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Trajectory-Based Embedding Relations

Michael Domaratzki*
School of Computing, Queen's University
Kingston, ON K7L 3N6 Canada
domaratz@cs.queensu.ca

Abstract. We generalize the embedding order to shuffle on trajectories and examine its properties. We give general characterizations of reflexivity, transitivity, anti-symmetry and other properties of binary relations, and investigate the associated decision problems. We also examine the property of convexity of a language with respect to these generalized embedding relations.

Keywords: embedding order, shuffle on trajectories, convexity.

1. Introduction

The embedding order on words (also known as the subword order) is defined by $x \leq_e y$ if x can be obtained from y by deleting some number of letters; it is one of many orders of interest on words, including the prefix, suffix and factor orders (see Section 2 for definitions). These orders have each received much attention in the literature, especially in relation to the theory of codes. For example (and without trying to be exhaustive), we note the work of Haines [6], Jullien [9], Shyr and Thierrin [20], and Guo *et al.* [5] on the embedding order and its relation to codes. In this paper, we define a class of binary relations which generalizes the embedding order, as well as the prefix, suffix and factor orders.

It is well known that the embedding order $\leq_e$ can be represented using shuffle: given two words x, y, $x \leq_e y$ iff $y \in x \sqcup\!\sqcup \Sigma^*$. We extend this definition to shuffle on trajectories, recently introduced by Mateescu *et al.* [16]. Particular cases of these generalized embedding relations are the prefix, suffix, factor and embedding orders, as well as many binary relations defined for the purposes of studying their associated classes of anti-chains. In this paper, we study the properties of this class of generalized

*Research supported by an NSERC PGS-B graduate scholarship.
Address for correspondence: School of Computing, Queen's University, Kingston, ON K7L 3N6 Canada

embedding relations. We focus on properties of binary relations that are of interest to the theory of codes. We also examine the question of decidability of these properties for particular classes of our embedding relations.

We also consider languages which are convex with respect to these generalized embedding orders. We examine the decidability of determining whether a languages is convex with respect to one of our generalized embedding relations.

2. Definitions

For additional background in formal languages and automata theory, please see Yu [23] or Hopcroft and Ullman [7]. Let Σ be a finite set of symbols, called *letters*. Then Σ^* is the set of all finite sequences of letters from Σ, which are called *words*. The empty word ϵ is the empty sequence of letters. The *length* of a word $w = w_1 w_2 \cdots w_n \in \Sigma^*$, where $w_i \in \Sigma$, is n, and is denoted $|w|$. Note that ϵ is the unique word of length 0. Given a word $w \in \Sigma^*$ and $a \in \Sigma$, $|w|_a$ is the number of occurrences of a in w. A *language* L is any subset of Σ^*. By $\overline{L}$, we mean $\Sigma^* - L$, the complement of L. If $L_1, \ldots, L_k \subseteq \Sigma^*$ are languages, we use the notation $\prod_{i=1}^{k} L_i = L_1 L_2 \cdots L_k$. Given two languages $L_1, L_2 \subseteq \Sigma^*$, the *(right-) quotient* of L_2 by L_1 is denoted L_2/L_1 and is given by

$$L_2/L_1 = \{x \in \Sigma^* \ : \ \exists y \in L_1 \text{ such that } xy \in L_2\}.$$

We refer the reader to the appropriate chapter of Rozenberg and Salomaa [18] for the definitions of the regular, linear context-free (LCFL) and context-free languages (CFLs). We denote the families of regular, linear context-free and context-free languages by REG, LCF and CF respectively.

We denote by $\mathbb{N}$ the set of natural numbers: $\mathbb{N} = \{0, 1, 2, \ldots\}$. Given alphabets Σ, Δ, a *morphism* is a function $h : \Sigma^* \to \Delta^*$ satisfying $h(xy) = h(x)h(y)$ for all $x, y \in \Sigma^*$. Given a morphism $h : \Sigma^* \to \Delta^*$ and a language $L \subseteq \Sigma^*$, then the image of L under h is given by $h(L) = \{h(x) \ : \ x \in L\}$, while if $L' \subseteq \Delta^*$, the inverse image of L' under h is defined by $h^{-1}(L') = \{x \in \Sigma^* \ : \ h(x) \in L'\}$.

The shuffle on trajectories operation is a method for specifying the ways in which two input words may be combined to form a result. Each trajectory $t \in \{0,1\}^*$ with $|t|_0 = n$ and $|t|_1 = m$ specifies the manner in which we can form the shuffle on trajectories of word of length n (as the left input word) and m (as the right input word). The word resulting from the shuffle along t will have a letter from the left input word in position i if the i-th symbol of t is 0, and a letter from the right input word in position i if the i-th symbol of t is 1.

We now recall the formal definition of shuffle on trajectories, originally given by Mateescu *et al.* [16]. Shuffle on trajectories is defined by first defining the shuffle of two words x and y over an alphabet Σ on a trajectory $t \in \{0,1\}^*$. We denote the shuffle of x and y along trajectory t by $x \sqcup\!\sqcup_t y$.

If $x = ax'$, $y = by'$ (with $a, b \in \Sigma$) and $t = et'$ (with $e \in \{0,1\}$), then

$$x \sqcup\!\sqcup_{et'} y = \begin{cases} a(x' \sqcup\!\sqcup_{t'} by') & \text{if } e = 0; \\ b(ax' \sqcup\!\sqcup_{t'} y') & \text{if } e = 1. \end{cases}$$

If $x = ax'$ ($a \in \Sigma$), $y = \epsilon$ and $t = et'$ ($e \in \{0,1\}$), then

$$x \sqcup\!\sqcup_{et'} \epsilon = \begin{cases} a(x' \sqcup\!\sqcup_{t'} \epsilon) & \text{if } e = 0; \\ \emptyset & \text{otherwise.} \end{cases}$$

If $x = \epsilon$, $y = by'$ ($b \in \Sigma$) and $t = et'$ ($e \in \{0,1\}$), then

$$\epsilon \sqcup\!\sqcup_{et'} y = \begin{cases} b(\epsilon \sqcup\!\sqcup_{t'} y') & \text{if } e = 1; \\ \emptyset & \text{otherwise.} \end{cases}$$

We let $x \sqcup\!\sqcup_\epsilon y = \emptyset$ if $\{x, y\} \neq \{\epsilon\}$. Finally, if $x = y = \epsilon$, then $\epsilon \sqcup\!\sqcup_t \epsilon = \epsilon$ if $t = \epsilon$ and $\emptyset$ otherwise.

It is not difficult to see that if $t = \prod_{i=1}^n 0^{j_i} 1^{k_i}$ for some $n \geq 0$ and $j_i, k_i \geq 0$ for all $1 \leq i \leq n$, then we have that

$$x \sqcup\!\sqcup_t y = \{\prod_{i=1}^n x_i y_i \ : \ x = \prod_{i=1}^n x_i, y = \prod_{i=1}^n y_i, \text{ with } |x_i| = j_i, |y_i| = k_i \text{ for all } 1 \leq i \leq n\}$$

if $|x| = |t|_0$ and $|y| = |t|_1$ and $x \sqcup\!\sqcup_t y = \emptyset$ if $|x| \neq |t|_0$ or $|y| \neq |t|_1$.

We extend shuffle on trajectories to sets $T \subseteq \{0,1\}^*$ of trajectories as follows:

$$x \sqcup\!\sqcup_T y = \bigcup_{t \in T} x \sqcup\!\sqcup_t y.$$

Further, for $L_1, L_2 \subseteq \Sigma^*$, we define

$$L_1 \sqcup\!\sqcup_T L_2 = \bigcup_{\substack{x \in L_1 \\ y \in L_2}} x \sqcup\!\sqcup_T y.$$

Note that, e.g., if $T = (0+1)^*$, then $\sqcup\!\sqcup_T$ is the usual shuffle operation $\sqcup\!\sqcup$.

We will also require the following definition, introduced independently by the author [2] and Kari and Sosík [12], called *deletion along trajectories*, which models deletion operations controlled by a set of trajectories.

Informally, a trajectory $t \in \{d, i\}^*$ will denote the manner in which a word x may be deleted from a word y. If $|y| = |t|$ and the symbols of x occur in order in the positions of y corresponding to the positions of t equal to d, the resulting word is the symbols of y which occur in the positions corresponding to t equal to i.

Let $x, y \in \Sigma^*$ be words with $x = ax'$, $y = by'$ ($a, b \in \Sigma$). Let $t \in \{i, d\}^*$ such that $t = et'$ with $e \in \{i, d\}$. Then we define $x \rightsquigarrow_t y$, the deletion of y from x along trajectory t, as follows:

$$x \rightsquigarrow_t y = \begin{cases} a(x' \rightsquigarrow_{t'} by') & \text{if } e = i; \\ x' \rightsquigarrow_{t'} y' & \text{if } e = d \text{ and } a = b; \\ \emptyset & \text{otherwise.} \end{cases}$$

Also, if $x = ax'$ ($a \in \Sigma$) and $t = et'$ ($e \in \{i, d\}$), then

$$x \rightsquigarrow_t \epsilon = \begin{cases} a(x' \rightsquigarrow_{t'} \epsilon) & \text{if } e = i; \\ \emptyset & \text{otherwise.} \end{cases}$$

If $x \neq \epsilon$, then $x \rightsquigarrow_\epsilon y = \emptyset$. Further, $\epsilon \rightsquigarrow_t y = \epsilon$ if $t = y = \epsilon$. Otherwise, $\epsilon \rightsquigarrow_t y = \emptyset$.

Let $T \subseteq \{i, d\}^*$. Then

$$x \rightsquigarrow_T y = \bigcup_{t \in T} x \rightsquigarrow_t y.$$

We extend this to languages as expected: Let $L_1, L_2 \subseteq \Sigma^*$ and $T \subseteq \{i, d\}^*$. Then

$$L_1 \rightsquigarrow_T L_2 = \bigcup_{\substack{x \in L_1 \\ y \in L_2}} x \rightsquigarrow_T y.$$

We now come to the main definition of the paper. For $T \subseteq \{0, 1\}^*$, define ω_T as follows: for all $x, y \in \Sigma^*$,

$$x \, \omega_T \, y \iff y \in x \sqcup\!\sqcup_T \Sigma^*.$$

We note that the analogous relation to ω_T for infinite words and ω-trajectories was defined by Kadrie *et al.* [11], and its properties were briefly investigated. While some results are similar (e.g., our Lemma 3.9), most of the results are fundamentally different, and Kadrie *et al.* do not investigate the analogous relations with the same amount of detail as below. Kadrie *et al.* also do not appear to be motivated by coding theory at all.

There has been much research into ω_T for particular $T \subseteq \{0, 1\}^*$, including the following four well-known examples:

(a) for $T = 0^*1^*$, ω_T is the prefix order, given by $x \, \omega_T \, y$ iff $y = xw$ for some $w \in \Sigma^*$;

(b) for $T = 1^*0^*$, ω_T is the suffix order, given by $x \, \omega_T \, y$ iff $y = wx$ for some $w \in \Sigma^*$;

(c) for $T = 1^*0^*1^*$, ω_T is the factor order, given by $x \, \omega_T \, y$ iff $y = y_1 x y_2$ for some $y_1, y_2 \in \Sigma^*$;

(d) for $T = (0 + 1)^*$, ω_T is the embedding order, given by $x \, \omega_T \, y$ iff $y \in x \sqcup\!\sqcup \Sigma^*$.

Several other binary relations have also been defined by, e.g., Kari and Thierrin [13], Long [14] and Thierrin and Yu [22] for the purpose of investigating their associated anti-chains. These relations and their associated anti-chains will be investigated in a companion paper [3].

We immediately note that if $T_1, T_2 \subseteq \{0, 1\}^*$ are sets of trajectories, there is not necessarily a set of trajectories T such that $\omega_T = \omega_{T_1} \cap \omega_{T_2}$, i.e., such that $x \, \omega_T \, y \iff (x \, \omega_{T_1} \, y) \wedge (x \, \omega_{T_2} \, y)$. For instance, for $P = 0^*1^*$ and $S = 1^*0^*$, the relation $\omega_P \cap \omega_S$ is given by $\leq_d$, where $x \leq_d y$ iff there exist $u, v \in \Sigma^*$ such that $y = xu = vx$. This relation cannot be represented by a set of trajectories (for a discussion of $\leq_d$, see Shyr [19, Ch. 8]).

3. Properties of ω_T

We now investigate some of the properties of the binary relations ω_T. In what follows, we will refer to T having a property P iff ω_T has property P.

3.1. Anti-Symmetry

First, we note that ω_T always gives an anti-symmetric binary relation:

Lemma 3.1. Let $T \subseteq \{0,1\}^*$. The relation ω_T is anti-symmetric.

Proof:
Let $x, y \in \Sigma^*$ be such that $x\ \omega_T\ y\ \omega_T\ x$. Then let $t_1, t_2 \in T$ and $\alpha, \beta \in \Sigma^*$ be such that $x \in y \sqcup\!\sqcup_{t_1} \alpha$ and $y \in x \sqcup\!\sqcup_{t_2} \beta$. By definition of shuffle on trajectories, $|x| = |y| + |\alpha|$ and $|y| = |x| + |\beta|$. Thus, $|\alpha| = |\beta| = 0$, i.e., $\alpha = \beta = \epsilon$. But now $x \in y \sqcup\!\sqcup_{t_1} \epsilon$, which implies that $x = y$, again by definition of shuffle on trajectories. □

3.2. Reflexiveness

Lemma 3.2. Let $T \subseteq \{0,1\}^*$. Then T is reflexive iff $0^* \subseteq T$.

Proof:
Let $0^* \subseteq T$. Then $x \in x \sqcup\!\sqcup_{0^{|x|}} \epsilon$, i.e., $x\ \omega_T\ x$. Thus ω_T is reflexive. For the converse, let $x \in x \sqcup\!\sqcup_T \Sigma^*$ for all $x \in \Sigma^*$. Then clearly $0^{|x|} \in T$ for all $x \in \Sigma^*$, which implies $0^* \subseteq T$. □

Corollary 3.1. Given a CF set $T \subseteq \{0,1\}^*$ of trajectories, it is decidable whether T is reflexive.

Proof:
Let $T' = T \cap 0^*$, which is a unary CFL, and thus regular. In fact, if T is effectively context-free, then T' is effectively regular. We can then test the equality $0^* = T'$. □

3.3. Positivity

A binary relation ρ on Σ^* is said to be positive if $\epsilon\ \rho\ x$ for all $x \in \Sigma^*$.

Lemma 3.3. Let $T \subseteq \{0,1\}^*$. Then T is positive iff $1^* \subseteq T$.

Proof:
Let $1^* \subseteq T$. Then $u \in \epsilon \sqcup\!\sqcup_{1^{|u|}} u$ for all $u \in \Sigma^*$, whereby $\epsilon\ \omega_T\ u$, as $1^{|u|} \in T$. The reverse implication is similarly established. □

Corollary 3.2. Given a CF set $T \subseteq \{0,1\}^*$ of trajectories, it is decidable whether T is positive.

3.4. ST-Strictness

Shyr and Thierrin [21] define the concept of a *strict* binary relation. To avoid confusion with the concept of a *strict ordering* (see, e.g., Choffrut and Karhumäki [1, Sect. 7.1]), we will call a binary relation ρ on Σ^* *ST-strict* if it satisfies the following four properties:

(a) ρ is reflexive;

(b) ρ is positive;

(c) for all $u, v \in \Sigma^*$, $u \,\rho\, v$ implies $|u| \le |v|$;

(d) for all $u, v \in \Sigma^*$, $u \,\rho\, v$ and $|u| = |v|$ implies $u = v$.

We now consider T such that ω_T is ST-strict. We first note that conditions (c) and (d) are satisfied by all T. Indeed, if $u \,\omega_T\, v$, then $v \in u \sqcup\!\sqcup_T \Sigma^*$, which implies that $|v| \ge |u|$. Further, if $|u| = |v|$, then $u \,\omega_T\, v$ implies that $v \in u \sqcup\!\sqcup_T \epsilon$, which implies that $u = v$.

Thus, as we already have necessary and sufficient conditions on T being reflexive and positive, the following results are immediate:

Corollary 3.3. Let $T \subseteq \{0,1\}^*$. Then T is ST-strict iff $0^* + 1^* \subseteq T$.

Corollary 3.4. Given a CF set $T \subseteq \{0,1\}^*$ of trajectories, it is decidable whether T is ST-strict.

3.5. Cancellativity

A binary relation ρ is said to be *left-cancellative* (resp., *right-cancellative*) if $uv \,\rho\, ux$ implies $v \,\rho\, x$ (resp., $vu \,\rho\, xu$ implies $v \,\rho\, x$). The relation ρ is *cancellative* if it is both left- and right-cancellative.

Given $T \subseteq \{0,1\}^*$, we define two sets of trajectories, $s(T), p(T) \subseteq \{0,1\}^*$, as follows:

$$\begin{aligned} p(T) &= \{t_1 1^j \;:\; t_1 t_2 \in T, 0 \le j \le |t_2|\}, \\ s(T) &= \{1^j t_2 \;:\; t_1 t_2 \in T, 0 \le j \le |t_1|\}. \end{aligned}$$

Lemma 3.4. Let $T \subseteq \{0,1\}^*$. Then T is left-cancellative (resp., right-cancellative) if $s(T) \subseteq T$ (resp., $p(T) \subseteq T$).

Proof:
We establish the result for left-cancellativity only; the other case is symmetric. Let $s(T) \subseteq T$. Then let $u, v, x \in \Sigma^*$ be such that $uv \,\omega_T\, ux$. Let $t \in T$ and $\alpha \in \Sigma^*$ be chosen so that $ux \in uv \sqcup\!\sqcup_t \alpha$. Write $t = t_1 t_2$ and $\alpha = \alpha_1 \alpha_2$ so that $ux \in (u \sqcup\!\sqcup_{t_1} \alpha_1)(v \sqcup\!\sqcup_{t_2} \alpha_2)$. Let $\beta_1, \beta_2 \in \Sigma^*$ be chosen so that $\beta_1 \in u \sqcup\!\sqcup_{t_1} \alpha_1$, $\beta_2 \in v \sqcup\!\sqcup_{t_2} \alpha_2$ and $ux = \beta_1 \beta_2$. As $|\beta_1| = |u| + |\alpha_1| \ge |u|$, there exists $\gamma \in \Sigma^*$ such that $u\gamma = \beta_1$ and $x = \gamma\beta_2$. Note that $|\gamma| \le |\beta_1| = |t_1|$. Thus,

$$x \in \gamma(v \sqcup\!\sqcup_{t_2} \alpha_2).$$

Let $t_3 = 1^{|\gamma|} t_2 \in s(T)$. By assumption, $t_3 \in T$. Further,

$$x \in v \sqcup\!\sqcup_{t_3} \gamma\alpha_2.$$

We conclude that $v \,\omega_T\, x$. □

Corollary 3.5. Let $T \subseteq \{0,1\}^*$. If $s(T) \cup p(T) \subseteq T$, then T is cancellative.

We now consider a condition of Jürgensen *et al.* [10]. Say that a binary relation ρ on Σ^* is *leviesque* if $uv \,\rho\, xy$ implies that $u \,\rho\, x$ or $v \,\rho\, y$, for all $u, v, x, y \in \Sigma^*$.

Lemma 3.5. Let $T \subseteq \{0,1\}^*$. If $s(T) \cup p(T) \subseteq T$, then T is leviesque.

Proof:
Let $rs\,\omega_T\,xy$. Then there exist $t \in T$ and $\alpha \in \Sigma^*$ such that $xy \in rs \sqcup\!\sqcup_t \alpha$. Then there exist factorizations $t = t_1 t_2$, $\alpha = \alpha_1\alpha_2$ such that $xy \in (r \sqcup\!\sqcup_{t_1} \alpha_1)(s \sqcup\!\sqcup_{t_2} \alpha_2)$. Let $\beta_1, \beta_2 \in \Sigma^*$ be such that $\beta_1 \in r \sqcup\!\sqcup_{t_1} \alpha_1$, $\beta_2 \in s \sqcup\!\sqcup_{t_2} \alpha_2$ and $xy = \beta_1\beta_2$. There are two cases:

(i) If $|x| \geq |\beta_1|$, then there exists $\gamma \in \Sigma^*$ such that $x = \beta_1\gamma$ and $\gamma y = \beta_2$. Note that $|\gamma| \leq |\beta_2| = |t_2|$. Consider that $x = \beta_1\gamma \in (r \sqcup\!\sqcup_{t_1 1^{|\gamma|}} \alpha_1\gamma)$. As $t_1 1^{|\gamma|} \in p(T) \subseteq T$, $r\,\omega_T\,x$.

(ii) If $|x| \leq |\beta_1|$, there exists $\gamma \in \Sigma^*$ such that $x\gamma = \beta_1$ and $y = \gamma\beta_2$. Note that $|\gamma| \leq |\beta_1| = |t_1|$. In this case, $y = \gamma\beta_2 \in (s \sqcup\!\sqcup_{1^{|\gamma|} t_2} \gamma\alpha_2)$. Thus, as $1^{|\gamma|} t_2 \in s(T) \subseteq T$, we have $s\,\omega_T\,y$.

Thus, $rs\,\omega_T\,xy$ implies $(r\,\omega_T\,x)$ or $(s\,\omega_T\,y)$. □

3.6. Compatibility

Let ρ be a binary relation on Σ^*. Then we say that ρ is *left-compatible* (resp., *right-compatible*) if, for all $u, v, w \in \Sigma^*$, $u\,\rho\,v$ implies that $wu\,\rho\,wv$ (resp., $uw\,\rho\,vw$). If ρ is both left- and right-compatible, we say it is *compatible*.

Lemma 3.6. Let $T \subseteq \{0,1\}^*$. Then T is right-compatible (resp., left-compatible) iff $T0^* \subseteq T$ (resp., $0^*T \subseteq T$).

Proof:
We establish the result for right-compatibility. The result for left-compatibility is symmetrical.

Let $T0^* \subseteq T$. Let $u, v, w \in \Sigma^*$ with $u\omega_T v$. Then there exist $t \in T$ and $\alpha \in \Sigma^*$ such that $v \in u \sqcup\!\sqcup_t \alpha$. As $t' = t0^{|w|} \in T$, $vw \in uw \sqcup\!\sqcup_{t'} \alpha$. Thus $uw\,\omega_T\,vw$.

Assume that $T0^*$ is not a subset of T. Then there exist $t \in T$ and $i \in \mathbb{N}$ such that $t0^i \notin T$. Let $j = |t|_0$ and $k = |t|_1$. Consider that $0^j\,\omega_T\,t$, as $t \in 0^j \sqcup\!\sqcup_t 1^k$. However, $0^j \cdot 0^i \not\omega_T\, t \cdot 0^i$, as $t0^i \in 0^{j+i} \sqcup\!\sqcup_T 1^k$ would imply that $t0^i \in T$. Thus, T is not right-compatible. □

The following corollary is immediate; it is identical to the condition that $T0^* \cup 0^*T \subseteq T$:

Corollary 3.6. Let $T \subseteq \{0,1\}^*$. Then T is compatible iff $0^*T0^* \subseteq T$.

Corollary 3.7. Given a regular set $T \subseteq \{0,1\}^*$ of trajectories, it is decidable whether T is (left- or right-) compatible.

To prove the undecidability of compatibility for LCFLs, we will apply a meta-theorem of Hunt and Rosenkrantz [8, Thm. 2.10], which we recall first:

Theorem 3.1. Let P be a predicate on LCF over Σ^* such that $P(\Sigma^*)$ holds and the set

$$\{L' \;:\; L' = x\text{Ł}, x \in \Sigma^+, L \in \text{LCF and } P(L)\}$$

is a proper subset of LCF. Then given a LCFG G, it is undecidable whether $P(L(G))$ holds.

Lemma 3.7. Given a set of trajectories $T \in$ LCF, it is undecidable whether T is (left- or right-) compatible.

Proof:
We prove only the case for left-compatibility; the other cases are similar and are left to the reader. First, we note that $T = \{0,1\}^*$ is left-compatible.

Let $T = \{0^n 1^n \ : \ n \geq 0\}$. We claim that there is no LCFL $T' \subseteq \{0,1\}^*$ and $t \in \{0,1\}^+$ such that $T = T'/t$. Assume that there were such T', t. Then as $\epsilon \in T = T'/t$, we must have $t \in T'$. As T' is left-compatible, we have that $0t \in T'$. Thus $0 \in T'/t = T$, a contradiction. Thus, the set

$$\{T \ : \ \exists \text{ left-compatible LCF } T' \subseteq \{0,1\}^*, t \in \{0,1\}^+ \text{ such that } T = T'/t\}$$

is a proper subset of the LCFLs. Therefore, we may apply Theorem 3.1, and it is undecidable whether a given LCF set of trajectories is left-compatible. □

3.7. Transitivity

We now consider conditions on T which will ensure that ω_T is a transitive relation. Transitivity is often, but not always, a property of the binary relations defining the classic code classes. For instance, both bi-prefix and outfix codes are defined by binary relations which are not transitive, and hence not partial orders. We now give necessary and sufficient conditions on a set T of trajectories defining a transitive binary relation, and examine the associated decidability problem.

First, we define three morphisms we will need. Let $D = \{x, y, z\}$ and $\varphi, \sigma, \psi : D^* \to \{0,1\}^*$ be the morphisms given by

$$\begin{array}{rclrclrcl} \varphi(x) &=& 0, & \sigma(x) &=& 0, & \psi(x) &=& 0, \\ \varphi(y) &=& 0, & \sigma(y) &=& 1, & \psi(y) &=& 1, \\ \varphi(z) &=& 1, & \sigma(z) &=& \epsilon, & \psi(z) &=& 1. \end{array}$$

Note that these morphisms are similar to the substitutions defined by Mateescu *et al.* [16], whose purpose is to give necessary and sufficient conditions on a set T of trajectories defining an associative operation. Indeed, our condition is a weakening of their conditions, which, intuitively, reflects the fact that any associative operation defines a transitive binary relation (note, however, that $T = 1^*0^*1^*$ is transitive but not associative).

Theorem 3.2. Let $T \subseteq \{0,1\}^*$. Then T is transitive iff

$$\psi(\varphi^{-1}(T) \cap \sigma^{-1}(T)) \subseteq T. \tag{1}$$

Proof:
($\Leftarrow$): Let T define a transitive binary relation. Let $w \in \psi(\varphi^{-1}(T) \cap \sigma^{-1}(T))$. Then there exist $t_1, t_2 \in T$ such that $w \in \psi(\varphi^{-1}(t_1) \cap \sigma^{-1}(t_2))$. Let $t \in \varphi^{-1}(t_1) \cap \sigma^{-1}(t_2)$ be chosen so that $w \in \psi(t)$.

Consider t_1. Let $n \in \mathbb{N}$ and $\alpha_i, \beta_i \in \mathbb{N}$ be chosen for $1 \leq i \leq n$ so that

$$t_1 = \prod_{i=1}^{n} 0^{\alpha_i} 1^{\beta_i}.$$

Note that

$$\varphi^{-1}(t_1) = \prod_{i=1}^{n} (x+y)^{\alpha_i} z^{\beta_i}.$$

As $t \in \varphi^{-1}(t_1)$, and $t \in \sigma^{-1}(t_2)$, by definition of σ, we must have that $t_2 = \prod_{i=1}^n s_i$ for $s_i \in \{0,1\}^*$ satisfying $|s_i| = \alpha_i$. Thus, we have that $|t_2| = |t_1|_0$. Furthermore, $t \in (x^{p_1} \sqcup\!\sqcup_{t_2} y^{p_2}) \sqcup\!\sqcup_{t_1} z^{p_3}$, where $p_1 = |t_2|_0$, $p_2 = |t_2|_1$ and $p_3 = |t_1|_1$. Consider now that $w \in \psi(t)$, so that

$$w \in (0^{p_1} \sqcup\!\sqcup_{t_2} 1^{p_2}) \sqcup\!\sqcup_{t_1} 1^{p_3}.$$

Clearly, $0^{p_1} \sqcup\!\sqcup_{t_2} 1^{p_2} = t_2$. Thus, $w \in t_2 \sqcup\!\sqcup_{t_1} 1^{p_3}$, as well. By definition, we then have that $0^{p_1}\, \omega_T\, t_2\, \omega_T\, w$. By the transitivity of T, $0^{p_1}\, \omega_T\, w$, i.e.,

$$w \in 0^{p_1} \sqcup\!\sqcup_T \{0,1\}^*.$$

Note that $|w|_1 = p_2 + p_3$ and $|w|_0 = p_1$. The only word v over $\{0,1\}$ such that $w \in 0^{p_1} \sqcup\!\sqcup_T v$ is $v = 1^{p_2+p_3}$ (regardless of T). That is, $w \in 0^{p_1} \sqcup\!\sqcup_T 1^{p_2+p_3}$. But from this, we must have that $w \in T$. Thus, we have that $\psi(\varphi^{-1}(T) \cap \sigma^{-1}(T)) \subseteq T$.

($\Rightarrow$): Assume that $\psi(\varphi^{-1}(T) \cap \sigma^{-1}(T)) \subseteq T$. Let $u, v, w \in \Sigma^*$ be such that $u\, \omega_T\, v$ and $v\, \omega_T\, w$. We wish to show that $u\, \omega_T\, w$. Let $t_1, t_2 \in T$ and $\theta_1, \theta_2 \in \Sigma^*$ be such that $w \in v \sqcup\!\sqcup_{t_1} \theta_1$ and $v \in u \sqcup\!\sqcup_{t_2} \theta_2$. Thus, $w \in (u \sqcup\!\sqcup_{t_2} \theta_2) \sqcup\!\sqcup_{t_1} \theta_1$. Note then that $|t_1|_0 = |t_2|$. Let $n \in \mathbb{N}$ and $\alpha_i, \beta_i \in \mathbb{N}$ be chosen for $1 \le i \le n$ so that

$$t_1 = \prod_{i=1}^n 0^{\alpha_i} 1^{\beta_i}.$$

Furthermore, let $t_2 = \prod_{i=1}^n s_i$ be so that $|s_i| = \alpha_i$ for all $1 \le i \le n$. For all $1 \le i \le n$, let η_i be the word obtained from s_i by replacing 0 with x and 1 with y, i.e., $\{\eta_i\} = \sigma^{-1}(s_i) \cap \{x,y\}^*$. Then let

$$t = \prod_{i=1}^n \eta_i z^{\beta_i}.$$

We can verify that $\varphi(t) = t_1$ and $\sigma(t) = t_2$. Thus, $t \in \varphi^{-1}(t_1) \cap \sigma^{-1}(t_2)$. Let $t' = \psi(t)$. By assumption, $t' \in T$, and we further note that

$$t' = \prod_{i=1}^n s_i 1^{\beta_i}.$$

We now define a morphism $h : D^* \to \{0,1\}^*$ given by $h(x) = \epsilon$, $h(y) = 0$ and $h(z) = 1$. Let $\theta \in \theta_2 \sqcup\!\sqcup_{h(t)} \theta_1$. Then we can verify that $w \in u \sqcup\!\sqcup_{t'} \theta \subseteq u \sqcup\!\sqcup_T \Sigma^*$. Thus, $u\, \omega_T\, w$ as required. □

Corollary 3.8. Given a regular set $T \subseteq \{0,1\}^*$ of trajectories, it is decidable whether T is transitive.

Proof:
Since the regular languages are closed under morphism and inverse morphism, and inclusion of regular languages is decidable, we can determine whether the inclusion (1) holds. □

The following decidability result also holds, since we can determine whether $T \supseteq 0^*$ and (1) holds if T is regular:

Corollary 3.9. Given a regular set $T \subseteq \{0,1\}^*$ of trajectories, it is decidable whether ω_T is a partial order.

We now turn to undecidability:

Theorem 3.3. Given a CF set $T \subseteq \{0,1\}^*$ of trajectories, it is undecidable whether T is transitive.

Proof:
Let $P = (u_1, u_2, \dots, u_n; v_1, v_2, \dots, v_n)$ be a PCP instance. Define

$$\begin{aligned} L_1 &= \{01^{i_1}01^{i_2}\cdots 01^{i_m}0^{n+1}1^{n+1}u_{i_m}u_{i_{m-1}}\cdots u_{i_1} \ : \ m \geq 1, 1 \leq i_p \leq n, 1 \leq p \leq m\}; \\ L_2 &= \{01^{i_1}01^{i_2}\cdots 01^{i_m}0^{n+1}1^{n+1}v_{i_m}v_{i_{m-1}}\cdots v_{i_1} \ : \ m \geq 1, 1 \leq i_p \leq n, 1 \leq p \leq m\}. \end{aligned}$$

Let $K = L_1 \cap L_2$. It is easy to see that P has a solution iff $K \neq \emptyset$. Let $T = \{0,1\}^* - K$. Thus, P has no solutions iff $T = \{0,1\}^*$. It is easily verified that that T is a CFL.

We now show that P has no solutions iff T is transitive. If P has no solutions, then clearly $T = \{0,1\}^*$ is transitive.

Assume that P has a solution. Then there is some word

$$t = 01^{i_1}01^{i_2}\cdots 01^{i_m}0^{n+1}1^{n+1}u_{i_m}u_{i_{m-1}}\cdots u_{i_1} \notin T.$$

Note that $(L_1 \cup L_2) \cap 0^2(0+1)^* = \emptyset$, since $m \geq 1$, and $i_p \geq 1$ for all $1 \leq p \leq m$. Thus, we have that

$$t_1 = 001^{i_1-1}01^{i_2}\cdots 01^{i_m}0^{n+1}1^{n+1}u_{i_m}u_{i_{m-1}}\cdots u_{i_1} \notin K \subseteq L_1 \cup L_2.$$

Thus $t_1 \in T$. Let $\alpha = |t_1|_0$. Note that as $n \geq 1$, certainly $|x|_1 \geq 2$ for all $x \in L_1 \cup L_2$. Thus, we have $t_2 = 010^{\alpha-2} \in T$.

Assume now that T is transitive, contrary to what we want to prove. By (1), as $t_1, t_2 \in T$, we must have that $\psi(\varphi^{-1}(t_1) \cap \sigma^{-1}(t_2)) \subseteq T$. But it is easy to verify that $t \in \psi(\varphi^{-1}(t_1) \cap \sigma^{-1}(t_2))$, which is a contradiction. Thus, T is not transitive.

Therefore P has a solution iff T is not transitive, and we conclude that it is undecidable whether T is transitive. □

Consider, by (1), or by direct observation, that if $\{T_i\}_{i \in I}$ is a family of transitive sets of trajectories, then the set $\cap_{i \in I} T_i$ is also transitive. Thus, we can define the transitive closure of a set T of trajectories as follows: for all $T \subseteq \{0,1\}^*$, let $tr(T) = \{T' \subseteq \{0,1\}^* \ : \ T \subseteq T', T' \text{ transitive}\}$. Note that $tr(T) \neq \emptyset$, as $\{0,1\}^* \in tr(T)$ for all $T \subseteq \{0,1\}^*$. Define $\hat{T}$ as

$$\hat{T} = \bigcap_{T' \in tr(T)} T'.$$

Then note that $\hat{T}$ is transitive and is the smallest transitive set of trajectories containing T. The operation $\hat{\cdot} : 2^{\{0,1\}^*} \to 2^{\{0,1\}^*}$ is indeed a closure operator (much like the closure operators on sets of trajectories constructed by Mateescu *et al.* [16] for, e.g., associativity and commutativity) in the algebraic sense, since $T \subseteq \hat{T}$, and $\hat{\cdot}$ preserves inclusion and is idempotent.

A partial order is said to be a *division ordering* [1] if it is positive and compatible.

Lemma 3.8. Let $T \subseteq \{0,1\}^*$ be a partial order. If T is a division ordering, then $T = (0+1)^*$.

Proof:
Let $O = 0^*1^*0^*$. As T is positive and compatible, then $T \supseteq 1^*$ and $T \supseteq 0^*T0^*$. Thus, $T \supseteq O$. As T is a partial order, then T is transitive. Thus, $T = \hat{T} \supseteq \hat{O} = H$. The result follows. $\square$

Consider the operator $\Omega_T : 2^{\{0,1\}^*} \to 2^{\{0,1\}^*}$ given by

$$\Omega_T(T') = T \cup T' \cup \psi(\sigma^{-1}(T') \cap \varphi^{-1}(T')).$$

By definition of Ω_T, any fixed point $\Omega_T(T_0) = T_0$ contains T and is transitive. Then we have

$$\emptyset \subseteq \Omega_T(\emptyset) = T \subseteq \Omega_T^2(T) \subseteq \Omega_T^3(T) \subseteq \cdots \subseteq \hat{T}.$$

Since the operations of ϵ-free morphism, inverse morphism, union and intersection are monotone and continuous [17], Ω_T is monotone and continuous and thus $\hat{T}$ is the least upper bound of $\{\Omega_T^i(\emptyset)\}_{i\geq 0}$. Thus, given T, we can find $\hat{T}$ by iteratively applying Ω_T to T, and in fact

$$\hat{T} = \bigcup_{i\geq 0} \Omega_T^i(T).$$

This observation allows us to construct $\hat{T}$, and, for instance, gives us the following result (a similar result for ω-trajectories is given by Kadrie *et al.* [11]):

Lemma 3.9. There exists a regular set of trajectories $T \subseteq \{0,1\}^*$ such that $\hat{T}$ is not a CFL.

Proof:
Consider $T = (01)^*$, corresponding to perfect or balanced literal shuffle. Then we note that $\hat{T} \cap 01^* = \{01^{2^n-1} \,:\, n \geq 1\}$. $\square$

Open Problem 3.1. Given $T \in$ REG (or $T \in$ CF), is it decidable whether $\hat{T} \in$ CF?

3.8. Monotonicity

A binary relation ρ on Σ^* is said to be *monotone* (see, e.g, Ehrenfeucht *et al.* [4, p. 315]) if $x \,\omega_T\, y$ and $u \,\omega_T\, v$ implies $xu \,\omega_T\, yv$ for all $x, y, u, v \in \Sigma^*$. Occasionally, the concept of monotonicity is included as a requirement in compatibility, but we separate the two concepts here for clarity. We note that monotone here is a condition on T, rather than the monotonicity of the operation $\sqcup\!\sqcup_T$, which holds for all T.

Lemma 3.10. Let $T \subseteq \{0,1\}^*$. Then T is monotone iff $T^2 \subseteq T$ iff $T = T^*$.

Proof:
The fact that $T^2 \subseteq T$ iff $T = T^*$ is obvious. Thus, we establish that T is monotone iff $T^2 \subseteq T$.

Assume that $T^2 \subseteq T$. Let $x_i \,\omega_T\, y_i$ for $i = 1, 2$. Let $t_i \in T$ and $\alpha_i \in \Sigma^*$ be chosen so that $y_i \in x_i \sqcup\!\sqcup_{t_i} \alpha_i$ for $i = 1, 2$. Then as $t_1t_2 \in T$, we have the fact that $y_1y_2 \in x_1x_2 \sqcup\!\sqcup_{t_1t_2} \alpha_1\alpha_2$ implies that $x_1x_2 \,\omega_T\, y_1y_2$. Thus T is monotone.

Assume that T is monotone. Let $t_1, t_2 \in T$ be arbitrary. Let $n_i = |t_i|_0$ and $m_i = |t_i|_1$ for $i = 1, 2$. Thus, we have that $0^{n_i} \,\omega_T\, t_i$ for $i = 1, 2$. By the monotonicity of T, $0^{n_1+n_2} \,\omega_T\, t_1t_2$. Thus, there exist $t \in T$ and $\alpha \in \{0,1\}^*$ such that $t_1t_2 \in 0^{n_1+n_2} \sqcup\!\sqcup_t \alpha$. But it is now clear that $\alpha = 1^{m_1+m_2}$ and $t = t_1t_2$. Thus $t_1t_2 \in T$ and $T^2 \subseteq T$. $\square$

The following corollary is immediate, since it is decidable whether $T^* = T$ for regular languages.

Corollary 3.10. Given a regular set T of trajectories, it is decidable whether T is monotone.

We also have the following undecidability result:

Lemma 3.11. Given a LCF set $T \subseteq \{0,1\}^*$ of trajectories, it is undecidable whether T is monotone.

Proof:
We apply Theorem 3.1. First, we note that $T = \{0,1\}^*$ is monotone. Further, we note that the LCF set of trajectories $T = \{0^n 1^n \ : \ n \geq 0\}$ is not expressible as $T = T'/t$ for any monotone $T' \subseteq \{0,1\}^*$ and $t \in \{0,1\}^+$ (Indeed, if this were the case, then as $\epsilon \in T$, $t \in T'$. As $T' = (T')^*$, we have that $t^2, t^3 \in T'$ and $t, t^2 \in T$. But the only way this can happen is if $t = \epsilon$). Therefore, it is undecidable whether a given LCF set of trajectories is monotone. □

We can now consider the monotone closure of a set T of trajectories, much in the same way we considered the transitive closure in Section 3.7. However, we do not need the same level of detail, since it is clear that the monotone closure of T is T^*. Thus, we have the following result:

Lemma 3.12. Let $T \subseteq \{0,1\}^*$ be a regular (resp., CF, CS, recursive) set of trajectories. Then the monotone closure of T is also a regular (resp., CF, CS, recursive) set of trajectories.

3.9. Well-Foundedness

A partial order ρ is said to be *well-founded* (see, e.g., Choffrut and Karhumäki [1, Sect. 7.1]) if every strictly descending chain under ρ is finite. We note that for relations defined by sets of trajectories, well-foundedness is implied by partial orders (and even by reflexive binary relations):

Theorem 3.4. Let $T \subseteq \{0,1\}^*$ be a partial order. Then ω_T is a well-founded partial order.

Proof:
Let T be a partial order. Then T is reflexive. Let $\{w_i\}_{i \geq 1}$ be a descending chain, i.e., $w_{i+1} \ \omega_T \ w_i$ for all $i \geq 1$. Then $|w_{i+1}| \leq |w_i|$ for all $i \geq 1$. Let $K = |w_1|$. Thus, $|w_i| \leq K$ for all $i \geq 1$. Thus, there exists $j \geq 1$ such that $|w_j| = |w_{j+1}|$. In particular, this implies that $w_j = w_{j+1}$, and so by the reflexivity of T, $w_j \ \omega_T \ w_{j+1}$. Thus, $\{w_i\}_{i \geq 1}$ is not an infinite strictly descending chain. □

4. Convexity

We call a language $L \subseteq \Sigma^*$ *T-convex* if, for all $y \in \Sigma^*$ and $x, z \in L$, $x \ \omega_T \ y$ and $y \ \omega_T \ z$ implies $y \in L$.

We will require the following result. Let $\tau : \{0,1\}^* \to \{i,d\}^*$ be given by $\tau(0) = i$ and $\tau(1) = d$. The following lemma is independently due to the author [2] and Kari and Sosík [12]:

Lemma 4.1. Let $T \subseteq \{0,1\}^*$. Then for all $x, y, z \in \Sigma^*$, $x \in y \sqcup\!\sqcup_T z$ iff $y \in x \rightsquigarrow_{\tau(T)} z$.

We now characterize when a language is T-convex using shuffle and deletion on trajectories.

Lemma 4.2. Let $T \subseteq \{0,1\}^*$. Then $L \subseteq \Sigma^*$ is T-convex iff $(L \sqcup\!\sqcup_T \Sigma^*) \cap (L \rightsquigarrow_{\tau(T)} \Sigma^*) \subseteq L$.

Proof:
Let L be a T-convex language. Consider an arbitrary word $x \in (L \sqcup\!\sqcup_T \Sigma^*) \cap (L \rightsquigarrow_{\tau(T)} \Sigma^*)$. Then there exist $y_1, y_2 \in L$ such that $x \in y_1 \sqcup\!\sqcup_T \Sigma^*$ and $x \in y_2 \rightsquigarrow_{\tau(T)} \Sigma^*$. By Lemma 4.1, we have that $y_2 \in x \sqcup\!\sqcup_T \Sigma^*$. Thus, $y_1 \, \omega_T \, x \, \omega_T \, y_2$. By the T-convexity of L, $x \in L$. Thus, the inclusion is established.

The reverse implication is similar. Let $(L \sqcup\!\sqcup_T \Sigma^*) \cap (L \rightsquigarrow_{\tau(T)} \Sigma^*) \subseteq L$. Let $y_1, y_2 \in L$ and $x \in \Sigma^*$ be such that $y_1 \, \omega_T \, x \, \omega_T \, y_2$. Then $x \in y_1 \sqcup\!\sqcup_T \Sigma^*$ and $y_2 \in x \sqcup\!\sqcup_T \Sigma^*$. Again, Lemma 4.1 implies that $x \in y_2 \rightsquigarrow_{\tau(T)} \Sigma^*$. Thus, $x \in L$, by our assumed inclusion, and L is T-convex. □

Corollary 4.1. Let $T \subseteq \{0,1\}^*$ be reflexive. Then $L \subseteq \Sigma^*$ is T-convex iff $(L \sqcup\!\sqcup_T \Sigma^*) \cap (L \rightsquigarrow_{\tau(T)} \Sigma^*) = L$.

Proof:
We show that if T is reflexive, then for all $L \subseteq \Sigma^*$,

$$L \subseteq (L \sqcup\!\sqcup_T \Sigma^*) \cap (L \rightsquigarrow_{\tau(T)} \Sigma^*). \tag{2}$$

If T is reflexive, then $0^* \subseteq T$ and $(L \sqcup\!\sqcup_T \Sigma^*) \supseteq (L \sqcup\!\sqcup_{0^*} \{\epsilon\}) = L$. Further, if $T \supseteq 0^*$ then $\tau(T) \supseteq i^*$ and $(L \rightsquigarrow_{\tau(T)} \Sigma^*) \supseteq (L \rightsquigarrow_{i^*} \{\epsilon\}) = L$. Thus, we have established (2). □

We now turn to decidability:

Corollary 4.2. Let $T \subseteq \{0,1\}^*$ be a regular set of trajectories. Given a regular language L, it is decidable whether L is T-convex.

Proof:
As L, T are regular, so are $L \sqcup\!\sqcup_T \Sigma^*$, $L \rightsquigarrow_{\tau(T)} \Sigma^*$ and $(L \sqcup\!\sqcup_T \Sigma^*) \cap (L \rightsquigarrow_{\tau(T)} \Sigma^*)$. Thus, the inclusion in Lemma 4.2 is decidable. □

The interesting question of the complexity of T-convex languages will be examined in the companion paper [3].

5. Further Work

We have yet to find necessary and sufficient conditions for a set of trajectories to be left-cancellative. Further work on this property is necessary in order to obtain decidability results.

We can also consider the class of codes defined not by shuffle on trajectories, but by *splicing on routes*, a generalization of shuffle on trajectories introduced by Mateescu [15]. We define splicing on routes informally as follows: we expand our trajectory alphabet to $\{0, 1, \overline{0}, \overline{1}\}$, with $0, 1$ retaining the same meaning, and $\overline{0}, \overline{1}$ meaning "discard the current symbol from the input word, and continue". For example, if $x = abc$, $y = def$ and $t = 01\overline{0}\overline{0}1\overline{1}$, then we denote the splicing of x and y along route t by $x \bowtie_t y$, and it is given by $x \bowtie_t y = ade$.

We note that additional interesting binary relations from the literature can be modeled using splicing on routes. For instance, we leave it to the reader to verify that if

$$T = 0^* + (\overline{0}1)^* 1^+$$

then the associated binary relation (defined in the same way as for shuffle on trajectories) is the length ordering, given by

$$x \leq y \iff (|x| < |y|) \text{ or } x = y.$$

In a companion paper [3] we will investigate anti-chains under ω_T. Particular instances of these anti-chains are the prefix-, suffix-, infix-, outfix-, shuffle- and hyper-codes. The results obtained in this paper are useful in proving properties about these generalized codes.

6. Conclusions

We have investigated the properties of the relation ω_T, which is a generalization of the embedding order. Characterizations of the properties of the relation ω_T are investigated, as well as their associated decision problems. We have also defined the notion of convex languages under the relation ω_T, and shown that given a regular set of trajectories and a regular language L, we can determine if L is convex with respect to T.

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References

[1] Choffrut, C., Karhumäki, J.: Combinatorics on Words, 329–438, In [18].

[2] Domaratzki, M.: Deletion along trajectories, *Accepted, Theor. Comput. Sci.*, 2004.

[3] Domaratzki, M.: Trajectory-based codes, *Accepted, Acta Inf.*, 2004.

[4] Ehrenfeucht, A., Haussler, D., Rozenberg, G.: On regularity of context-free languages, *Theor. Comp. Sci.*, **23**, 1983, 311–332.

[5] Guo, Y., Shyr, H., Thierrin, G.: E-Convex infix codes, *Order*, **3**, 1986, 55–59.

[6] Haines, L.: On free monoids partially ordered by embedding, *J. Comb. Th.*, **6**, 1969, 94–98.

[7] Hopcroft, J. E., Ullman, J. D.: *Introduction to Automata Theory, Languages, and Computation*, Addison-Wesley, 1979.

[8] Hunt, H., Rosenkrantz, D.: Computational parallels between the regular and context-free languages, *SIAM J. Comput.*, **7**(1), 1978, 99–114.

[9] Jullien, P.: Sur un théorème d'extension dans la théorie des mots, *CR Acad. Sc. Paris (Série A)*, **266**, 1968, 851–854.

[10] Jürgensen, H., Shyr, H., Thierrin, G.: Codes and compatible partial orders on free monoids, *Algebra and Order: Proceedings of the First International Symposium on Ordered Algebraic Structures, Luminy–Marseilles 1984* (S. Wolfenstein, Ed.), Heldermann Verlag, 1986.

[11] Kadrie, A., Dare, V., Thomas, D., Subramanian, K.: Algebraic properties of the shuffle over ω-trajectories, *Inf. Proc. Letters*, **80**(3), 2001, 139–144.

[12] Kari, L., Sosík, P.: *Language deletions on trajectories*, Technical Report 606, Computer Science Department, University of Western Ontario, 2003.

[13] Kari, L., Thierrin, G.: k-catenation and applications: k-prefix codes, *J. Inf. Opt. Sci.*, **16**(2), 1995, 263–276.

[14] Long, D.: On two infinite hierarchies of prefix codes, *Proceedings of the Conference on Ordered Structures and Algebra of Computer Languages* (K. Shum, P. Yuen, Eds.), World Scientific, 1993.

[15] Mateescu, A.: Splicing on routes: a framework of DNA computation, *Unconventional Models of Computation* (C. Calude, J. Casti, M. Dinneen, Eds.), Springer, 1998.

[16] Mateescu, A., Rozenberg, G., Salomaa, A.: Shuffle on trajectories: Syntactic constraints, *Theor. Comp. Sci.*, **197**, 1998, 1–56.

[17] Okhotin, A.: Personal communication, September 2003.

[18] Rozenberg, G., Salomaa, A., Eds.: *Handbook of Formal Languages, Vol. I*, Springer-Verlag, 1997.

[19] Shyr, H.: *Free Monoids and Languages*, Hon Min Book Company, Taichung, Taiwan, 2001.

[20] Shyr, H., Thierrin, G.: Hypercodes, *Inf. and Cont.*, **24**(1), 1974, 45–54.

[21] Shyr, H., Thierrin, G.: Codes and binary relations, *Séminaire d'Algébre Paul Dubreil, Paris 1975–1976* (A. Dold, B. Eckmann, Eds.), Lecture Notes in Mathematics 586, Springer-Verlag, 1977.

[22] Thierrin, G., Yu, S. S.: Shuffle relations and codes, *J. Inf. Opt. Sci.*, **12**(3), 1991, 441–449.

[23] Yu, S.: Regular Languages, 41–110, in [18].