Friendship 3-Hypergraphs

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Abstract

A *friendship 3-hypergraph* is a 3-hypergraph in which any 3 vertices, u, v and w, occur in pairs with a unique fourth vertex x; i.e., uvx, uwx, vwx are 3-hyperedges. Sós found friendship 3-hypergraphs coming from Steiner Triple Systems. Hartke and Vandenbussche showed that any friendship 3-hypergraph can be decomposed into sets of K_4^3's. We think of this as a set of 4-tuples and call it a *friendship design*. We define a geometric friendship design to be a resolvable friendship design that can be embedded into an affine geometry. Refining the problem from friendship designs to geometric designs allows us state some more structure theorems about these geometric friendship designs and decreases the state space when searching for these designs. Hartke and Vandenbussche discovered 5 new examples of friendship designs which happen to be geometric. We find that there are exactly three (known) non-isomorphic geometric friendship designs on 16 vertices. We also improve the known upper and lower bounds on the number of edges in a friendship 3-hypergraph. Finally we show that no friendship 3-hypergraph exists on 11 or 12 points.

1 Introduction

Before we discuss friendship 3-hypergraphs, we need to discuss friendship graphs. This is a well-known and beautiful area of graph theory; see [4] and [1]. A *friendship* graph is a graph in which any two vertices have exactly one common neighbour.

There are two types of friendship graphs: 1) the universal friendship graph - The graph consists of $(n-1)/2$ cycles of length 3, all joined at one vertex, called the *universal friend.* The graphs are called *windmill graphs* or 2) the regular friendship graph which has the same number of edges incident with any vertex. The first graph exists only for an odd number of vertices and the latter graph does not exist except for the trivial case of a triangle. (one-vane windmill graph). The proofs can be found in [4] and [1]. This effectively ends the discussion on friendship graphs. So researchers generalized the concept in many ways.

We are interested in the generalization by Sós [3]. A *friendship 3-hypergraph* is a 3-hypergraph in which any 3 vertices (elements), u, v and w, occur in pairs with a unique fourth vertex x; i.e., uvx, uwx, vwx are

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3-hyperedges. The element x is said to *complete* the elements u, v and w. A 3-hypergraph has exactly 3 vertices on all its hyperedges.

If one thinks of the 3-hyperedges as sets (blocks) of 3 elements, then the problem of finding friendship 3-hypergraphs resembles a design theory problem. But how many of the results about friendship graphs will generalize to this setting? Is it more like graph theory or more like design theory?

The first question is "Is there a universal friendship 3-hypergraph?". So we must define a universal friend 3-hypergraph. A friendship 3-hypergraph that has a vertex (*universal friend*) that appears in a hyper-edge with each pair of the other vertices is called a *universal friend 3-hypergraph.* We need the following definition to construct universal 3-hypergraphs. A *Steiner Triple System* on n elements is a set of triples (sets of 3 elements) from an n-set such that each pair of elements occurs exactly once in some triple.

Theorem 1.1 *(Sós [3]) There exists a universal friend 3-hypergraph if and only if* $n = 2, 4$ *mod 6,*

Proof : For $n-1 = 1, 3$ modulo 6, there exists a Steiner Triple System on $n-1$ elements. To this collection of 3-sets add the universal friend ∞ to each three set getting a 4-set. Now replace each four-set with the four subsets of size 3 from each 4-set. Clearly ∞ is the only vertex that completes u, v and w where u, v and w are from the Steiner System because the pairs occur exactly once in the Steiner Triple System. Also u, v and ∞ have a unique completion w where u,v and w are a triple in the Steiner System. So the construction yields a friendship 3-hypergraph on n vertices. Clearly, if there is a universal friend in a friendship 3-hypergraph on n vertices then the quads not containing the universal friend must form a Steiner Triple System on $n-1$ elements. □

Sós [3] asked whether there were any other friendship 3-hypergraphs? This was answered in the positive by Hartke and Vandenbussche [2]. They formulated the problem as an integer programming problem. Using integer programming software, they found friendship 3-hypergraphs on 8 vertices (unique), 16 vertices(≥ 3 non-isomorphic hypergraphs) and 32 vertices (≥ 1 non-isomorphic hypergraph). The 3-hypergraphs were regular; i.e., all vertices appeared the same number of times. Further the 3 friendship hypergraphs on 16 vertices had 108, 114 and 272 hyperedges (or triples).

Sòs [3] asked her question in a conference exploring the connections between graph theory, design theory and geometries. Sós has shown a connection between graph theory and design theory versions of this hypergraph problem. In Section 2, we show a connection between friendship 3-hypergraphs and geometries by showing that the friendship 3-hypergraphs found by Hartke and Vandenbussche can be embedded into affine planes. We do a complete search of the affine plane on 16 points to show that no other friendship 3-hypergraphs come from this geometry. In Section 3 we improve the lower and upper bounds on the number of hyperedges in a friendship 3-hypergraph. Finally in Section 4 we show that there are no friendship 3-hypergraphs on 11 or 12 vertices.

2 Geometric Friendship Designs

Before we can show the connection between friendship 3-hypergraphs and geometries, we need a definition and two easy but important results from Hartke and Vandenbussche [2]. A K_4^3 is the set of four subsets of size three from a set of 4 elements.

Lemma 2.1 *([2]) Every hyperedge of a friendship 3-hypergraph must be contained in a unique* K_4^3*.*

Theorem 2.2 *([2]) The hyperedges of the friendship 3-hypergraph can be partitioned into* K_4^3*'s.*

This means that the set of hyperedges (3-sets) of a friendship 3-hypergraph on n vertices can be written down as a set of subsets of size 4 from a set of size n. We call the subsets of size 4 *quads*. This makes the study seem more like design theory. We call the set of 4-sets obtained this way from a friendship 3-hypergraph a *friendship design.* We refer to elements in quads. The *friendship property* for friendship designs becomes that for any 3 elements, u, v and w, there is a fourth element x such that uvx, uwx, vwx occur either in a quad $uvwx$ or in 3 distinct quads. Clearly there is a one-to-one relationship between friendship 3-hypergraphs and friendship designs. We now give some examples. As usual, we do not write in the brackets and all the commas.

eg. The universal friend design on 8 elements which has 7 quads

$\infty 013$ $\infty 124$ $\infty 235$ $\infty 346$ $\infty 450$ $\infty 561$ $\infty 602$

eg. The friendship design on 8 elements which has 8 quads

0123 0145 0167 0246 1357 2345 2367 4567

which can be resolved into some of the planes of $AG(2,3)$:

0123	4567
0145	2367
0167	2345
0246	1357

The friendship designs of Hartke and Vandenbussche [2] can be partitioned into groups of disjoint quads (resolution classes) that partition the vertex set. Call such a friendship design that can be so partitioned a *resolvable friendship design.* It is surprisingly easy to show that each of the 5 friendship designs that were found by Hartke and Vandenbussche [2] can be embedded in an AG(2,n). Just write the elements in the examples listed in their paper as a number in binary, low order bits on the right, and check the properties. We call a resolvable friendship design that can be embedded into an affine design a *geometric friendship design.* So what we want to do next is to search for all geometric friendship designs but first we need to define some concepts in the geometry.

Let F be a field of order 2 and let V be a vector space over F of dimension $n+1$; i.e., $n+1$ tuples. There are 2^{n+1} vectors. Let $\mathcal{P}(V)$ be the projective space of dimension n, i.e., the set of all one dimensional subspaces of V. Elements of $\mathcal{P}(V)$are called points of the projective space. Thus, if $a \in V, a \neq 0$, then $\{0, a\}$ is a point of $\mathcal{P}(V)$. We will denote this point by a itself. There are $2^{n+1}-1$ points in $\mathcal{P}(V)$. Let H be a hyperplane in $\mathcal{P}(V)$. Then H has $2^n - 1$ points. $\mathcal{A}(V) = \mathcal{P}(V) \backslash H$ is the affine space of 2^n points.

A plane in $\mathcal{P}(V)$ has 7 points and there are 3 points on a line. A plane in $\mathcal{A}(V)$ has 4 points and there are 2 points on a line of $\mathcal{A}(V)$. Suppose W is a plane in $\mathcal{A}(V)$, then $W = W'' \cap \mathcal{A}(V)$ where W'' is a plane in $\mathcal{P}(V)$ and $W' = W'' \cap H$ is a line in $\mathcal{P}(V)$. If y is any point in W, then W'' is the unique plane in $\mathcal{P}(V)$ containing y and W'. Of course, any plane of $\mathcal{P}(V)$ is the Fano plane.

Any line in $\mathcal{P}(V)$ with points u and v will have $u+v$ as its third point. Also the unique plane in $\mathcal{A}(V)$ that contains u, v, and w will have $u+v+w$ as its fourth point. So if $W = \{u, v, s, t\}$ is a plane in $\mathcal{A}(V)$, then $W' = \{u+v, u+s, u+t\}$. The set of all planes in $\mathcal{A}(V)$ forms a $3-(2^n, 4, 1)$-design on $((2^n(2^n-1)(2^n-2)/24 = (2^{n-3}(2^n-1)(2^n-2)/3$ points. Also given any line in L in H, there are exactly 2^{n-2} planes W of $\mathcal{A}(V)$ such that $W' = L$ and these 2^{n-2}planes form a parallel class of $\mathcal{A}(V)$.

A *triangle* in $\mathcal{P}(V)$ is a set of 3 mutually intersecting, but not concurrent lines. Each triangle $T = \{L_1, L_2, L_3\}$ has 3 vertices v_1, v_2, v_3 which are common to one pair of the three pairs of lines. Each line of the triangle has 2 vertices from v_1, v_2, v_3 and a point left over which will be called the *midpoint* of the line. The midpoint of L_i will be denoted by w_i. The set $m_T = \{w_1, w_2, w_3\}$ of midpoints of the triangle T is a line in $\mathcal{P}(V)$ and will be called the *midpoint line* of the triangle T. Also there is is a unique point in the plane P_T of $\mathcal{P}(V)$ containing the triangle T, which is neither a vertex nor a midpoint in the triangle T. We will denote this unique point by c_T and call it the center of triangle T. Clearly, $c_T = v_1 + v_2 + v_3$. a SET s of lines of $\mathcal{P}(V)$ is said to have the *nomidpoint property*, if for every triangle T of lines in S, the midpoint line m_T is not in S. Now suppose x is a point of $\mathcal{P}(V)$ and $L = \{y_1, y_2, y_3\}$ is a line of $\mathcal{P}(V)$ such that $x \notin L$, then we will denote by $P(x, L)$ the unique plane in $\mathcal{P}(V)$ containing both the point x and the line L. Further, we will denote by $S(x, L)$ the set $\{xy_1, xy_2, x(y_1 + y_2), L\}$ which are the 4 lines of the plane $P(x, L)$.

Let S be a set of lines of a plane W of $\mathcal{P}(V)$. Then we have the following:
(i) If $|S| \leq 3$, then S has the nomidpoint property.
(ii) If $S = 4$ then one of the following is true:
(a) There exists a point x and a line L of W such that $x \notin L$ and $S = S(x, L)$. In this case, S has the nomidpoint property.
(b) There exists a point $y \in W$ such that S is the set of 4 lines in W not on y and S does not have the midpoint property.
(iii) If $|S| \geq 4$ then S does not have the nomidpoint property.

Let S be a set of lines of H. Then we will say that S has the *unique midpoint property*, if the following conditions hold:
i) S has the nomidpoint property.
ii) For every line L of H, $L \notin S$, there is a unique triangle T of lines in S such that L is the midpoint line of T.

Now Suppose S is a set of lines of H satisfying the unique midpoint property . Let L be a line in H not in S. We will denote by $T_{L,S}$ the unique triangle in S for which L is the midpoint line. Let $E(S)$ be the set of all planes W of $\mathcal{A}(V)$ such that the line $W \cap H = W'$ is in S. Then $E(S)$ has $|S|$ parallel classes and $2^{n-2}|S|$ subsets of $\mathcal{A}(V)$.

We can now state out theorem.

Theorem 2.3 *If a set of lines, S, obeys the unique midpoint property then E(S) is a geometric friendship design.*

Proof Let u, v and w be 3 distinct points of $\mathcal{A}(V)$ and $W = P_{aff}(u, v, w) = \{u, v, w, u + v + w\}$. If, $W' = \{u + v, u + w, v + w\}$ is an edge in S then clearly $\{u, v, w, u + v + w\}$ is in $E(S)$ and $u + v + w$ is friend to u, v, w. If $W' = \{u + v, u + w, v + w\}$ is not an edge in S, so W is not a plane of $E(S)$. Then using the second part of the definition of the unique midpoint property it follows that W' is the midpoint line of the unique triangle T_W which resides in S. Let c_W be the center of the triangle T_W. We will denote by s_W the unique point $c_W + (u + v + w)$ of $\mathcal{A}(V)$. Clearly, s_W is in $\mathcal{A}(V)$. Then $c_W = s_W + u + v + w$. Then the vertex on the line joining $s_W + U + v + w$ to $u + v$ must be $s_W + U + v + w + u + v = s_W + w$. Similarly the other vertices of T_W are $s_W + u$ and $s_W + v$. All 3 vertices are in H and the lines joining them in pairs are the lines of T_W and these lines are in S. The lines are $\{s_W + u, u + w, s_W + w\}$, $\{s_W + u, u + v, s_W + v\}$ and $\{s_W + w, v + w, s_W + v\}$. Then the planes, $P_{aff}(s_W uv)$, $P_{aff}(s_W uw)$ and $P_{aff}(s_W vw)$ are in $E(S)$. Thus s_W is a friend of u, v, w. In either case, u, v, w has a friend.

Now suppose u, v and w are 3 distinct points of $\mathcal{A}(V)$, $W = P_{aff}(uvw)$ and s is a friend of u, v, w. If $T = \{s + u, s + v, s + w\}$ is a line in H then $(s + u) + (s + v) + (s + w) = 0$ and $s = u + v + w$ and so $T = \{s + u, s + v, u + v\} = W'$. Thus W is a plane of $E(S)$. Now Suppose that T is not a line in H. Since s is a friend of W, the planes, $P_{aff}(suv)$, $P_{aff}(suw)$ and $P_{aff}(svw)$ are in $E(S)$ and T is the set of vertices of a triangle T of lines in S. Also $W' = \{u + v, u + w, v + w\}$ is the midpoint line of T. So $W' \notin S$. So $W = P_{aff}(uvw) \notin E(S)$. Using the second part of the midpoint property, $T = T_W$ and the center $c_W = (s + u) + (s + v) = (s + w) = s + (u + v + w)$. So $s = s_W$ is a friend of u, v, w. So either $u + v + w$ or s_W, as defined in the previous paragraph, is the unique friend of u, v, w. $\square$

The converse of Theorem 2.3 is easy to prove and is now presented.

Theorem 2.4 *If a geometric friendship design exists in an $AG(2,n)$ then there exists, in a hyperplane H of $PG(2,n)$, a set S that has the midpoint property.*

Proof Consider the set of planes in geometric friendship graph. The graph can be decomposed into resolution classes. Let S be the set of lines defined by the resolution classes in $\mathcal{A}(V)$ as follows: Let $L \in S$ be the line $W'' \cap H$ where W'' is the unique plane in H that contains a plane of the resolution class. The rest of the proof closely follows the ideas of Theorem 2.3 and are left out. $\square$

These ideas are important when writing a backtrack algorithm to find all geometric friendship designs. The recursion is 17 levels deep instead of 68 levels deep. This makes a huge difference in the time complexity. So we did a complete search for geometric friendship designs on 16 points.

Theorem 2.5 *There are only 3 non-isomorphic geometric friendship designs on 16 elements.*

The program first found all lines in H and used them as an index into a one-dimensional array. The array entry was set to 1 when its line entered S and to 2 if it was the midpoint line of a triangle in S. So to look for the next line of S, the program would find the first (lexographically least) line with an undefined entry, change it to 1, check which triangles were formed and put a 2 in the appropriate place in the array. If a 2 was already there, then the program backtracked. It was a very "clean" backtrack to program.

Although, the program found over a thousand geometric friendship designs, the list held only the three obviously non-isomorphic designs found by Hartke and Vandenbussche [2]. We tried 32 points but that was too big to finish and we only found the design exhibited by Hartke and Vandenbussche [2].

3 Bounds

Hartke and Vandenbussche [2] found some easy bounds on the size of friendship 3-hypergraphs. With some work they can be improved upon. But first we need this lemma.

Lemma 3.1 *(Hartke and Vandenbussche [2]) Every pair of elements in a friendship design occurs in at least one quad.*

Hartke and Vandenbussche [2]proved a lower bound for the number of quads in a friendship design on n elements of $\lceil \frac{n(n-2)}{8} \rceil$. We will slightly improve this although the odd and even n case differ. For the odd case, we need the following technical lemma.

Lemma 3.2 *In a friendship design on n elements where n is odd, if two elements both occur only once with a third element, these three elements can not occur in the same quad.*

Proof Assume that two elements, say 2 and 3 occur once in a quad with some element, say 1. Further assume 1, 2 and 3 occur together in some quad of the friendship design, say 1234. Then the triple 1,2,x,where $5 \leq x \leq n$, must have a completion. But the pair 12 occurs only once so the completion must be 3 or 4. Since 13 can not occur in any other quads, the completion must be 4. This forces the triples $14x$. These $n-4$ triples force the existence of $(n-3)/2$ quads containing the pair 14. Along with the quad 1234, this implies that there are $(n-3)/2+1 = (n-1)/2$ quads containing 14 pairs. This contradicts that there can only be $\lceil (n-2)/2 \rceil = (n-3)/2$ quads containing any particular pair of elements. $\square$

It would be nice to prove this lemma for even n, but the existing friendship designs on even n get in the way. You have to show that if 1, 2 and 3 exist as in the previous lemma, then a universal friendly design or one of the designs of Hartke and Vandenbussche exist. This we have not been able to do. So we carry on with the odd number of elements case. We need a definition.

Let $N(n)$ be the number of quads in any friendship design on n elements.

Lemma 3.3 *$N(6s+1) \geq (4s-1)(6s+1)/4$, $N(6s+3) \geq (4s+1)(6s+3)/4$ and $N(6s+5) \geq (4s+2)(6s+5)/4$*

Proof Let $n = 6s+1, s \geq 1$. Assume that an element, say 1 occurs in $4s-2$ quads. Then there are $12s-6$ elements. including multiplicities, that occur with 1. Then, on average, an element occurs with 1, $3(4s-2)/6s = 2 - 6/6s$ times. There can not be more than $4s-2$ elements that occur once with element 1 because of Lemma 3.2. The frequency of the elements in the quads containing 1 can be summarized as $6+x$ elements occur once with 1 and $6s-6-2x+y$ elements occur twice with 1 and $x-y$ elements that, in total, occur $x+2(x-y)$ times with 1. Let $2, 3, \ldots, (x+7)$ be the elements that occur exactly once with 1. By Lemma 3.2, there can only be one of $2, 3, \ldots, (x+7)$ in a quad with 1. Further, if 2 is in a quad with 1, say $12ab$, then 2, a and b can occur in, at most, 1+2+2+(x-2)-2 = $x+3$ quads containing the element 1 and one of $2, a, b$. We get this as element 2 occurs only once with 1. We assume that a and b occur at least twice with 1 and that the frequency over 2 i.e. $x-2$, is divide up between a and b. Then we subtract 2 as quad $12ab$ has been counted 3 times. We put the a's and b's in quads containing element 1 and one of $3, 4, \ldots, x+4$. This means that quad $12ab$ intersects quad $1(x+8)cd$ in element 1 only. Then triple $12(x+8)$ has no completion as the elements that occur with 12 are not the elements that occur with $1(x+8)$. This contradiction means that any element in the design must occur at least $4s-1$ times. Then there are at least (4s-1)(6S+1)/4 quads. For $n = 6s+3$ and $6s+5$ the proofs are essentially the same. □

For n odd, our lower bound is roughly $n^2/6$ For n even, we can not prove this result. We can only slightly improve the lower bound for n even.

Lemma 3.4 *The number of quads in a non-universal friendship design on n, is at least $n^2/8$ where n is even.*

Proof Hartke and Vandenbussche [2] proved that any element in a friendship design on n elements occurs in at least $(n-2)/2$ quads. Let us assume some element 1 occurs in exactly $(n-2)/2)$ quads. There are $3(n-2)/2$ elements (counting multiplicities) that occur with 1. Then the average occurrence with 1 is $\frac{3(n-2)}{2(n-1)}$ $= 1 + \frac{n+2}{2n-2}$. So there are at least $(n+2)/2$ elements that occur with 1 exactly one time. This means that there are two elements, say 2 and 3 that occur exactly once with element 1 and all are in the quad, say 1234. As in the proof of Lemma 3.2, we get quads $1234, 1456, 1478, \ldots, 14(n-1)n$. There are no more quads containing the element 1. Consider the completion of 1, a and b where $5 \leq a \neq b \leq n$. If $1ab$ is in a quad then the completion is 4. If $1ab$ is not in a quad then only 4 could be a completion as all quads containing a 1 are determined and the quads containing $1a$ and $1b$ have only 4 in common. So there must be the triple $4ab$ in a quad. So

triples $4ab$ where $a, b \in \{1, 2, 3, 5, \ldots, n\}$, $a \neq b$ are all in quads somewhere forcing the quads to be a universal friend graph. But the design is a non-universal friendship design so every element must occur in at least $n/2$ quads. Then there are $n^2/2$ elements in the design and $n^2/8$ quads in the design. □

Hartke and Vandenbussche [2] proved that that the upper bound on the number of quads in a friendship design on n elements is $\binom{n}{3}/4$ or roughly $\frac{n^3}{24}$. We will improve this to roughly $\frac{n^3}{36}$.

Lemma 3.5 *The number of quads in a friendship design on n elements is at most $\frac{\binom{n}{3}(2n-6)}{4(3n-10)}$.*

Proof Consider a triple, say 123. Let the triple appear in some quad say, 1234. Consider the other quads containing the pairs 12, 13 or 23. The other quads must be distinct or we get a repeated triple. The other quads must contain two elements from $5-n$. The other quads containing 12 (or 13 or 23) must contain at most one element from $5-n$. Also an element from $5-n$ can appear in at most 2 of the other quads as otherwise the triple 1,2 ,3 has two completions. So the number of other quads is at most $\frac{2(n-5)}{2} = n-4$. So the pairs, 12, 13 and 23, appear at most $n-4+3 = n-1$ times.

Let the triple 1,2,3 not occur in any quad. In this case there is no quad $123x$. But there is an element (the completion of 1,2,3) that appears in 3 distinct quads containing 12, 13 or 23. Every other element can appear at most twice in these quads. So the number of quads containing 12, 13 or 23 and the number of

n	old lower bound $n(n-2)/8$	new lower bound Lemmata 3.4, 3.3	new upper bound $\frac{\binom{n}{3}(2n-6)}{4(3n-10)}$	old upper bound $\binom{n}{3}/4$	# blocks in designs
4	1	2	1	1	1
5	2	3	2	3	-
6	3	5	4	4	-
7	5	6	7	6	-
8	6	8	10	14	7,8
9	8	12	14	21	-
10	10	13	21	30	12
11	13	17	28	41	-
12	15	18	38	55	?
13	18	23	49	71	?
14	21	25	62	91	26,?
15	25	34	78	113	?
16	28	32	95	140	35,52,56,68,?
32	120	128	836	1240	155,344,?

Table 1: Bounds and Number of Blocks on Quads in Friendship Designs

times the three pairs appear is is at most $\lceil\frac{2(n-4)+3}{2}\rceil = n-3$.

Let b be the number of quads in the friendship design. Then there are $4b$ distinct triples in the design whose 3 pairs occur at most $n-1$ times and $\binom{n}{3}-4b$ triples that do not occur in the design whose pairs occur at most $n-3$ times. But each pair is in $(n-2)$ triples and so every pair is counted at least that many times . So the number of pairs (counting multiplicities is $\frac{4b(n-1)+(\binom{n}{3}-4b)(n-3)}{n-2}$. Since there are 6 pairs in a quad, the number of quads is at most $\frac{4b(n-1)+(\binom{n}{3}-4b)(n-3)}{6(n-2)}$ which must be greater than b. This gives that $b \leq \frac{\binom{n}{3}(2n-6)}{4(3n-10)}$. $\square$

To give an overview of the preceding results, we present Table 1. The known lower bound is quadratic and the known upper bound is cubic. Above the line the dashes indicate that there was no friendship design found by Hartke and Vandenbussche [2]. They used an integer program formulation and solved it using C-PLEX. Our programs confirm these results. However we were able to completely search the case where $n = 11, 12$.

4 Our Computer Results

Running a straightforward backtracking program, we were able to reproduce all the complete searches that Hartke and Vandenbussche did for $n \leq 10$. But this approach was too slow for $n = 11, 12$. The problem is that the search tree is very bushy. That is, there are many choices from each node in the search space. Also we don't get to prune the tree, until we are many levels down the search tree. So we developed a multi-stage algorithm.

Our Algorithm - 7 Steps

Step 1: We let M be the maximum number of times a pair of elements can occur in a quad in a friendship design. Further we assume that there is a pair 12 that occurs M times. So we start M at $(n-2)/2$ and go down to 2 in decrements of 1. Without loss of generality we can fill in the quads containing the pair 12. They are $1234, 1256, 1278, \ldots, 12(n-1)n$. We call this the *starter set*.

Step 2: From the starter set generate all possible sets that contain quads containing a 1 that does not cause two completions for some triple of elements. We will assume that these are the only quads in the set containing element 1. We check to see if every element occurs in a block with element 1.
eg. $n = 8, M = 3$
1234 1256 1278
1234 1256 1278 1357
1234 1256 1278 1358
1234 1256 1278 1357 1368
etc.

Step 3: Eliminate isomorphic copies. Call the result a 1-set.

Step 4: For each of the 1-sets generate a list of 'forces'.
eg. $n = 9$ $M = 3$
1234
1256
1278
1357
Pair 13 occurs with elements 2,4,5,7 and pair 16 occurs with elements 2 and 5. Since we are assuming no more quads containing a 1, the only completion for 1,3,6 is the element 2. So the pair 36 must occur with a 2 in a quad. So the triple 236 must occur in some quad. The only possibilities are 2346, 2356, 2367, 2368 and 2369. But 2346 has a 3-intersection with the first quad, 2356 has a 3-intersection with the second quad. This leaves 2367 and 2368 and 2369 as forces. That is one of these 3 quads must be in the design. We get a list of these "forces" for a 1-set. If we can pick a quad from each 'force', then we have a candidate. If not, we have a dead end we can eliminate this possibility. A 1-set may generate 0, 1, 2 or many candidates. All the candidates are grouped together. We go through the 'forces' in the order of fewer choices to more choices. The whole idea is to reduce the bushiness of the search tree.

Step 5: Eliminate isomorphic candidates.

Step 6: Throw the candidates into a normal backtracking program that will see if they lead to solutions. The program starts its search to add on quads at 1234.

Step 7: Eliminate isomorphic solutions.

We now give some statistics on the program for n=10, 11 and 12. We give the time taken by various steps and the number of configurations found.

Time in seconds

n \ M	2	3	4	5
10	3	14	1	.
11	66	86	11	.
12	1893	16310	17382	816

Number of configurations
$n = 10$

Case \ M	2	3	4	5
Non-isomorphic 1-sets	9	75	46	.
All Candidates	384	973	359	.
Non-isomorphic Candidates	4	131	108	.
Solutions	0	0	1	.
Non-isomorphic Solutions	0	0	1	.

$n = 11$

Case \ M	2	3	4	5
Non-isomorphic 1-sets	12	461	1045	.
All Candidates	0	170514	18351	.
Non-isomorphic Candidates	0	11830	14504	.
Solutions	0	0	0	.
Non-isomorphic Solutions	0	0	0	.

$n = 12$

Case \ M	2	3	4	5
Non-iso. 1-sets	21	3414	39935	11410
All Candidates	0	71630269	76553127	3105440
Non-iso. Cand.	0	5825458	70819810	2802491
Solutions	0	0	0	0
Non-iso. Sol.	0	0	0	0

5 Conclusions and Conjectures

We have defined geometric friendship designs and then characterized in geometric terms. This allowed us to prove, by computer, that there are only 3 non-isomorphic geometric friendship designs on 16 vertices. We also proved that there is no friendship 3-hypergraph on either 13 or 14 points. Also the upper and lower bounds on the number of edges in a friendship 3-hypergraph were improved. But still relatively little is known about friendship designs, but nevertheless we can look at the results and make many interesting conjectures. The first parallels the result from friendship graphs that state there are no friendship graphs on an even number of nodes. The second conjecture extends it.

Conjecture 1 *([2]) There are no friendship 3-hypergraphs on an odd number of points.*

Conjecture 2 *There are no friendship 3-hypergraphs on* $n = 0 \mod 6$ *points.*

The next conjecture is an analogue of the result that if a friendship graph is not a universal graph then it is regular. For friendship designs, the analogue is each pair of elements occurs the same number of times. This makes the friendship design into a BIBD. We define a *balanced incomplete block design* (BIBD) to be a pair $(\mathcal{V}, \mathcal{B})$ where $\mathcal{V}$ is a v-set and $\mathcal{B}$ is a collection of b k-subsets of $\mathcal{A}$ such that each element of $\mathcal{A}$ is contained in exactly r blocks and any 2-subset of $\mathcal{A}$ is contained in exactly λ blocks. It is easy to prove that for $\lambda \leq 4$, no friendship design is a BIBD. Also, for a fixed v, if λ is less than some linear value of n or greater than some quadratic value of n, then no friendship design is a BIBD. But in between those values nothing is known.

Conjecture 3 *No BIBD is a friendship design.*

The next conjecture is perpendicular to the result in graph theory.

Conjecture 4 *There are an infinite number of friendship 3-hypergraphs that are not universal friend 3-hypergraphs.*

The next two conjectures fits the known facts.

Conjecture 5 *All friendship 3-hypergraphs are either a universal friend 3-hypergraph or are regular.*

Conjecture 6 *All non-universal friendship graphs are on* 2^n *points.*

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