Antimagic labelings of power of cycles graphs

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Abstract

An antimagic labeling of a graph with n vertices and m edges is a bijection from the set of edges to the integers $1, 2, ..., m$ such that all n vertex sums are pairwise distinct. For a cycle C_n of length n, the k^{th} *power of* C_n, denoted by C_n^k, is the supergraph formed by adding an edge between all pairs of vertices of C_n with distance at most k. Antimagic labelings for C_n^k are given where $k = 2, 3, 4$.

1 Introduction

In this paper, all graphs are finite, undirected, and simple. Let $G = (V, E)$ be a graph with n vertices and m edges. Suppose the edges of G are labeled using distinct values from $\{1, 2, .., m\}$. For each vertex v, define its *vertex sum* be the sum of the labels of the edges incident on v. A labeling is an *antimagic labeling* of G if all n vertex sums are pairwise distinct. If a graph has an antimagic labeling, then the graph is *antimagic*. For a vertex v, denote its vertex-sum by S_v.

In 1990, Hartsfield and Ringel [3] introduced the notion of antimagic labelings and antimagic graphs. They conjectured that every connected graph, other than K_2, is antimagic. In 2004, Alon et al. [1] validated this conjecture for graphs having minimum degree $\Omega(\log n)$. They also showed that graphs with maximum degree at least $n - 2$ are antimagic, as well as complete k-partite graphs, for any $k \geq 2$. In 2005, Hefetz [4] showed that a graph with 3^k vertices admitting a K_3-factor is antimagic. Also in 2005, Wang [6] showed that the Cartesian product of a finite number of cycles is antimagic. In addition, Wang showed that the Cartesian product of an antimagic regular graph and a cycle is antimagic. In 2008, Wang and Hsiao [7] showed that toroidal grids are antimagic.

Suppose $C_n = (V, E)$ is a cycle of length n and k is a positive integer. The k^{th} *power of* C_n, denoted by C_n^k, is the supergraph formed by adding an edge between all pairs of vertices of C_n with distance at most k. In 2010, Lee, Lin, and Tsai [5] showed that if n is odd, the power of cycles graph C_n^2 is antimagic. Other results can be found in the dynamic survey by Gallian [2].

In this report, I extended the work of Lee, Lin and Tsai [5] by giving an alternate proof of their result on C_n^2, where n is odd. I also showed that, for n even, C_n^2 is antimagic by constructing an antimagic labeling for C_n^2 antimagic. Then, I extended the antimagic labelings for C_n^2 to obtain antimagic labelings for C_n^3, whenever $n \geq 6$. Finally, I showed that the antimagic labelings for C_n^3, where n is odd, extends to antimagic labelings for C_n^4.

2 The Graph C_n^2

In this section, I will show that C_n^2 is antimagic for all $n \geq 4$. Note that when $n = 3$, $C_n^2 = C_n$. I begin by providing an antimagic labeling of C_n^2 that differs from the one given in [5].

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Theorem 2.1 *([5]) If $n > 3$ is an odd integer, then C_n^2 is antimagic.*

Proof : The vertices of C_n^2 will be $V = \{0, 1, 2, ..., n-1\}$. I note that C_n^2 has $2n$ edges. Define a bijection function $L : E \to \{1, 2, ..., 2n\}$ that labels the edges of the graph as follows:

$$L(\{i,j\}) = \begin{cases} i+1 & : \quad 0 \le i \le n-2 \text{ and } j = i+1 \\ n & : \quad i = n-1 \text{ and } j = 0 \\ n+1 & : \quad i = n-1, j = 1 \\ 2n & : \quad i = n-2, j = 0 \\ n+i+2 & : \quad 0 \le i \le n-3 \text{ and } j = i+2 \end{cases}$$

I claim that the labeling L is an antimagic labeling of C_n^2. Observe that $S_1 = 1+2+(n+1)+(n+3) = 2n+7$ and $S_2 = 2+3+(n+2)+(n+4) = 2n+11$, which is 4 greater than S_1. In fact, it is easy to verify that for $1 \le i \le n-3$, $S_{i+1} = S_i + 4$. Since S_1 is odd, then so is every S_i, for $1 \le i \le n-2$. In addition, they are pairwise distinct. The vertex $n-1$ has vertex-sum $S_{n-1} = (n-1) + n + (n+1) + (2n-1) = 5n-1$ which is even. Finally, vertex 0 has vertex-sum $S_0 = 1 + n + 2n + (n+2) = 4n+3$, which is odd. All that remains is to show S_0 does not appear in the set of vertex sums $\{S_1, S_2, .., S_{n-2}\}$. To see this, note that if S_0 is the same as the vertex-sum of some vertex in $\{1, 2, 3, ..., n-2\}$, then $S_0 - S_1$ must be divisible by 4. But this difference is $4n+3-(2n+7) = 2n-4 = 2(n-2)$. As n is odd, then $n-2$ is odd. Therefore $2(n-2)$ is not divisible by 4 which implies $S_0 \notin \{S_1, S_2, ..., S_{n-2}\}$. Therefore, all the vertex-sums of this labeling are pairwise distinct. $\square$

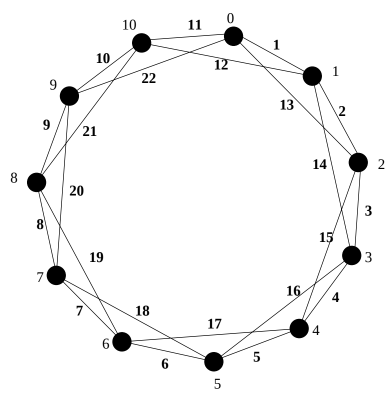

Figure 1: antimagic labeling of C_2^{11}

Observe that since n is odd, the labeling L, as given in the proof of Theorem 2.1, for the edges of C_n is antimagic. Figure 1 shows the antimagic labeling of C_{11}^2 using the labeling given in the proof of Theorem 2.1. Consider the graph C_n^2, where the number of vertices is even. I now describe a construction for an antimagic labeling of C_n^2 which can be extended to an antimagic labeling for C_n^3.

Theorem 2.2 *If $n > 6$ is an even integer , then C_n^2 is antimagic.*

Proof : The vertices of C_n^2 will be $V = \{0, 1, 2, ..., n-1\}$. Let E denote the edges of the graph. Define a bijection $L : E \to \{1, 2, ..., 2n\}$ that labels the edges of the graph as follows:

$$L(\{i,j\}) = \begin{cases} 2 & : \quad i = 0, j = 1 \\ 1 & : \quad i = 1, j = 2 \\ n-1 & : \quad i = n-3, j = n-2 \\ n-2 & : \quad i = n-2, j = n-1 \\ n & : \quad i = n-1, j = 0 \\ i+1 & : \quad j = i+1 \text{ and } i \notin \{0, 1, n-3, n-2, n-1\} \\ n+1 & : \quad i = n-1, j = 1 \\ 2n & : \quad i = n-2, j = 0 \\ n+i+2 & : \quad 0 \le i \le n-3 \text{ and } j = i+2 \end{cases}$$

By definition of the labeling L, $S_0 = 2+n+2n+(n+2) = 4n+4$, $S_1 = 1+2+(n+1)+(n+3) = 2n+7$, $S_2 = 1+3+(n+2)+(n+4) = 2n+10$, $S_3 = 3+4+(n+3)+(n+5) = 2n+15$. It can be verified that $S_{i+1} = S_i + 4$ for $3 \leq i \leq n-5$. Since S_3 is odd, S_i is odd for $3 \leq i \leq n-4$. In addition, they are pairwise distinct. Also, $S_{n-3} = 6n-8$, $S_{n-2} = 6n-5 = S_{n-4}+8$ and $S_{n-1} = 5n-2$. Note that S_1, S_{n-2} are both odd. In fact $S_1 = S_3 - 8$ and $S_{n-2} = S_{n-4}+8$. This implies S_1, S_2 are pairwise distinct and does not belong in the set of vertex-sums $\{S_3, S_4, ..., S_{n-4}\}$. By the labeling L, $S_2 < S_0 < S_{n-1} < S_{n-3}$ and they are all even. Since $n > 6$, we don't have the scenario where $S_{n-3} = S_3$. Therefore all the vertex sums are distinct. $\square$

Figure 2 shows the antimagic labeling of C_{11}^2 using the labeling given in the proof of Theorem 2.2. Theorems 2.1 and 2.2 gives antimagic labelings C_2^n, for all n, except when $n = 4, 6$. Figures 3 and 4 shows that C_2^4 and C_2^6 are antimagic, respectively. This along with Theorems 2.1 and 2.2 gives the following result.

Corollary 2.3 *For every $n \geq 4$, C_n^2 is antimagic.*

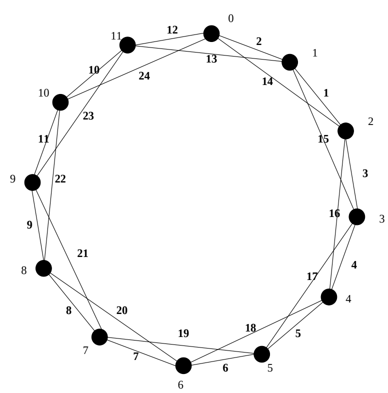

Figure 2: Antimagic labeling of C_{12}^2

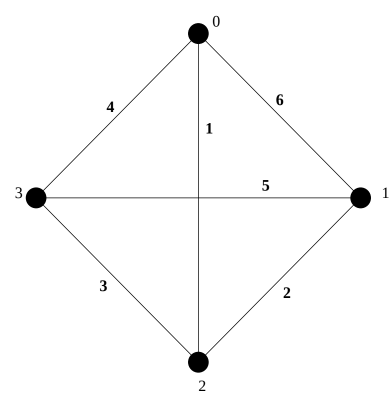

Figure 3: antimagic labeling of C_4^2

3 The Graph C_n^3

In the previous section, we constructed an antimagic labeling of C_n^2, for every $n \geq 4$. In this section, we will extend those constructions to give antimagic labelings for C_n^3. We will consider the two cases of n odd and n even separately.

Theorem 3.1 *If $n \geq 7$ is an odd integer, then C_n^3 has an antimagic labeling.*

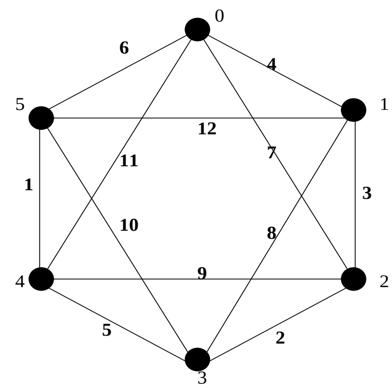

Figure 4: antimagic labeling of C_6^2

Proof : Recall that that the labeling L, which was used to prove Theorem 2.1, has the following properties. I will use S_i^L to denote the vertex-sum of vertex i under the labeling L, of C_n^2.

1. $S_0^L = 4n + 3$,
2. $S_1^L = 2n + 7$,$S_2^L = 2n + 11$,$S_3^L = 2n + 15$
3. $S_{i+1}^L = S_i + 4$ for $1 \leq i \leq n - 3$,
4. $S_{n-1}^L = 5n - 1$.

In addition, recall that every vertex-sum S_i^L is odd except for S_{n-1}^L, which is even. We now show how to extend the antimagic labeling L for C_n^2 to an antimagic labeling M for C_n^3 such that $M|C_n^2 = L$. For each edge $e \in C_n^2$, assign $M(e) = L(e)$. For the edge $e = \{i, i+3\}$ where $0 \leq i < n-3$, assign $M(e) = 2n+i+1$. For the edge $e = \{n-3, 0\}$, we assign $M(e) = 3n-2$. For the edge $e = \{n-2, 1\}$, we assign $M(e) = 3n-1$. Finally, for the edge $e = \{n-1, 2\}$, we assign $M(e) = 3n$. This gives a labeling M for C_n^3, which extend the labeling L. We now show that it is an antimagic labeling of C_n^3.

Consider the vertices $0, 1, 2, n-1$. They have vertex-sums $S_0 = (4n+3) + (2n+1) + (3n-2) = 9n+2$, $S_1 = (2n+7) + (2n+2) + (3n-1) = 7n+8$, $S_2 = (2n+11) + (2n+3) + (3n) = 7n+14$, and $S_{n-1} = (5n-1) + (3n-3) + (3n) = 11n-4$. Since n is odd, these four vertex-sums are odd. Since $n \geq 7$, these four vertex-sums are distinct. As $M(e) = 2n+i+1$ for edges of the form $e = \{i, i+3\}$, where $0 \leq i < n-3$ and $S_{i+1}^L = S_i^L + 4$, for $3 \leq i \leq n-3$, then $S_{i+1} = S_i + 6$, for $3 \leq i \leq n-3$. Therefore, it suffices to show that S_3 is even. But $S_3 = (2n+15) + (2n+1) + (2n+4) = 6n+20$, which is even.

□

Figure 5 shows the labeling of the edges of $C_n^3 \setminus C_n^2$ as given in the proof of Theorem 3.1.

We now consider the case where n is even. Again, we will extend the antimagic labeling L stated in the proof of Theorem 2.2.

Theorem 3.2 *If $n > 6$ is a even number that is not a multiple of 6, then C_n^3 has an antimagic labeling.*

Proof : Consider the labeling L used in the proof of Theorem 2.2. Recall that it has the following properties, where we use S_i^L to denote the vertex-sum of vertex i under the labeling L.

1. $S_0^L = 4n + 4$, $S_1^L = 2n + 7$, $S_2^L = 2n + 10$, $S_3^L = 2n + 15$,
2. $S_{i+1}^L = S_i + 4$ for $3 \leq i \leq n - 5$,
3. $S_{n-3}^L = 6n - 8$, $S_{n-2}^L = S_{n-4} + 8$, and $S_{n-1}^L = 5n - 2$.

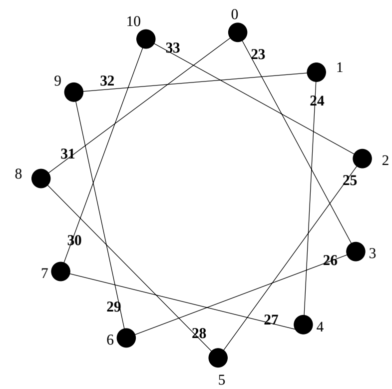

Figure 5: antimagic labeling of edges of $C_{11}^3 \setminus C_{11}^2$

We now show how to extend the antimagic labeling L for C_n^2, as given by the construction in the proof of Theorem 2.2, to an antimagic labeling M for C_n^3 such that $M|C_n^2 = L$. For each edge $e \in C_n^2$, assign $M(e) = L(e)$. For the edge $e = i, i+3$, where $0 \leq 0 \leq i < n-3$, we assign $M(e) = 2n+i+1$. For the edge $e = \{n-3, 0\}$, we assign $M(e) = 3n-2$. For the edge $e = \{n-2, 1\}$, we assign $M(e) = 3n-1$. Finally, for the edge $e = \{n-1, 2\}$, we assign $M(e) = 3n$. This gives a labeling M for C_n^3. We now show that it is an antimagic labeling of C_n^3.

By the definition of M,$S_0 = (4n+4)+(2n+1)+(3n-2) = 9n+3$, $S_2 = (2n+10)+(2n+3)+(3n) = 7n+13$, $S_{n-3} = (6n-8)+(3n-2)+(3n-5) = 12n-15$ and $S_{n-1} = (5n-2)+(3n-3)+(3n) = 11n-5$. These numbers are odd and pairwise distinct because of the assumptions on n is even and $n > 6$. Also, $S_1 = (2n+7)+(2n+2)+(3n-1) = 7n+8$ and $S_3 = (2n+15)+(2n+1)+(2n+4) = 6n+20$, which are both even and are distinct, as $n \neq 12$. As $M(e) = 2n+i+1$ for edges of the form $e = \{i, i+3\}$, where $0 \leq i < n-3$, and $S_{i+1}^L = S_i^L + 4$, for $3 \leq i \leq n-5$, it follows that $S_{i+1} = S_i + 6$, for $3 \leq i \leq n-3$. Therefore, S_i is even, for $3 \leq i \leq n-4$, and they are pairwise distinct. Thus, S_{n-2}, which equals $S_{n-4}+12$, is also even as $S_{n-2} = 12n-10$. So it remains to show that S_1 is not in the set $\{S_3, S_4, ..., S_{n-4}, S_{n-2}\}$. Consider $S_1 - S_3 = n-12$. If the vertex-sum S_1 appears again (as one of $\{S_3, S_4, ..., S_{n-4}, S_{n-2}\}$, then $n-12$ must is be a multiple of 6. But $n-12$ is a multiple of 6 if and only if n is a multiple of 6. As we assume n is not a multiple of 6, then the vertex sum S_1 occurs only once. Therefore all the vertex-sums for the labeling M are distinct. □

Figure 6 shows the labeling of the edges of $C_n^3 \setminus C_n^2$ as given in the proof of Theorem 3.1. As it turns out, a slight modification to labeling M of Theorem 3.2 gives an antimagic labeling for C_n^3 where $n > 6$ is even and a power of 6.

Theorem 3.3 *If $n > 6$ is a even number that is a multiple of 6, then C_n^3 has an antimagic labeling.*

Proof : In the labeling M given in the proof of Theorem 3.2, make the following two modifications.

1. For the edge $e = \{n-2, 1\}$, we assign $M(e) = 3n$, and
2. for the edge $e = \{n-1, 2\}$, we assign $M(e) = 3n-1$.

With this modification, we have $S_0 = 9n+3$ (odd), $S_1 = 7n+9$ (odd), $S_2 = 7n+12$ (even),$S_{n-3} = 12n-15$ (odd) $S_{n-2} = 12n-9$ (odd), $S_{n-1} = 11n-6$ (even). The vertex sums S_i, for $3 \leq i \leq n-4$ have the same values as in the proof of theorem 3.2 and therefore are all even and pairwise distinct. The values S_0, S_1, S_{n-3}, and S_{n-2} are all odd and distinct. All that remains to show is that S_2 and S_{n-1} are not the vertex sums of some other vertex. Clearly $S_2 \neq S_{n-1}$. To show that S_2 and S_{n-1} does not appear in $\{S_3, S_4, ..., S_{n-4}\}$, it suffices to show that $S_2 - S_3$ and $S_{n-4} - S_{n-1}$ are not divisible by 6. If $S_2 - S_3 = n-8$ is divisible by 6, then n must be of the form $n = 6k+2$. As we assume that n is a multiple of 6, $n-8$ cannot be divisible by 6. Similarly, if $S_{n-4} - S_{n-1} = n-16$ is divisible by 6, then n must be of the form $6k+4$. As we assume that n is a multiple of 6, $n-16$ cannot be divisible by 6. Thus, all the vertex sums are distinct, and M is a antimagic labeling for $n > 6$ and a multiple of 6.

□

Figures 6 and 7 gives the antimagic labelings of the edges of $C_{12}^3 \setminus C_{12}^2$ and $C_{16}^3 \setminus C_{16}^2$ respectively. Figure 8 gives an antimagic labeling for C_6^3. Theorems 3.2 and 3.3 along with Figure 8 implies that C_6^3 is antimagic, for all $n \geq 6$.

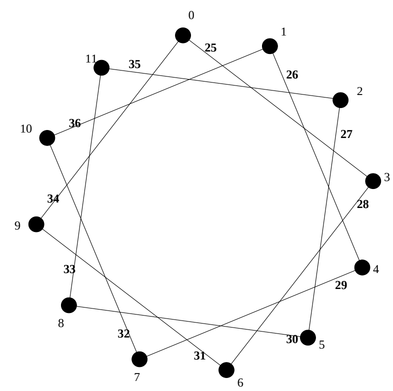

Figure 6: antimagic labeling of edges of $C_{12}^3 \setminus C_{12}^2$

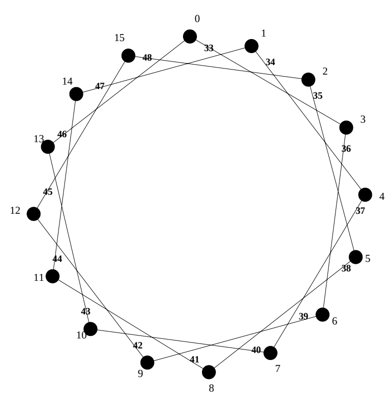

Figure 7: antimagic labeling of edges of $C_{16}^3 \setminus C_{16}^2$

Corollary 3.4 *: For $n \geq 6$, C_n^3 is antimagic.*

4 The graph C_n^4

In this section, I will prove that C_n^4 has an antimagic labeling. I will do this by extending the labeling given in Section 2 for C_n^3.

Theorem 4.1 *If $n \geq 7$ is an odd integer, then C_n^4 has an antimagic labeling.*

Proof I will show how the extend the labeling M for C_n^3, as given in the proof of Theorem 3.1, to an antimagic labeling N for C_n^4 such that $N|C_n^3 = M$. For each edge $e \in C_n^3$, assign $N(e) = M(e)$. For the edge $e = \{i, i+4\}$, where $0 \leq i < n-4$, assign $N(e) = 3n+i+1$. For the edge $e = \{n-4, 0\}$, assign $N(e) = 4n-3$. For the edge $e = \{n-3, 1\}$, assign $N(e) = 4n-2$. For the edge $e = \{n-2, 2\}$, assign $M(e) = 4n-1$. Finally, for the edge $e = \{n-1, 3\}$, we assign $N(e) = 4n$. I claim that N is an antimagic labeling of C_n^4.

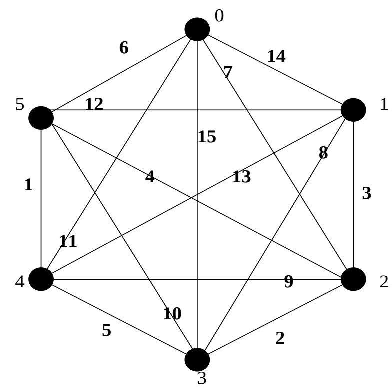

Figure 8: antimagic labeling of C_6^3

Based on the labeling N, $S_0 = 16n$, $S_1 = 14n + 8$, $S_2 = 14n + 16$, $S_3 = 13n + 24$, $S_4 = 12n + 32$, $S_{i+1} = S_i + 8$, for $4 \leq i \leq n-3$, and $S_{n-1} = 19n - 8$. It is easy to see that S_0, S_1, S_2, S_4 are even. As n is even, the vertex sums S_0, S_1, S_2 and S_4 are pairwise distinct. As S_4 is even, so is S_i, for $4 \leq i \leq n-2$ and these vertex sums are distinct. As S_3 and S_{n-1} are odd, they are distinct from all the other vertex sums. They are also different from each other. It remains to show that the vertex sums S_0, S_1, S_2 are not one of the vertex sums $S_4, S_5, ..., S_{n-2}$. For S_0, $S_0 - S_4 = 4(n-8)$ is divisible by 8 if and only if n is even. As n is odd, the vertex sum S_0 is unique. For S_1, $S_1 - S_4 = 2(n-12)$ is divisible by 8 implies n is even. So S_1 is unique also. For S_2, $S_2 - S_4 = 2(n-8)$ is divisible by 8 implies n is even. So S_2 is also unique. Therefore, all the vertex sums are distinct and N is an antimagic labeling of C_n^4.

□

Theorem 4.2 *Let $n \geq 8, n \neq 12, 14$ be an even integer. Then C_n^4 is antimagic.*

Proof We begin be handling the special case where $n = 8$. To show that C_8^4 is antimagic, start with the labeling M for C_n^3, as given in the proof of Theorem 3.2. Now label the edge $\{0, 4\}$ with 15, the edge $\{1, 5\}$ with 16, the edge $\{2, 6\}$ with 18, and the edge $\{3, 7\}$ with 17. It is easy to show that this is an antimagic labeling for C_8^4.

We now suppose that $n > 8$. I will show how the extend the labeling M for C_n^3, as given in the proof of Theorem 3.3, to an antimagic labeling N for C_n^4 such that $N|C_n^3 = M$. Note that when n is not a multiple of 6, the labeling M may not be an antimagic labeling of C_n^3. For each edge $e \in C_n^3$, assign $N(e) = M(e)$. For the edge $e = \{i, i+4\}$, where $0 \leq i < n-4$, assign $N(e) = 3n + i + 1$. For the edge $e = \{n-4, 0\}$, assign $N(e) = 4n - 3$. For the edge $e = \{n-3, 1\}$, assign $N(e) = 4n - 2$. For the edge $e = \{n-2, 2\}$, assign $M(e) = 4n$. Finally, for the edge $e = \{n-1, 3\}$, we assign $N(e) = 4n - 1$. I claim that N is an antimagic labeling of C_n^4.

Based on the labeling labeling N, $S_0 = 16n+1$, $S_1 = 14n+9$, $S_2 = 14n+15$, $S_3 = 13n+23$, $S_4 = 12n+32$, $S_{i+1} = S_i + 8$, for $4 \leq i \leq n-5$, $S_{n-3} = 20n - 23$, $S_{n-2} = 20n - 14$ and $S_{n-1} = 19n - 11$. It is easy to see that S_4 is even, and therefore $S_4, S_5, S_6, ..., S_{n-4}$ are all even and distinct. In since $S_1 = S_3$ only when $n = 14$ and $S_{n-3} = S_{n-1}$ only when $n = 12$, $S_0, S_1, S_2, S_3, S_{n-3}, S_{n-1}$ are all odd and pairwise distinct. It remains to show that S_{n-2} is not in the set $S = \{S_4, S_5, S_6, ..., S_{n-4}\}$. This is true, since $S_{n-2} - S_{n-4} = (20n - 14) - (20n - 32) = 18 > 0$. Therefore, all the vertex sums are distinct. □

At this point, we could handle the remaining cases $n = 12, 14$ separately. Instead, we give another general construction that will deal with these two cases.

Theorem 4.3 *Let $n > 8$ be an even integer of the form $8k, 8k+4$ or $8k+6$. Then C_n^4 is antimagic.*

Proof I will show how the extend the labeling M for C_n^3, as given in the proof of Theorem 3.2, to an antimagic labeling N for C_n^4 such that $N|C_n^3 = M$. Note that when n is a multiple of 6, the labeling M may not be an antimagic labeling of C_n^3. For each edge $e \in C_n^3$, assign $N(e) = M(e)$. For the edge $e = \{i, i+4\}$, where $0 \leq i < n-4$, assign $N(e) = 3n + i + 1$. For the edge $e = \{n-4, 0\}$, assign $N(e) = 4n - 3$. For the edge $e = \{n-3, 1\}$, assign $N(e) = 4n$. For the edge $e = \{n-2, 2\}$, assign $M(e) = 4n - 1$. Finally, for the edge $e = \{n-1, 3\}$, we assign $N(e) = 4n - 2$. I claim that N is an antimagic labeling of C_n^4.

Based on the labeling labeling N, $S_0 = 16n + 1$, $S_1 = 14n + 10$, $S_2 = 14n + 15$, $S_3 = 13n + 22$, $S_4 = 12n + 32$, $S_{i+1} = S_i + 8$, for $4 \leq i \leq n-5$, $S_{n-3} = 20n - 21$, $S_{n-2} = S_{n-4} + 16$ and $S_{n-1} = 19n - 11$. It is easy to see that S_4 is even, and therefore $S_5, S_6, ..., S_{n-4}, S_{n-2}$ are all even and pairwise distinct. In addition, S_0, S_2, S_{n-3} and S_{n-1} are all odd and pairwise distinct, as $n > 8$. It remains to show that S_1 and S_3, which are both even and distinct from each other, cannot be in the set $S = \{S_4, S_5, S_6, ..., S_{n-4}, S_{n-2}\}$. To see this, suppose S_1 is in the set S. Then $S_1 - S_4 = 14n + 10 - (12n + 32) = 2(n - 11)$ must be divisible by 8. But since n is even, this is not possible. Therefore, S_1 is not in the set S. Now, suppose S_3 is in the set S. Then, $S_3 - S_4 = 13n + 22 - (12n + 32) = n - 10$ must be divisible by 8. But $n - 10$ is divisible by 8 if and only if n is of the form $8k + 2$. Since we assumed n is not of this form, S_3 cannot be in the set S. Therefore, all the vertex sums are distinct.

□

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