log n) \).

- For insert, we do search and add an expected \( \Theta(1) \) number of nodes; search time dominates.
- Similarly, for delete, search time dominates.
Summary of Skip Lists

- Expected space usage: $O(n)$
- Expected height: $O(\log n)$
- Skip-Search: $O(\log n)$ expected time
- Skip-Insert: $O(\log n)$ expected time
- Skip-Delete: $O(\log n)$ expected time
- Skip lists are fast and simple to implement in practice