Sarvate–Beam Triple Systems for $v = 5$ and $v = 6$

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Abstract

A *Sarvate–Beam Triple System* $SB(v, 3)$ is a set V of v elements and a collection of 3–subsets of V such that each distinct pair of elements in V occurs in i blocks, for every i in the list $1, 2, \ldots, \binom{v}{2}$. In this paper, we completely enumerate all Sarvate-Beam Triple Systems for $v = 5$ and $v = 6$. (In the case $v = 5$, we extend a previous result of R. Stanton [8].)

1 Sarvate–Beam Designs

The present problem under consideration has its roots in papers published since 2007 by D. Sarvate and W. Beam, R. Stanton and others. In these papers, Sarvate and Beam introduced a new type of combinatorial object called an *adesign*.

Definition 1.
An adesign $AD(v, k)$ *is a set* V *of* v *elements and a collection of* k*–subsets of* V *(called* blocks*) such that each distinct pair of elements in* V *occurs in a different number of blocks. A* strict *adesign* $SAD(v, k)$ *is an adesign such that exactly one pair of elements occurs* i *times for every* i *in the list* $1,\ 2,\ \ldots,\ \binom{v}{2}$.

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Definition 1 was given by Sarvate and Beam [3], although the term *frequency* was used by Dukes [2] to refer to the number of blocks containing each distinct pair of points from V. We note that this *distinct frequency condition* distinguishes an adesign from a balanced incomplete block design (BIBD). The following definition was also given by Sarvate and Beam [3]:

Definition 2.
An aPBD(v, K) *is a set* V *of* v *elements and a collection of subsets of* V *such that every pair of distinct elements of* V *occurs a distinct number of times, and the size of any block is in* K*.*

The following definition appears in [1]:

Definition 3.
A strict *aPBD* $SaPBD(v, K)$ *is a set* V *of* v *elements and a collection of subsets of* V *such that every pair of distinct elements of* V *occurs exactly once from the list* 1, 2, ..., $\binom{v}{2}$*, and the size of any block is in* K*.*

The definition of "strict adesign" was renamed by Stanton [4] as a *Sarvate–Beam design* (or *SB design*). Stanton [4] also introduced the term "SB Triple System", referring to a $SAD(v, 3)$. He generalized his terminology in [6] to a "SB Quad System", which is a $SAD(v, 4)$. Dukes [2] improved the notation, and labels it $SB(v, k)$. Thus, we have $SB(v, 3)$ for SB Triple Systems and $SB(v, 4)$ for SB Quad Systems. The notation can further be used to denote a $SaPBD(v, K)$ by $SB(v, K)$.

2 Preliminaries

Sarvate and Beam [3] proved the following:

Theorem 4.
For a $SB(v, 3)$*, it must be true that* $v \equiv 0, 1 \pmod{3}$*.*

Using Stanton's [4] notation, some examples of SB designs are provided: (Here, the notation $x(abc)$ means x copies of the block $\{a, b, c\}$.)

Example 5 (Sarvate, Beam [3])**.**
$124 + 2(134) + 4(234)$ *is a* $SB(4, 3)$*.*

Remark 6.
This was proved by Stanton [4] to be unique up to isomorphism.

Example 7 (Sarvate, Beam [3])**.**
$124 + 2(135) + 2(145) + 234 + 4(235) + 5(245) + 7(345)$ *is an* $AD(5, 3)$*.*

Remark 8.
Note that it is not a strict adesign. Hence, it does not contradict Theorem 4.

Sarvate and Beam [3] gave examples of $SB(6,3)$ and $SB(7,3)$. Stanton [4] also gave examples of $SB(6,3)$ and $SB(7,3)$, neither of which are isomorphic to Sarvate and Beam's examples.

Stanton [8] addressed the $v \equiv 2 \pmod 3$ case for $SB(v,3)$ by allowing a single pair $\{1,2\}$ in the designs. Thus, even though he still calls them "SB Triple Systems", they are technically $SB(v,\{2,3\})$. Stanton [8] also gave examples of $SB(5,\{2,3\})$:

Example 9 (Stanton [8])**.**
$10(12)+135+3(145)+2(234)+4(235)+5(245)$ *is a* $SB(5,\{2,3\})$.

Stanton [4] goes on to make a new definition:

Definition 10.
An SB design is called restricted *if only blocks beginning with 1 or 2 (up to isomorphism) are allowable.*

Stanton [4] gave examples of restricted $SB(6,3)$ and $SB(7,3)$. Also, Stanton [5] gave an example of a restricted $SB(8,\{2,3\})$.

Lastly, Stanton [7] outlined a general procedure for finding bounds on possible values of v for restricted SB Triple Systems. He proved the following:

Theorem 11.
When $k \leq 3$, the only restricted $SB(v,k)$ that possibly exist are for $v \in \{4,5,6,7,8\}$.

3 SB Triple Systems for $v = 5$

Stanton analyzed $SB(5,\{2,3\})$ with the frequency of the pair $\{1,2\}$ being 1:

Theorem 12 (Stanton [8])**.**
There are 20 nonisomorphic $SB(5,\{2,3\})$ with $f(12)=1$ (10 of which are restricted).

He remarked that it is possible for us to have $f(12) \in \{1,4,7,10\}$. These are all different designs, since the frequencies of the pair $\{1,2\}$ are different. Hence, we enumerate all $SB(5,\{2,3\})$ with $f(12) \in \{4,7,10\}$.

Similar to the analysis of restricted SB Quad Systems by Stanton [6], we assign the frequency x to the block 12, a_1 to the block 134, a_2 to the block 135, a_3 to the block 145, b_1 to the block 234, b_2 to the block 235, b_3 to the block 245 and d_1 to the block 345.

We then label the frequency of the pair $\{a, b\}$ by $f(ab)$, noting that $f(12) = x$, $f(13) = a_1+a_2$, $f(14) = a_1+a_3$, $f(15) = a_2+a_3$, $f(23) = b_1+b_2$, $f(24) = b_1 + b_3$, $f(25) = b_2 + b_3$, $f(34) = a_1 + b_1 + d_1$, $f(35) = a_2 + b_2 + d_1$ and $f(45) = a_3 + b_3 + d_1$. We consider the existence of a $SB(5, \{2, 3\})$ to be equivalent to finding an ordered 8–tuple $(1, a_1, a_2, a_3, b_1, b_2, b_3, d_1)$ satisfying the above conditions and such that $1 \leq f(ab) \leq 10 = \binom{5}{2}$ with each $f(ab)$ a distinct integer.

We also introduce parameters $A = \sum a_i$, $B = \sum b_i$ and $D = \sum d_i = d_1$. The sum of all the frequencies $f(a, b)$ must be the sum of the integers 1 through 10. That is, $x + 3A + 3B + 3D = 55$. Hence, $A + B + D = \frac{55-x}{3}$. (Thus we see that $x \in \{1, 4, 7, 10\}$.) We also see that $2A \geq 1 + 2 + 3 = 6$, or $A \geq 3$. Further, we assume that $A \leq B$.

3.1 Case $f(12) = 4$

In this case, $A + B + D = 17$. This leaves 6 cases of possibilities for $SB(5, \{2, 3\})$ with $f(12) = 4$: $(A, B + D) \in \{(3, 14), (4, 13), (5, 12), (6, 11), (7, 10), (8, 9)\}$.

Each of these cases requires checking large numbers of possible 8–tuples that can produce $SB(5, \{2, 3\})$ with $f(12) = 4$. It is a simple matter to write a computer code to check all of the conditions for each possible 8–tuple. When we did this, we got the following results:

$(A, B + D)$	(3, 14)	(4, 13)	(5, 12)	(6, 11)	(7, 10)	(8, 9)
solutions	96	0	0	24	0	12
nonisomorphic	8	0	0	2	0	1

These solutions correspond to the 8–tuples $(1, 0, 1, 2, 2, 4, 4, 3)$, $(1, 0, 1, 2, 2, 5, 3, 4)$, $(1, 0, 1, 2, 3, 4, 2, 5)$, $(1, 0, 1, 2, 3, 6, 2, 3)$, $(1, 0, 1, 2, 4, 2, 3, 5)$, $(1, 0, 1, 2, 4, 5, 1, 4)$, $(1, 0, 1, 2, 4, 6, 3, 1)$, $(1, 0, 1, 2, 6, 4, 1, 3)$, $(1, 0, 1, 5, 7, 0, 3, 1)$, $(1, 0, 1, 5, 8, 1, 2, 0)$ and $(1, 0, 1, 8, 5, 1, 2, 0)$. Thus we have shown the following:

Theorem 13.
There are 11 nonisomorphic $SB(5, \{2, 3\})$ with $f(12) = 4$ (2 of which are restricted).

3.2 Case $f(12) = 7$

In this case, $A + B + D = 16$. This leaves 6 cases of possibilities for $SB(5, \{2, 3\})$ with $f(12) = 7$: $(A, B + D) \in \{(3, 13), (4, 12), (5, 11), (6, 10), (7, 9), (8, 8)\}$.

Each of these cases requires checking large numbers of possible 8–tuples that can produce $SB(5, \{2, 3\})$ with $f(12) = 7$. It is a simple matter to write a computer code to check all of the conditions for each possible 8–tuple. When we did this, we got the following results:

$(A, B + D)$	(3, 13)	(4, 12)	(5, 11)	(6, 10)	(7, 9)	(8, 8)
solutions	96	36	60	0	12	0
nonisomorphic	8	3	5	0	1	0

These solutions correspond to the 8–tuples $(1, 0, 1, 2, 1, 4, 5, 3)$, $(1, 0, 1, 2, 1, 5, 3, 4)$, $(1, 0, 1, 2, 2, 3, 6, 2)$, $(1, 0, 1, 2, 4, 2, 6, 1)$, $(1, 0, 1, 2, 4, 5, 0, 4)$, $(1, 0, 1, 2, 6, 4, 0, 3)$, $(1, 0, 1, 2, 7, 2, 3, 1)$, $(1, 0, 1, 2, 7, 3, 1, 2)$, $(1, 0, 1, 3, 2, 4, 6, 0)$, $(1, 0, 1, 3, 2, 7, 3, 0)$, $(1, 0, 1, 3, 6, 0, 2, 4)$, $(1, 0, 1, 4, 0, 6, 2, 3)$, $(1, 0, 1, 4, 8, 1, 2, 0)$, $(1, 0, 2, 3, 0, 6, 4, 1)$, $(1, 0, 2, 3, 1, 7, 3, 0)$, $(1, 1, 2, 4, 1, 8, 0, 0)$ and $(1, 1, 2, 4, 5, 1, 2, 0)$. Thus we have shown the following:

Theorem 14.
There are 17 nonisomorphic $SB(5, \{2, 3\})$ with $f(12) = 7$ (5 of which are restricted).

3.3 Case $f(12) = 10$

In this case, $A + B + D = 15$. This leaves 5 cases of possibilities for $SB(5, \{2, 3\})$ with $f(12) = 10$: $(A, B + D) \in \{(3, 12), (4, 11), (5, 10), (6, 9), (7, 8)\}$.

Each of these cases requires checking large numbers of possible 8–tuples that can produce $SB(5, \{2, 3\})$ with $f(12) = 10$. It is a simple matter to write a computer code to check all of the conditions for each possible 8–tuple. When we did this, we got the following results:

$(A, B + D)$	(3, 12)	(4, 11)	(5, 10)	(6, 9)	(7, 8)
solutions	96	36	48	0	12
nonisomorphic	8	3	4	0	1

These solutions correspond to the 8–tuples $(1,0,1,2,2,3,5,2)$, $(1,0,1,2,4,1,3,4)$, $(1,0,1,2,4,2,5,1)$, $(1,0,1,2,4,5,3,0)$, $(1,0,1,2,5,1,4,2)$, $(1,0,1,2,5,3,4,0)$, $(1,0,1,2,6,2,3,1)$, $(1,0,1,2,6,3,1,2)$, $(1,0,1,3,1,5,4,1)$, $(1,0,1,3,2,4,5,0)$, $(1,0,1,3,2,6,3,0)$, $(1,0,1,4,6,0,3,1)$, $(1,0,1,4,7,1,2,0)$, $(1,0,2,3,1,6,3,0)$, $(1,0,2,3,7,0,1,2)$ and $(1,1,2,4,1,7,0,0)$. Thus we have shown the following:

Theorem 15.
There are 16 nonisomorphic $SB(5,\{2,3\})$ with $f(12)=10$ (7 of which are restricted).

Theorems 12, 13, 14 and 15 can be combined into one summary result:

Theorem 16.
There are 64 nonisomorphic $SB(5,\{2,3\})$ (24 of which are restricted).

The computer code used to produce these results can be found at the website `http://www.suu.edu/faculty/hein/professional.html` (We also furnish the results of the code on the website.) It is written in C++, and produces 5 files — 4 are auxiliary files used by the program, and a file called `u523solns.dat` with all 64 solutions. (The nonisomorphic solutions were also produced using Brenden McKay's program *nauty*). The total runtime for the code on our local server is 0.366 second.

4 SB Triple Systems for $v = 6$

Again similar to the analysis of restricted SB Quad Systems by Stanton [6], we assign the frequency 1 to the block 123, a_1 to the block 134, a_2 to the block 135, a_3 to the block 136, a_4 to the block 145, a_5 to the block 146, a_6 to the block 156, b_1 to the block 234, b_2 to the block 235, b_3 to the block 236, b_4 to the block 245, b_5 to the block 246, b_6 to the block 256, d_1 to the block 345, d_2 to the block 346, d_3 to the block 356 and d_4 to the block 456.

We again label the frequency of the pair $\{a,b\}$ by $f(ab)$, noting that $f(12)=1$, $f(13)=1+a_1+a_2+a_3$, $f(14)=a_1+a_4+a_5$, $f(15)=a_2+a_4+a_6$, $f(16)=a_3+a_5+a_6$, $f(23)=1+b_1+b_2+b_3$, $f(24)=b_1+b_4+b_5$, $f(25)=b_2+b_4+b_6$, $f(26)=b_3+b_5+b_6$, $f(34)=a_1+b_1+d_1+d_2$, $f(35)=a_2+b_2+d_1+d_3$, $f(36)=a_3+b_3+d_2+d_3$, $f(45)=a_4+b_4+d_1+d_4$, $f(46)=a_5+b_5+d_2+d_4$ and $f(56)=a_6+b_6+d_3+d_4$. We consider the existence of a $SB(6,3)$ to be equivalent to finding an ordered 17–tuple $(1,a_1,a_2,a_3,a_4,a_5,a_6,b_1,b_2,b_3,b_4,b_5,b_6,d_1,d_2,d_3,d_4)$ satisfying the above conditions and such that $1 \le f(ab) \le 15 = \binom{6}{2}$ with each $f(ab)$ a distinct integer.

We again use the parameters $A = \sum a_i$, $B = \sum b_i$ and $D = \sum d_i$. The sum of all the frequencies $f(a,b)$ must be the sum of the integers 1 through 15. That is, $3 + 3A + 3B + 3D = 120$. Hence, $A + B + D = 39$. We also see that $1 + 2A \geq 2 + 3 + 4 + 5 = 14$, or $A \geq 6$. We again assume that $A \leq B$. This leaves 14 cases of possibilities for $SB(6,3)$: $(A, B + D) \in$ $\{(6,33), (7,32), (8,31), (9,30), (10,29), (11,28), (12,27), (13,26), (14,25),$ $(15,24), (16,23), (17,22), (18,21), (19,20)\}$.

Each of these cases requires checking large numbers of possible 17–tuples that can produce $SB(6,3)$. It is a simple matter to write a computer code to check all of the conditions for each possible 17–tuple. When we did this, we got the following results:

Theorem 17.
There are 16,444,260 nonisomorphic $SB(6,3)$ *(48,843 of which are restricted).*

The nonisomorphic solutions were again produced using Brenden McKay's program *nauty*. The set of nonisomorphic $SB(6,3)$ may be found at the website `http://www.cs.umanitoba.ca/~lipakc/research/sbsystems.html`

5 Open Problems

Stanton [4, 5, 6, 7] states several open questions:

- Find the number of nonisomorphic $SB(7,3)$.
- Find the number of nonisomorphic restricted $SB(7,3)$.
- Find the number of nonisomorphic restricted $SB(8,\{2,3\})$.
- Find the number of nonisomorphic $SB(9,4)$.
- Find the number of nonisomorphic restricted $SB(9,4)$.
- Find the number of nonisomorphic $SB(v,4)$ for $v > 9$.

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