Degree sequence conditions for partial Steiner triple systems

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A partial Steiner triple system (PSTS) of order $n$ is a collection of 3-element subsets of the vertex set \{1, 2, \ldots, n\} called triples that pairwise intersect in at most one vertex. If $\mathcal{H}$ is a PSTS and $x$ is a vertex, then the degree of $x$ is $d_x$ and is the number of triples in $\mathcal{H}$ that contain $x$. The sequence $D = (d_1, d_2, \ldots, d_n)$ is called the degree sequence of the PSTS $\mathcal{H}$, and we assume without loss that $d_1 \geq d_2 \geq \ldots \geq d_n$.

**Theorem**  Let $D = (d_1, d_2, \ldots, d_n)$ be the degree sequence of a PSTS $\mathcal{H}$, where $d_1 \geq d_2 \geq \cdots \geq d_n$. Then $\sum_i d_i \equiv 0 \pmod{3}$, and the following conditions hold for $k = 1, 2, \ldots, n$.

\[
\sum_{i=1}^{k} d_i \leq \frac{3}{2} \binom{k}{2} + \frac{1}{2} \sum_{j=k+1}^{n} \min\{k, d_j\}, \quad \text{if } k \leq \frac{n}{2}
\]

\[
\sum_{i=1}^{k} d_i \leq \binom{k}{2} + \frac{1}{2} (n-k) \left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2} \sum_{j=k+1}^{n} \min\{k, d_j\}, \quad \text{if } k > \frac{n}{2}.
\]

**Proof.** Let $V_k = \{1, 2, \ldots, k\}$ and let $\overline{V}_k = \{k + 1, k + 2, \ldots, n\}$. A triple $T$ is an $(i, 3-i)$ triple if $|T \cap V_k| = i$ and $|T \cap \overline{V}_k| = 3-i$. Let $N_i$ be the number of $(i, 3-i)$ triples, $i = 0, 1, 2, 3$. Also let $N_i(x)$ be the number of $(i, 3-i)$ triples that contain the vertex $x$. Summing $N_i(x)$ over all $x \in V_k$ counts each $(i, 3-i)$ triple $i$ times, thus for $i = 0, 1, 2, 3$ we have

\[
\sum_{x \in V_k} N_i(x) = i \cdot N_i.
\]

1
Similarly, summing over \( y \in \overline{V}_k \) we obtain for \( i = 0, 1, 2, 3 \)
\[
\sum_{y \in \overline{V}_k} N_i(y) = (3 - i) \cdot N_i.
\]  
(2)

The number of points of intersection with triples and \( V_k \) is
\[
\sum_{x \in V_k} d_x = 3N_3 + 2N_2 + N_1 = 3N_3 + \frac{3}{2}N_2 + \frac{1}{2}N_2 + N_1
\]
\[
= 3N_3 + \frac{3}{2}N_2 + \sum_{y \in \overline{V}_k} \left( \frac{1}{2}N_2(y) + \frac{1}{2}N_1(y) \right)
\]
\[
= 3N_3 + \frac{3}{2}N_2 + \frac{1}{2} \sum_{y \in \overline{V}_k} (N_2(y) + N_1(y))
\]

This last follows from Equation 2. For \( y \in \overline{V}_k \), we have
\[
N_2(y) + N_1(y) \leq N_2(y) + N_1(y) + N_0(y) = d_y.
\]

Counting the number of points in \( V_k \) that are in triples that contain \( y \) we see that
\[
N_2(y) + N_1(y) \leq 2N_2(y) + N_1(y) \leq k,
\]
because each type \((i, 3 - i)\) triple contains \( i \) points of \( V_k \) and any two triples that contain a fixed point \( y \) cannot intersect in another point. Thus
\[
\sum_{x \in V_k} d_x \leq 3N_3 + \frac{3}{2}N_2 + \frac{1}{2} \sum_{y \in \overline{V}_k} \min\{d_y, k\}
\]

Every type \((2, 1)\) triple contains one of the \( \binom{k}{2} \) possible pairs in \( V_k \) and every type \((3, 0)\) contains 3. In a PSTS no pair is covered twice, thus \( 3N_3 + N_2 \leq \binom{k}{2} \), and hence
\[
\sum_{x \in V_k} d_x \leq \binom{k}{2} + \frac{1}{2}N_2 + \frac{1}{2} \sum_{y \in \overline{V}_k} \min\{d_y, k\}.
\]

Now for \( y \in \overline{V}_k \) we have \( N_2(y) \leq \lfloor \frac{k}{2} \rfloor \), so summing over \( y \in \overline{V}_k \) we see that Equation 2 gives \( N_2 \leq (n-k) \lfloor k/2 \rfloor \). For \( x \in V_k \) we have \( N_2(x) \leq (k-1) \). Thus using Equation 1 to sum over \( x \in V_k \) we obtain \( 2N_2 \leq k(k-1) \). Consequently
\[
N_2 \leq \min\left\{ \frac{k(k-1)}{2}, (n-k) \left\lfloor \frac{k}{2} \right\rfloor \right\} = \begin{cases} \binom{k}{2}, & \text{if } k \leq \frac{n}{2} \\ (n-k) \left\lfloor \frac{k}{2} \right\rfloor, & \text{if } k > \frac{n}{2} \end{cases}
\]
These 2 observations yield

\[
\sum_{x \in V_k} d_x \leq \begin{cases} 
\frac{3}{2} \binom{k}{2} + \frac{1}{2} \sum_{y \in V_k} \min\{k, d_y\}, & \text{if } k \leq \frac{n}{2} \\
\frac{1}{2} (n-k) \left\lfloor \frac{k}{2} \right\rfloor + \frac{1}{2} \sum_{y \in V_k} \min\{k, d_y\}, & \text{if } k > \frac{n}{2}.
\end{cases}
\]

We conjecture that the conditions in the theorem are also sufficient. It should be noted that the condition obtained when \( k = 1 \) is that \( 2d_1 < n \). This is obviously necessary as the triangles containing a given point must otherwise be disjoint. In [1] the authors show that \( 2r < n \) and \( rn \equiv 0 \pmod{3} \) are necessary and sufficient for the existence of a partial Steiner triple system with degree sequence \( (r, r, r, \ldots, r) \) \( n \) times. This latter result also follows from the results in [2]. A partial Steiner triple system is said to be equitable if \( |d_x - d_y| \leq 1 \) for any two points \( x \) and \( y \). In [2] it is shown that if there exists a partial Steiner triple system of order \( n \) with \( b \) triples, then there exists an equitable partial Steiner triple system of order \( n \) with \( b \) triples. Thus, by taking a maximum packing of triples on \( n \) points, deleting the appropriate number of triples, and applying this result, one can obtain a partial Steiner triple system in which all the vertices have degree \( r \), whenever \( 2r < n \) and \( rn \equiv 0 \pmod{3} \).

References
