Antimagic labelings of power of cycles graphs

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April 13, 2011

Abstract

An antimagic labeling of a graph with $n$ vertices and $m$ edges is a bijection from the set of edges to the integers $1, 2, ..., m$ such that all $n$ vertex sums are pairwise distinct. For a cycle $C_n$ of length $n$, the $k^{th}$ power of $C_n$, denoted by $C_k^n$, is the supergraph formed by adding an edge between all pairs of vertices of $C_n$ with distance at most $k$. Antimagic labelings for $C_k^n$ are given where $k = 2, 3, 4$.

1 Introduction

In this paper, all graphs are finite, undirected, and simple. Let $G = (V, E)$ be a graph with $n$ vertices and $m$ edges. Suppose the edges of $G$ are labeled using distinct values from $\{1, 2, ..., m\}$. For each vertex $v$, define its vertex sum be the sum of the labels of the edges incident on $v$. A labeling is an antimagic labeling of $G$ if all $n$ vertex sums are pairwise distinct. If a graph has an antimagic labeling, then the graph is antimagic. For a vertex $v$, denote its vertex-sum by $S_v$.

In 1990, Hartsfield and Ringel [3] introduced the notion of antimagic labelings and antimagic graphs. They conjectured that every connected graph, other than $K_2$, is antimagic. In 2004, Alon et al. [1] validated this conjecture for graphs having minimum degree $\Omega(\log n)$. They also showed that graphs with maximum degree at least $n - 2$ are antimagic, as well as complete $k$-partite graphs, for any $k \geq 2$. In 2005, Hefetz [4] showed that a graph with $3^k$ vertices admitting a $K_3$-factor is antimagic. Also in 2005, Wang [6] showed that the Cartesian product of a finite number of cycles is antimagic. In addition, Wang showed that the Cartesian product of an antimagic regular graph and a cycle is antimagic. In 2008, Wang and Hsiao [7] showed that toroidal grids are antimagic.

Suppose $C_n = (V, E)$ is a cycle of length $n$ and $k$ is a positive integer. The $k^{th}$ power of $C_n$, denoted by $C_k^n$, is the supergraph formed by adding an edge between all pairs of vertices of $C_n$ with distance at most $k$. In 2010, Lee, Lin, and Tsai [5] showed that if $n$ is odd, the power of cycles graph $C_2^n$ is antimagic. Other results can be found in the dynamic survey by Gallian [2].

In this report, I extended the work of Lee, Lin and Tsai [5] by giving an alternate proof of their result on $C_2^n$, where $n$ is odd. I also showed that, for $n$ even, $C_2^n$ is antimagic by constructing an antimagic labeling for $C_2^n$ antimagic. Then, I extended the antimagic labelings for $C_2^n$ to obtain antimagic labelings for $C_3^n$, whenever $n \geq 6$. Finally, I showed that the antimagic labelings for $C_3^n$, where $n$ is odd, extends to antimagic labelings for $C_4^n$.

2 The Graph $C_2^n$

In this section, I will show that $C_2^n$ is antimagic for all $n \geq 4$. Note that when $n = 3$, $C_2^3 = C_3$. I begin by providing an antimagic labeling of $C_2^n$ that differs from the one given in [5].

*Research supported by NSERC Discovery Grant 250389-06
Theorem 2.1 ([5]) If \( n > 3 \) is an odd integer, then \( C_n^2 \) is antimagic.

**Proof**: The vertices of \( C_n^2 \) will be \( V = \{0, 1, 2, ..., n-1\} \). I note that \( C_n^2 \) has 2\( n \) edges. Define a bijection function \( L : E \rightarrow \{1, 2, ..., 2n\} \) that labels the edges of the graph as follows:

\[
L(i, j) = \begin{cases} 
  i + 1 & : 0 \leq i \leq n - 2 \text{ and } j = i + 1 \\
  n & : i = n - 1 \text{ and } j = 0 \\
  n + 1 & : i = n - 1, j = 1 \\
  2n & : i = n - 2, j = 0 \\
  n + i + 2 & : 0 \leq i \leq n - 3 \text{ and } j = i + 2 
\end{cases}
\]

I claim that the labeling \( L \) is an antimagic labeling of \( C_n^2 \). Observe that \( S_1 = 1 + 2 + (n+1) + (n+3) = 2n + 7 \) and \( S_2 = 2 + 3 + (n+2) + (n+4) = 2n + 11 \), which is 4 greater than \( S_1 \). In fact, it is easy to verify that for 1 \( \leq i \leq n - 3 \), \( S_{i+1} = S_i + 4 \). Since \( S_1 \) is odd, then so is every \( S_i \), for 1 \( \leq i \leq n - 2 \). In addition, they are pairwise distinct. The vertex \( n - 1 \) has vertex-sum \( S_{n-1} = (n-1) + n + (n+1) + (2n-1) = 5n - 1 \) which is even. Finally, vertex 0 has vertex-sum \( S_0 = 1 + n + 2n + (n+2) = 4n + 3 \), which is odd. All that remains is to show \( S_0 \) does not appear in the set of vertex sums \( \{S_1, S_2, ..., S_{n-2}\} \). To see this, note that if \( S_0 \) is the same as the vertex-sum of some vertex in \( \{1, 2, 3, ..., n-2\} \), then \( S_0 - S_1 \) must be divisible by 4. But this difference is \( 4n + 3 - (2n + 7) = 2n - 4 = 2(n-2) \). As \( n \) is odd, then \( n - 2 \) is odd. Therefore \( 2(n-2) \) is not divisible by 4 which implies \( S_0 \not\in \{S_1, S_2, ..., S_{n-2}\} \). Therefore, all the vertex-sums of this labeling are pairwise distinct. 

![Figure 1: antimagic labeling of \( C_n^{11} \)](image)

Observe that since \( n \) is odd, the labeling \( L \), as given in the proof of Theorem 2.1, for the edges of \( C_n \) is antimagic. Figure 1 shows the antimagic labeling of \( C_n^{11} \) using the labeling given in the proof of Theorem 2.1. Consider the graph \( C_n^2 \), where the number of vertices is even. I now describe a construction for an antimagic labeling of \( C_n^2 \) which can be extended to an antimagic labeling for \( C_n^3 \).

**Theorem 2.2** If \( n > 6 \) is an even integer, then \( C_n^2 \) is antimagic.

**Proof**: The vertices of \( C_n^2 \) will be \( V = \{0, 1, 2, ..., n-1\} \). Let \( E \) denote the edges of the graph. Define a bijection \( L : E \rightarrow \{1, 2, ..., 2n\} \) that labels the edges of the graph as follows:

\[
L(i, j) = \begin{cases} 
  2 & : i = 0, j = 1 \\
  1 & : i = 1, j = 2 \\
  n - 1 & : i = n - 3, j = n - 2 \\
  2n & : i = n - 2, j = 0 \\
  i + 1 & : j = i + 1 \text{ and } i \not\in \{0, 1, n-3, n-2, n-1\} \\
  n + 1 & : i = n - 1, j = 1 \\
  0 & : i = n - 1, j = 0 \\
  n + i + 2 & : 0 \leq i \leq n - 3 \text{ and } j = i + 2 
\end{cases}
\]
By definition of the labeling $L$, $S_0 = 2 + n + 2n + (n + 2) = 4n + 4$, $S_1 = 1 + 2 + (n + 1) + (n + 3) = 2n + 7$, 
$S_2 = 1 + 3 + (n + 2) + (n + 4) = 2n + 10$, $S_3 = 3 + 4 + (n + 3) + (n + 5) = 2n + 15$. It can be verified that $S_{i+1} = S_i + 4$ for $3 \leq i \leq n - 5$. Since $S_3$ is odd, $S_i$ is odd for $3 \leq i \leq n - 4$. In addition, they are pairwise distinct. Also, $S_{n-3} = 6n - 8$, $S_{n-2} = 6n - 5 = S_{n-4} + 8$ and $S_{n-1} = 5n - 2$. Note that $S_1, S_{n-2}$ are both odd. In fact $S_1 = S_3 - 8$ and $S_{n-2} = S_{n-4} + 8$. This implies $S_1, S_2$ are pairwise distinct and does not belong in the set of vertex-sums $\{S_3, S_4, ..., S_{n-1}\}$. By the labeling $L$, $S_2 < S_0 < S_{n-1} < S_{n-3}$ and they are all even. Since $n > 6$, we don’t have the scenario where $S_{n-3} = S_3$. Therefore all the vertex sums are distinct.

Figure 2 shows the antimagic labeling of $C_{12}^2$ using the labeling given in the proof of Theorem 2.2. Theorems 2.1 and 2.2 gives antimagic labelings $C_n^2$, for all $n$, except when $n = 4, 6$. Figures 3 and 4 shows that $C_2^4$ and $C_2^6$ are antimagic, respectively. This along with Theorems 2.1 and 2.2 gives the following result.

**Corollary 2.3** For every $n \geq 4$, $C_n^2$ is antimagic.

![Figure 2: Antimagic labeling of $C_{12}^2$](image1)

![Figure 3: Antimagic labeling of $C_4^2$](image2)

3 The Graph $C_n^3$

In the previous section, we constructed an antimagic labeling of $C_n^2$, for every $n \geq 4$. In this section, we will extend those constructions to give antimagic labelings for $C_n^3$. We will consider the two cases of $n$ odd and $n$ even separately.

**Theorem 3.1** If $n \geq 7$ is an odd integer, then $C_n^3$ has an antimagic labeling.
Proof: Recall that that the labeling $L$, which was used to prove Theorem 2.1, has the following properties. I will use $S^L_i$ to denote the vertex-sum of vertex $i$ under the labeling $L$, of $C^2_n$.

1. $S^L_0 = 4n + 3$,
2. $S^L_1 = 2n + 7, S^L_2 = 2n + 11, S^L_3 = 2n + 15$
3. $S^L_{i+1} = S_i + 4$ for $1 \leq i \leq n - 3$,
4. $S^L_{n-1} = 5n - 1$.

In addition, recall that every vertex-sum $S^L_i$ is odd except for $S^L_{n-1}$, which is even. We now show how to extend the antimagic labeling $L$ for $C^2_n$ to an antimagic labeling $M$ for $C^3_n$ such that $M|C^2_n = L$. For each edge $e \in C^3_n$, assign $M(e) = L(e)$. For the edge $e = \{i, i+3\}$ where $0 \leq i < n - 3$, assign $M(e) = 2n + i + 1$. For the edge $e = \{n-3, 0\}$, we assign $M(e) = 3n - 2$. For the edge $e = \{n-2, 1\}$, we assign $M(e) = 3n - 1$. Finally, for the edge $e = \{n-1, 2\}$, we assign $M(e) = 3n$. This gives a labeling $M$ for $C^3_n$, which extend the labeling $L$. We now show that it is an antimagic labeling of $C^3_n$.

Consider the vertices $0, 1, 2, n - 1$. They have vertex-sums $S_0 = (4n + 3) + (2n + 1) + (3n - 2) = 9n + 2, S_1 = (2n + 7) + (2n + 2) + (3n - 1) = 7n + 8, S_2 = (2n + 11) + (2n + 3) + (3n) = 7n + 14, and $S_{n-1} = (5n - 1) + (3n - 3) + (3n) = 11n - 4$. Since $n \geq 7$, these four vertex-sums are odd. Since $M(e) = 2n + i + 1$ for edges of the form $e = \{i, i+3\}$, where $0 \leq i < n - 3$ and $S^L_{i+1} = S^L_i + 4$, for $3 \leq i \leq n - 3$, then $S_{i+1} = S_i + 6$, for $3 \leq i \leq n - 3$. Therefore, it suffices to show that $S_3$ is even. But $S_3 = (2n + 15) + (2n + 1) + (2n + 4) = 6n + 20$, which is even.

Figure 5 shows the labeling of the edges of $C^3_n \setminus C^2_n$ as given in the proof of Theorem 3.1.

We now consider the case where $n$ is even. Again, we will extend the antimagic labeling $L$ stated in the proof of Theorem 2.2.

Theorem 3.2 If $n > 6$ is a even number that is not a multiple of 6, then $C^3_n$ has an antimagic labeling.

Proof: Consider the labeling $L$ used in the proof of Theorem 2.2. Recall that it has the following properties, where we use $S^L_i$ to denote the vertex-sum of vertex $i$ under the labeling $L$.

1. $S^L_0 = 4n + 4, S^L_1 = 2n + 7, S^L_2 = 2n + 10, S^L_3 = 2n + 15$,
2. $S^L_{i+1} = S_i + 4$ for $3 \leq i \leq n - 5$,
3. $S^L_{n-3} = 6n - 8, S^L_{n-2} = S_{n-4} + 8$, and $S^L_{n-1} = 5n - 2$. 

Figure 4: antimagic labeling of $C^2_6$
Theorem 3.3 If \( n > 6 \) is an even number that is a multiple of 6, then \( C_n^3 \) has an antimagic labeling.

Proof: In the labeling \( M \) given in the proof of Theorem 3.2, make the following two modifications.

1. For the edge \( e = \{n - 2, 1\} \), we assign \( M(e) = 3n \), and
2. for the edge \( e = \{n - 1, 2\} \), we assign \( M(e) = 3n - 1 \).

With this modification, we have \( S_0 = 9n + 3 \) (odd), \( S_1 = 7n + 9 \) (odd), \( S_2 = 7n + 12 \) (even), \( S_{n-3} = 12n - 15 \) (odd) \( S_{n-2} = 12n - 9 \) (odd), \( S_{n-1} = 11n - 6 \) (even). The vertex sums \( S_i \), for \( 3 \leq i \leq n - 4 \) have the same values as in the proof of theorem 3.2 and therefore are all even and pairwise distinct. The values \( S_0, S_1, S_{n-3}, \) and \( S_{n-2} \) are all odd and distinct. All that remains to show is that \( S_2 \) and \( S_{n-1} \) are not the vertex sums of some other vertex. Clearly \( S_2 \neq S_{n-1} \). To show that \( S_2 \) and \( S_{n-1} \) do not appear in \( \{S_3, S_4, \ldots, S_{n-4}, S_{n-2}\} \), it suffices to show that \( S_2 - S_1 \) and \( S_{n-4} - S_{n-1} \) are not divisible by 6. If \( S_2 - S_1 = n - 8 \) is divisible by 6, then \( n \) must be of the form \( n = 6k + 2 \). As we assume that \( n \) is a multiple of 6, \( n - 8 \) cannot be divisible by 6. Similarly, if \( S_{n-4} - S_{n-1} = n - 16 \) is divisible by 6, then \( n \) must be of the form \( 6k + 4 \). As we assume

Figure 5: antimagic labeling of edges of \( C_{11}^3 \setminus C_{11}^2 \)
that \( n \) is a multiple of 6, \( n - 16 \) cannot be divisible by 6. Thus, all the vertex sums are distinct, and \( M \) is an antimagic labeling for \( n > 6 \) and a multiple of 6.

Figures 6 and 7 gives the antimagic labelings of the edges of \( C_{12}^3 \setminus C_{12}^2 \) and \( C_{16}^3 \setminus C_{16}^2 \), respectively. Figure 8 gives an antimagic labeling for \( C_{6}^3 \). Theorems 3.2 and 3.3 along with Figure 8 implies that \( C_{6}^3 \) is antimagic, for all \( n \geq 6 \).

![Figure 6: antimagic labeling of edges of \( C_{12}^3 \setminus C_{12}^2 \)](image)

Corollary 3.4 : For \( n \geq 6 \), \( C_{n}^3 \) is antimagic.

4 The graph \( C_{n}^{4} \)

In this section, I will prove that \( C_{n}^{4} \) has an antimagic labeling. I will do this by extending the labeling given in Section 2 for \( C_{n}^{3} \).

Theorem 4.1 If \( n \geq 7 \) is an odd integer, then \( C_{n}^{4} \) has an antimagic labeling.

Proof I will show how the extend the labeling \( M \) for \( C_{n}^{3} \), as given in the proof of Theorem 3.1, to an antimagic labeling \( N \) for \( C_{n}^{4} \) such that \( N|C_{n}^{3} = M \). For each edge \( e \in C_{n}^{3} \), assign \( N(e) = M(e) \). For the edge \( e = \{i, i + 4\} \), where \( 0 \leq i < n - 4 \), assign \( N(e) = 3n + i + 1 \). For the edge \( e = \{n - 4, 0\} \), assign \( N(e) = 4n - 3 \). For the edge \( e = \{n - 3, 1\} \), assign \( N(e) = 4n - 2 \). For the edge \( e = \{n - 2, 2\} \), assign \( M(e) = 4n - 1 \). Finally, for the edge \( e = \{n - 1, 3\} \), we assign \( N(e) = 4n \). I claim that \( N \) is an antimagic labeling of \( C_{n}^{4} \).
Based on the labeling $N$, $S_0 = 16n, S_1 = 14n + 8, S_2 = 14n + 16, S_3 = 13n + 24, S_4 = 12n + 32, S_{i+1} = S_i + 8$, for $4 \leq i \leq n - 3$, and $S_{n-1} = 19n - 8$. It is easy to see that $S_0, S_1, S_2, S_4$ are even. As $n$ is even, the vertex sums $S_0, S_1, S_2$ and $S_4$ are pairwise distinct. As $S_4$ is even, so is $S_1$, for $4 \leq i \leq n - 2$ and these vertex sums are distinct. As $S_1$ and $S_{n-1}$ are odd, they are distinct from all the other vertex sums. They are also different from each other. It remains to show that the vertex sums $S_0, S_1, S_2$ are not one of the vertex sums $S_4, S_5, ..., S_{n-2}$. For $S_0$, $S_0 - S_4 = 4(n - 8)$ is divisible by 8 if and only if $n$ is even. As $n$ is odd, the vertex sum $S_0$ is unique. For $S_1$, $S_1 - S_4 = 2(n - 12)$ is divisible by 8 implies $n$ is even. So $S_1$ is unique also. For $S_2$, $S_2 - S_4 = 2(n - 8)$ is divisible by 8 implies $n$ is even. So $S_2$ is also unique. Therefore, all the vertex sums are distinct and $N$ is an antimagic labeling of $C_n^3$.

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Theorem 4.2 Let $n \geq 8, n \neq 12, 14$ be an even integer. Then $C_n^4$ is antimagic.

**Proof** We begin by handling the special case where $n = 8$. To show that $C_8^4$ is antimagic, start with the labeling $M$ for $C_8^3$, as given in the proof of Theorem 3.2. Now label the edge $(0, 4)$ with 15, the edge $(1, 5)$ with 16, the edge $(2, 6)$ with 18, and the edge $(3, 7)$ with 17. It is easy to see that this is an antimagic labeling for $C_8^3$.

We now suppose that $n > 8$. I will show how the extend the labeling $M$ for $C_n^3$, as given in the proof of Theorem 3.3, to an antimagic labeling $N$ for $C_n^4$ such that $N|C_n^3 = M$. Note that when $n$ is not a multiple of 6, the labeling $M$ may not be an antimagic labeling of $C_n^4$. For each edge $e \in C_n^3$, assign $N(e) = M(e)$. For the edge $e = \{i, i + 4\}$, where $0 \leq i < n - 4$, assign $N(e) = 3n + i + 1$. For the edge $e = \{n - 4, 0\}$, assign $N(e) = 4n - 3$. For the edge $e = \{n - 3, 1\}$, assign $N(e) = 4n - 2$. For the edge $e = \{n - 2, 2\}$, assign $M(e) = 4n$. Finally, for the edge $e = \{n - 1, 3\}$, we assign $N(e) = 4n - 1$. I claim that $N$ is an antimagic labeling of $C_n^4$.

Based on the labeling labeling $N$, $S_0 = 16n + 1, S_1 = 14n + 9, S_2 = 14n + 15, S_3 = 13n + 23, S_4 = 12n + 32, S_{i+1} = S_i + 8$, for $4 \leq i \leq n - 5, S_{n-3} = 20n - 23, S_{n-2} = 20n - 14$ and $S_{n-1} = 19n - 11$. It is easy to see that $S_4$ is even, and therefore $S_4, S_5, S_6, ..., S_{n-4}$ are all even and distinct. In since $S_1 = S_3$ only when $n = 14$ and $S_{n-3} = S_{n-1}$ only when $n = 12$, $S_0, S_1, S_2, S_3, S_{n-3}, S_{n-1}$ are all odd and pairwise distinct. It remains to show that $S_{n-2}$ is not in the set $S = \{S_4, S_5, S_6, ..., S_{n-4}\}$. This is true, since $S_{n-2} - S_{n-4} = (20n - 14) - (20n - 32) = 18 > 0$. Therefore, all the vertex sums are distinct.

At this point, we could handle the remaining cases $n = 12, 14$ separately. Instead, we give another general construction that will deal with these two cases.

Theorem 4.3 Let $n > 8$ be an even integer of the form $8k, 8k + 4$ or $8k + 6$. Then $C_n^4$ is antimagic.

**Proof** I will show how the extend the labeling $M$ for $C_n^3$, as given in the proof of Theorem 3.2, to an antimagic labeling $N$ for $C_n^4$ such that $N|C_n^3 = M$. Note that when $n$ is a multiple of 6, the labeling $M$ may not be an antimagic labeling of $C_n^3$. For each edge $e \in C_n^3$, assign $N(e) = M(e)$. For the edge $e = \{i, i + 4\}$,
where \(0 \leq i < n - 4\), assign \(N(e) = 3n + i + 1\). For the edge \(e = \{n - 4, 0\}\), assign \(N(e) = 4n - 3\). For the edge \(e = \{n - 3, 1\}\), assign \(N(e) = 4n\). For the edge \(e = \{n - 2, 2\}\), assign \(M(e) = 4n - 1\). Finally, for the edge \(e = \{n - 1, 3\}\), we assign \(N(e) = 4n - 2\). I claim that \(N\) is an antimagic labeling of \(C^4_n\).

Based on the labeling \(N\), \(S_0 = 16n + 1\), \(S_1 = 14n + 10\), \(S_2 = 14n + 15\), \(S_3 = 13n + 22\), \(S_4 = 12n + 32\), \(S_{i+1} = S_i + 8\), for \(4 \leq i \leq n - 5\), \(S_{n-3} = 20n - 21\), \(S_{n-2} = S_{n-4} + 16\) and \(S_{n-1} = 19n - 11\).

It is easy to see that \(S_4\) is even, and therefore \(S_5, S_6, ..., S_{n-4}, S_{n-2}\) are all even and pairwise distinct. In addition, \(S_0, S_2, S_{n-3}\) and \(S_{n-1}\) are all odd and pairwise distinct, as \(n > 8\). It remains to show that \(S_1\) and \(S_3\), which are both even and distinct from each other, cannot be in the set \(S = \{S_1, S_5, S_6, ..., S_{n-4}, S_{n-2}\}\).

To see this, suppose \(S_1\) is in the set \(S\). Then \(S_1 - S_4 = 14n + 10 - (12n + 32) = 2(n - 11)\) must be divisible by 8. But since \(n\) is even, this is not possible. Therefore, \(S_1\) is not in the set \(S\). Now, suppose \(S_3\) is in the set \(S\). Then, \(S_3 - S_4 = 13n + 22 - (12n + 32) = n - 10\) must be divisible by 8. But \(n - 10\) is divisible by 8 if and only if \(n\) is of the form \(8k + 2\). Since we assumed \(n\) is not of this form, \(S_3\) cannot be in the set \(S\).

Therefore, all the vertex sums are distinct.

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References


